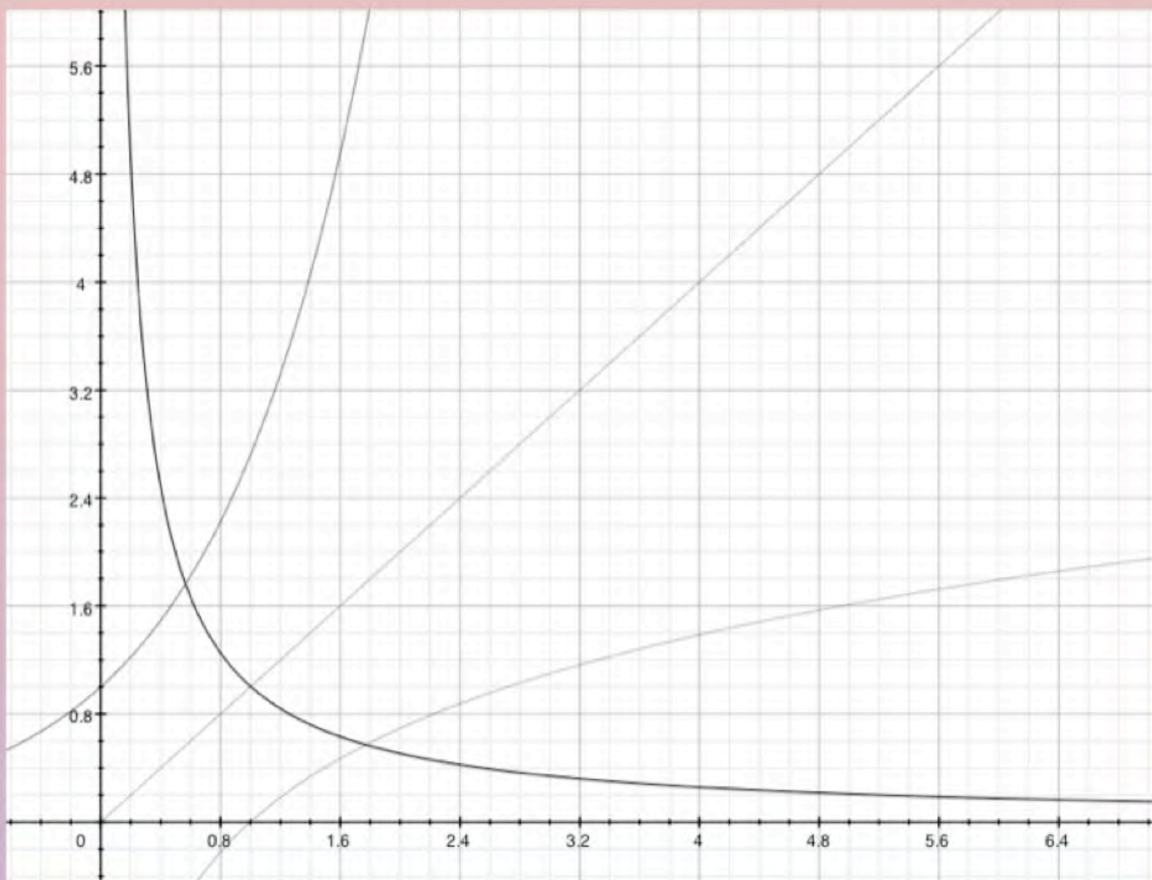


Logarithmic derivatives sans the diminishing differential

plus the proof for derivatives of inverse functions

by Michael Howell

Although the “unit hyperbola” is $x^2 - y^2 = 1$, the most basic hyperbola is $y = 1/x$. Hyperbolic functions are intricately connected with exponents and natural logarithms—for reasons far more fundamental than integrals and derivatives, as we have already observed.

The area under $1/x$—for intervals beginning at $x = 1$—is $\ln(x)$. This long-established fact can also be proved without infinite series. The fundamental reason is that the area increases arithmetically (1, 2, 3…) while the x-value (abscissa) increases geometrically (a, a^2, a^3…). Wikipedia sums it up in the article “Hyperbolic angle”:

The quadrature of the hyperbola was first accomplished by Gregoire de Saint-Vincent in 1647 in his momentous *Opus geometricum quadrature circuli et sectionum coni*. As David Eugene Smith wrote in 1925:

> [He made the] quadrature of a hyperbola to its asymptotes, and showed that as the area increased in arithmetic series the abscissas increased in geometric series.
>
> *History of Mathematics*, pp. 424,5 v. 1

By definition, the logarithms of geometric progression increase arithmetically. The term *logarithm* shares a common root with the term *arithmetic*, and now you know why.

