Eur. J. Phys. **21** (2000) 341–345. Printed in the UK PII: S0143-0807(00)10472-6

Energy consideration in the two-capacitor problem

Sami M Al-Jaber and Subhi K Salih
Physics Department, An-Najah National University, Nablus, West Bank

Received 20 December 1999, in final form 21 March 2000

Abstract. A derivation of the specific electrostatic energy loss for two interacting capacitors is given. The connecting wires are assumed to be superconducting. The method of derivation is based purely on the Poynting vector involved in the process without any reference to the nature of the mechanism that may be involved in achieving a final equilibrium state.

1. Introduction

We consider one of the best known examples in introductory physics textbooks which every physicist learns about in his/her first year undergraduate course, namely, that of two interacting capacitors. A capacitor is charged by a power source, which is then removed. The charged capacitor is then connected to an uncharged capacitor. This system will attain a final equilibrium state, which is determined by the conservation of charge and the requirement for a common final potential difference across the two capacitors. The calculation of the initial and the final electrostatic energies of the system has appeared in standard introductory physics textbooks [1, 2]. The result shows that a specific fraction of the original system energy has been dissipated, where this fraction depends on the two capacitance values. The question which usually troubles the students is: where does the loss in the electrostatic energy go? The usual simple answer to this is that it goes to heat energy regardless of any resistance in the circuit. However, a more troublesome question is that of how to calculate the energy loss without, of course, subtracting the initial energy from the final one. When one takes stock and starts to strive for a better understanding of the problem, one immediately realizes the need for sound answers to the above question. The present authors have felt themselves compelled to shed some light on the physics involved in this problem.

We stress the fact that the loss in the electrostatic energy is a prerequisite for obtaining a final equilibrium state, i.e. a final common potential for the two capacitors. The reader must realize this fact since a final equilibrium state will never be achieved from a transient solution without the action of a damping mechanism. The energy loss is the same regardless of the nature of the damping mechanism which need not necessarily be a circuit resistance, R. That is why one always obtains the energy loss even in the limit as R is chosen arbitrarily close to zero [3]. That is also why the present authors disagree with the usual explanation given for the energy loss as Joule heating.

The purpose of this paper is to present a derivation of the energy loss in the two-capacitor problem without any reference to the nature of the damping mechanism involved in the process. Our derivation is straightforward and based purely on the Poynting theorem in electrodynamics. In the methods used here we propose a model which consists of a parallel-plate capacitor with a circular geometry, where the connectors are superconducting wires with cylindrical symmetry.

0143-0807/00/040341+05$30.00

In section 2, we first consider a simpler but illuminating problem that consists of a capacitor being charged by a power supply, and we derive the energy loss. In section 3 we