As for measuring the area under the curve with infinite series, it is certainly objectionable to follow the standard procedure of shrinking a series of rectangles to width = 0. But this is not the only systematic way to create an infinite series of rectangles under the curve. The alternative is a fractal analysis. Fractals avoid the entire issue of one dimension of the problem disappearing. Just add big rectangles to smaller rectangles—rather than having all the rectangles shrink.

The reason $\ln(x)$ does not find the area under $1/x$ for values less than 1 is the following:
The interval (0, 1) contains the slope leading up to the singularity at $x = 0$. This is the behavior of an *improper integral*. An integral that applies everywhere on the x-axis will require a primordial form of renormalization.

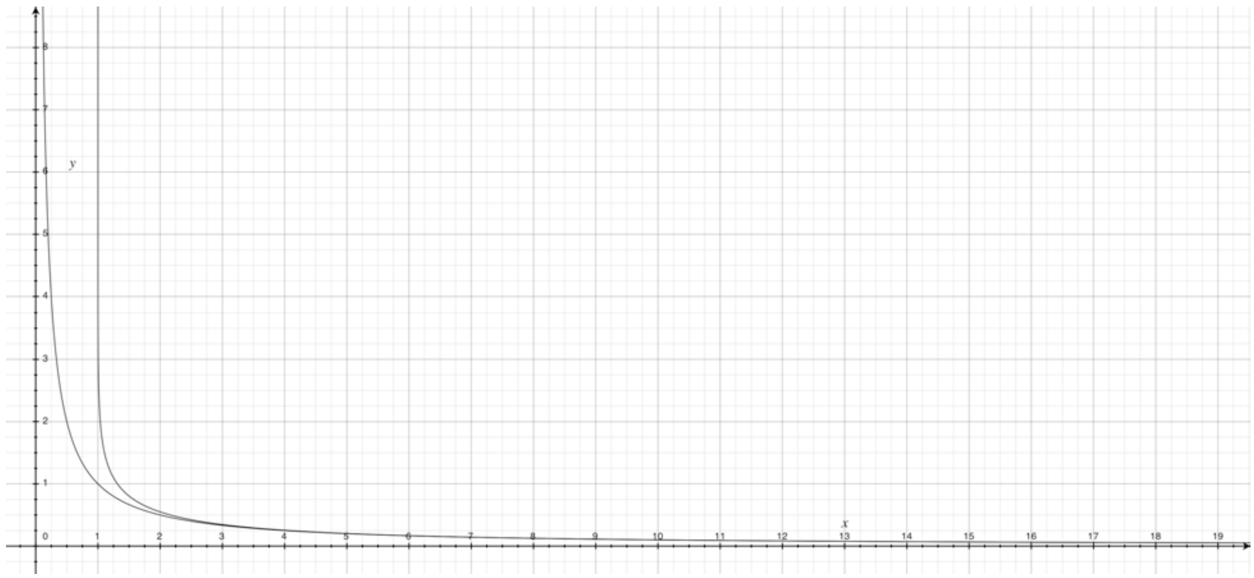
Clearly, the standard model for linear differential equations applies for at least *some* geometric situations it attempts to address. The standard integral for $1/x$ is correct under at least some circumstances. But does $1/x$ give the proper *tangent* of $\ln(x)$? In other words, does it properly average the series of slopes from $x+1$ to $x-1$?

Regardless of standard calculus, the slope of $\ln x$ is still intimately connected with $1/x$. For reminder’s sake, Mr. Mathis believes the slope of $\ln x$ is simply the following:

$$y = \frac{(\ln(x+1)-\ln(x-1))}{2}$$

For high values of x, Mathis’ slope approaches the standard derivative. Mathis accepts this asymptotic relationship but believes that mathematicians got seduced into producing a false derivative. I agree that they must have derived the equation improperly—I do not excuse them. But I can show that the derivative is true nevertheless.

For good measure, let's compare Mathis' graph with the graph of 1/*x*:



At *x* = 21, only very close magnification can tell the "simplistic" slope formula apart from the standard derivative.

At *x* = 80, even my simple graphing program has a hard time resolving the difference. The "simple man's" slope of ln *x* does not just get closer to the standard slope (1/*x*) in absolute terms. The *relative* difference becomes vanishingly small as well. Mathis does not dispute this.

The diminishing-interval scheme of standard calculus is physically, mathematically, and intuitively untenable because one of the dimensions disappears when it shouldn't. The typical proof for a derivative or antiderivative involves a sort of renormalization.

Richard Feynman called renormalization a “dippy” process (http://milesmathis.com/quant.html) despite his promotion of it. As Miles Mathis puts it (power.html):

Renormalization was perfected by a famous physicist named Richard Feynman, and he is notorious for calling his own creation “hocus-pocus” that was “not mathematically legitimate.” He also called it a shell-game. What does renormalization do? It removes zeroes and infinities from equations that are imploding or exploding.

The way around this conundrum is so simple that the only things blocking its favor have to be deconstructionism and self-aggrandizing physicists. The fake genius presents an inscrutable world the masses can never comprehend. The true genius unlocks the doors for the masses.

Again, the infinite series for $\ln x$ does not have to be in the form of rectangles that disappear at the limit. It can be composed in terms of a fractal. And this fractal can be produced in terms of normal polynomials—whose derivatives and integrals Mathis has already corroborated.

Expanding the exponential function into a power series is a straightforward process:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + O(x^7)$$

$$e^{-x} \qquad 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$$

Doing the same for the inverse—the logarithmic function—is somewhat trickier. An inverse function is a second measurement of the original function, so we should not be too surprised that the unfolding process of the inverse function will often be more complicated.

In Wikipedia, under “Natural logarithm,” this identity is given:

$$\ln(x) = \ln\left(\frac{1+y}{1-y}\right) \quad = 2\,y\left(\frac{1}{1} + \frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \frac{1}{9}y^8 + \cdots\right)$$

To put it more clearly…

$$2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \frac{y^9}{9} + \ldots\right)$$

The relationship between x and y is easy to see, even if Wolfram Alpha requires one to find it manually.

Input:

$$188 = \frac{1+y}{1-y}$$

Solution:

$$y = \frac{187}{189}$$

Input:

$$189 = \frac{1+y}{1-y}$$

Solution:

$$y = \frac{94}{95}$$

$$y = \frac{x-1}{x+1}$$

Upon substituting (1+*u*)/(1–*u*) for *x*, Alpha calculates this derivative:

$$(u-1)^2\left(u^8+u^6+u^4+u^2+1\right)=y'\left(\frac{u+1}{1-u}\right)$$

Except for missing that factor of 2, the polynomial inside the parentheses looks exactly like the expected derivative for the original equation. Unfortunately, substituting terms of *y* for the *x*-term puts polynomials into the denominator, and we want to avoid that. It prevents the straight derivation from regular polynomials and requires us to postulate a new kind of "chain rule."

That extra term in front of our expected derivative—$(u-1)^2$—is what I like to call the "unfolding operator." It turns the basic series into the form that applies to normal use.

Since an exponential curve is simply a fractal form of variable acceleration, it should be possible to express the inverse function—the natural-log curve—in nearly as straightforward a fashion. After some fussing with the information available on the Internet, I finally figured out a power function that works for all $x > 0$:

$$y = -\left[\sum_{n=0}^{\infty}(-1)^n\frac{\left[\frac{1}{Q}-1\right]^{n+1}}{n+1}\right]-\sum_{k=1}^{\infty}\left[-\frac{1}{Q}\right]^k\frac{(x-Q)^k}{k}$$

The derivative is

$$y = -\sum_{k=1}^{\infty}\left[-\frac{1}{Q}\right]^k(x-Q)^{k-1}$$

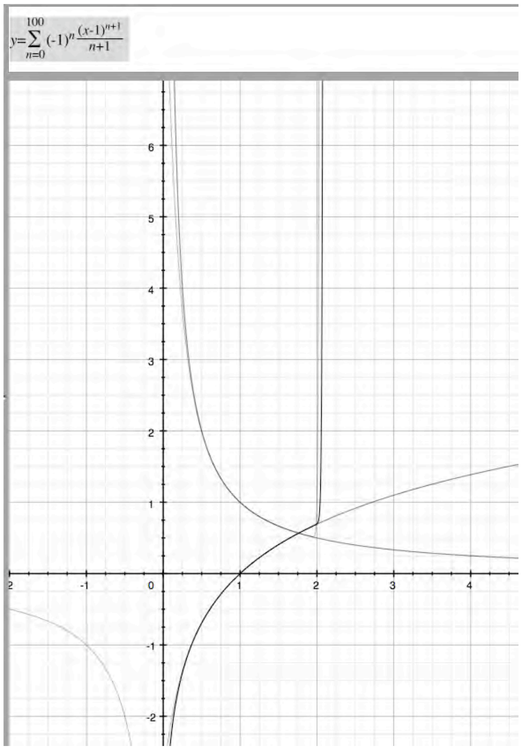
That term in the big brackets disappears because it is a constant. The (-1/*Q*) is also a constant, so it does not affect the basic derivative for each term in the sum. These two series corroborate that the derivative of $\ln x$ really is 1/*x*. And they use only terms whose standard derivatives are upheld by Mathis' work (milesmathis.com/calcsimp.html).

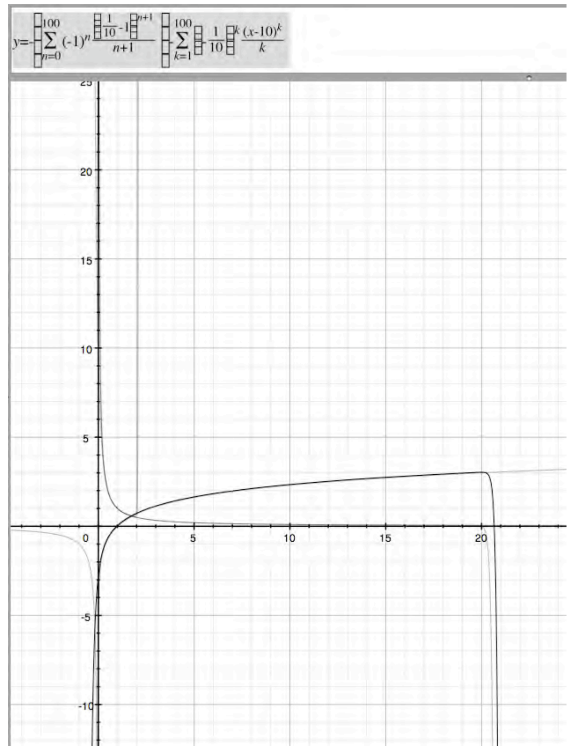
I started with the standard Taylor series of $\ln x$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n} \qquad =(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots \qquad 0<x<2$$

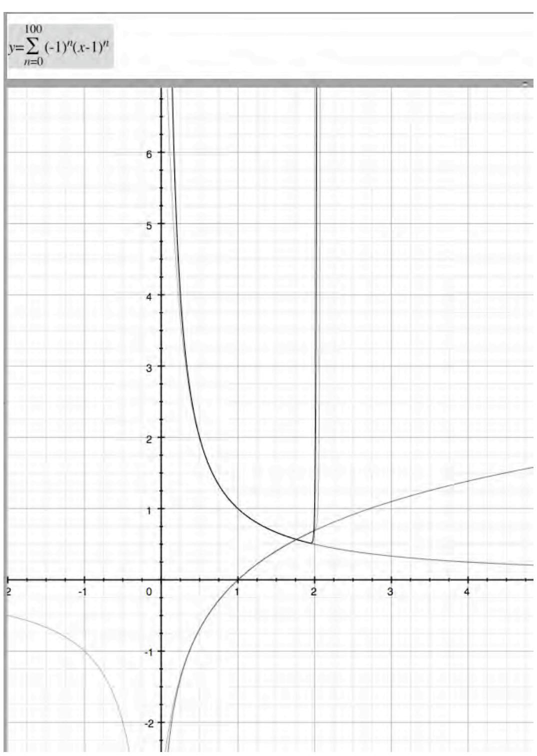
The more general form uses *a* in place of 1. Accommodating the Taylor Series for ln(*x*) will provide convergence for $0 < x < 2a$ (www.understandingcalculus.com/chapters/23/23-1.php). The modified series must include ln(*a*) as one of the terms. Since the original series converges for all numbers less than 1, we can find logarithms for all reciprocals of the desired $x > 1$. For the term ln(*a*) in the modified series, simply use the negative result of ln(1/*a*) = –ln(*a*).

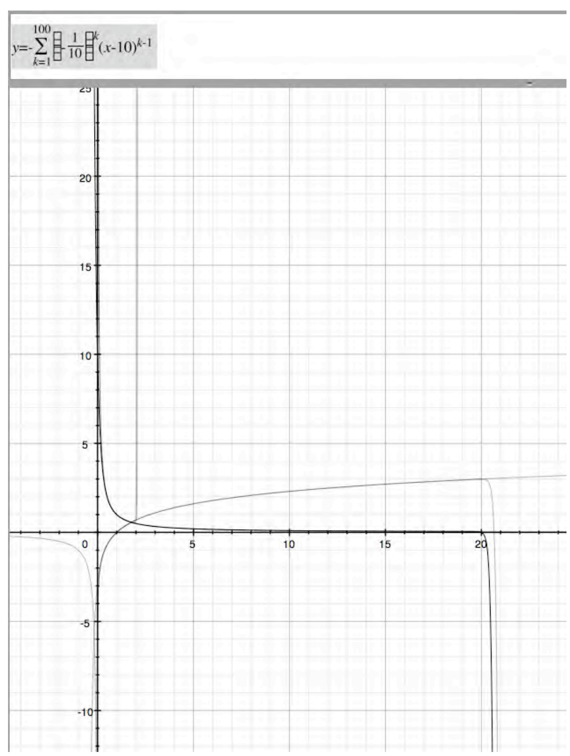
These graphs demonstrate how to unfold the logarithmic function from the Taylor series:





Unfolding the derivative works the same basic way:





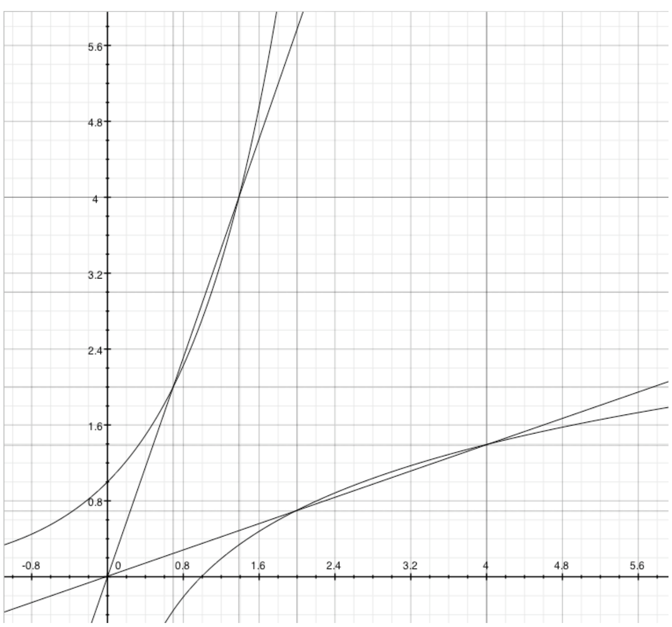
Perhaps some of you still don't appreciate why the derivative of the exponential function is simply itself while the inverse function is not. Again, you have to look at the properties of the graph. Precisely *because* $\ln x$ is e^x turned on its side, the horizontal asymptote at $y = 0$ turns into a singularity (vertical asymptote) at $x = 0$. Furthermore, notice that the slope of the exponential function approaches vertical (∞) as the x-value heads into the great beyond. When the function is turned on its side, the slope approaches horizontal (0). Between the two asymptotes of the derivative is a sharp turn where $y = x = 1$. This corresponds to the hard turn of $\ln x$.

Although the graph of an inverse function is simply the original curve turned on its side, the behavior of such functions is a surprisingly fascinating field of study. Measuring x relative to y is generally a rather different operation from measuring y relative to x.

The standard derivative of an inverse function is

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

To compare the slope of a function directly to its inverse, the interval Δx used to find the slope of the original curve must become the interval Δy when addressing the inverse function. Mathis said, as you might recall, that shrinking the interval is liable to change the curvature in an undesirable way for proving a derivative. However the interval Δx used to measure the slope of one curve must become the Δy of the turned-over curve—and that will mean that the inverse function uses (in general) a different Δx for its own slope. So even according to Mathis' definition of slope, it seems there are two valid intervals for measuring it.



When comparing the slope of e^x directly to its inverse, the interval Δx changes over the course of the curve if a <u>constant interval Δx for the slope of $\ln x$</u> is being used as the reference. (In other words, the Δy stays constant, this time, for e^x.)

Recall that Mathis' slope for $\ln(x)$ is

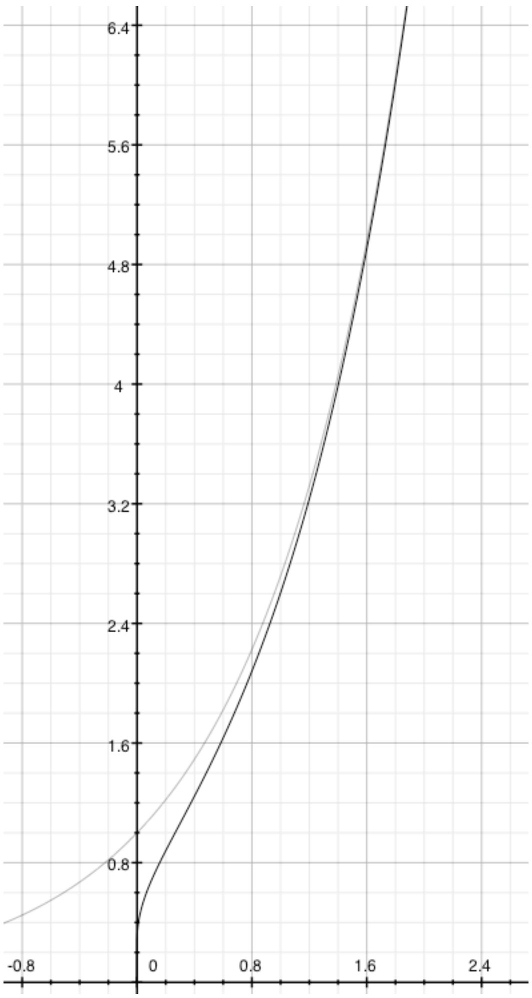
$$y = \frac{(\ln(x+1) - \ln(x-1))}{2}$$

Recall how this slope approached $1/x$ for large values of x. Now notice that the standard derivative of an inverse function is the reciprocal of $f'(f^{-1}(x))$. The inverse function is plugged into the derivative of the original function. In Mathis' slope for $\ln(x)$, let us now replace x with e^x.

Our formula becomes

$$y=\left(\frac{(\ln(e^x+1)-\ln(e^x-1))}{2}\right)^{-1}$$

Here is the resulting curve:



For high values of x, this graph approaches the exponential curve. I have already demonstrated that the derivative of e^x is its own self. I now prove, in very simple terms, that the derivative of an inverse function is $1/f'(f^{-1}(x))$—backing up this standard formula as well.

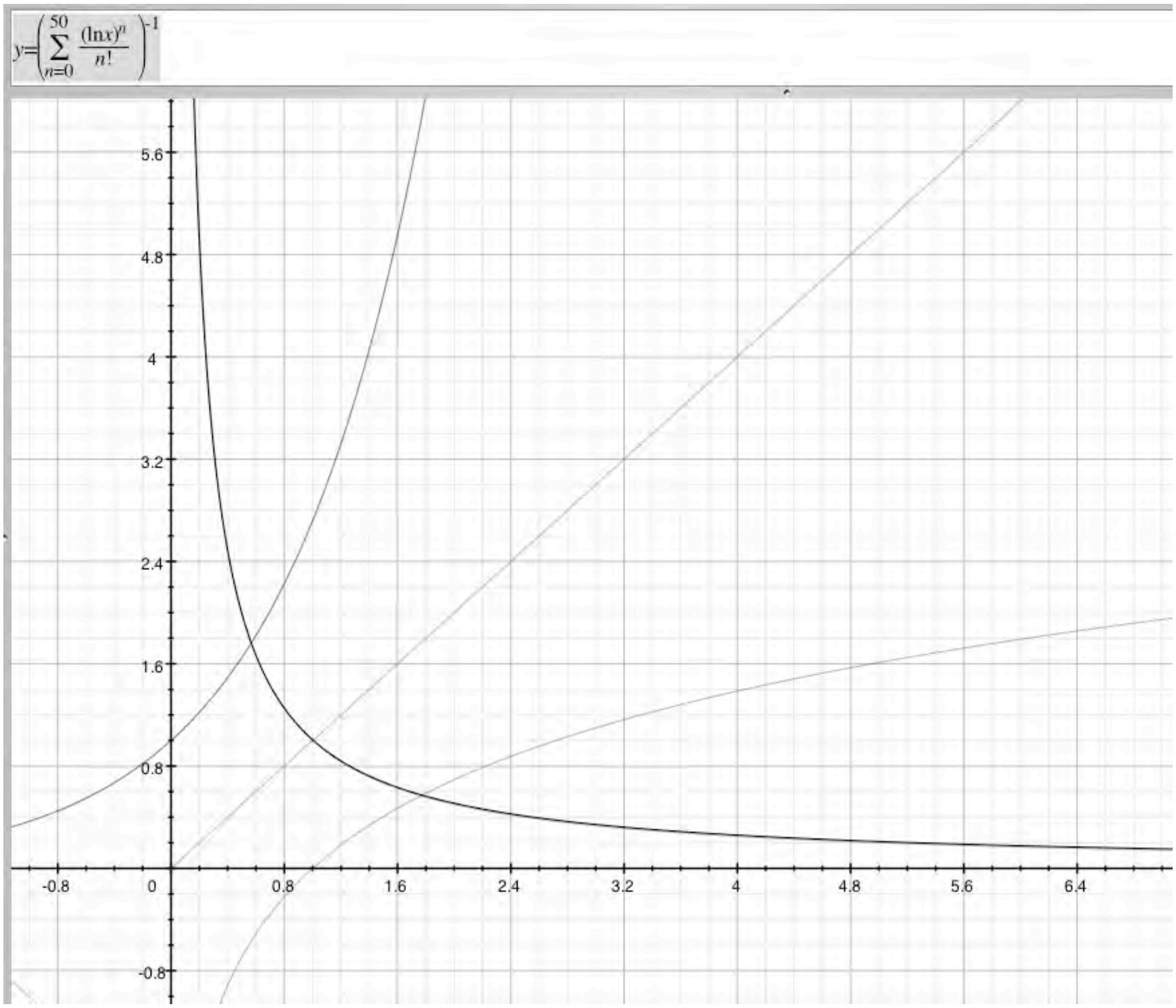
The website I consulted for this formula (oregonstate.edu/instruct/mth251/cq/Stage6/Lesson/inverseDeriv.html) offers this proof (or, more precisely, the lack thereof):

We shall not discuss the proof of this theorem here, other than to say that **the proof is relatively difficult**. A first step in the proof is to show that the inverse of a continuous function is continuous *[come on—it's just the curve turned on its side!]*; this proof in turn requires an application of the Intermediate Value Theorem (see Stage 4) *[I think we need inverse value theorem]* and actually reveals a fairly deep and subtle property of the real numbers.

Yes, it *is* a subtle property of numbers, but I can penetrate it—it's these people who apparently can't. Sometimes it pays just to leave the blackboard and go to the graph paper.

Some mathematicians have so little faith in what the plain numbers tell them that it seems a wonder they are part of mathematics at all. How do they ever find the faith to believe that math works—or science? I have concluded that this is the root of deconstructionism: the lack of faith that what one detects can inform him of absolute truth. This sort of insecurity easily breeds the sentiment that humans don't *find* truth—but rather *create* truth. In this way, no fabrication is seen as a lie—fabrication becomes the expected way of building one's own knowledge.

On the next page, I plug $\ln x$ into the series expansion of e^x. Since the exponential function really is its own derivative, it should be easy to get the derivative of its inverse.



When each function is the other's inverse, the slope of one curve can be mapped with the results for the other. The transform is a simple reciprocal relationship. How neatly this process works for exponential and logarithmic functions is the ultimate expression of the linearity of exponential change. (That line which resembles $y = x$ is the result of plugging $\ln x$ into e^x.)

A differential equation corroborates this simple graphical analysis:

$f\big(g(x)\big) = x$ definition of inverse function, where $g(x) = f^{-1}(x)$

$$\frac{df(g)}{dx} = 1 \qquad \frac{df(g)}{dg}\cdot\frac{dg}{dx} = 1 \qquad \frac{df(g)}{dg} = \frac{dx}{dg}$$

$$\because\ f\big(g(x)\big) = x \qquad \frac{dx}{dg} = \frac{dx}{dg} \qquad \because\ f'\big(g(x)\big) = \frac{1}{g'(x)} \quad OR \quad \left[f^{-1}(x)\right]' = \frac{1}{f'\big(f^{-1}(x)\big)}$$

So to those professors who think the derivative of an inverse function is too hard to prove, give me a break.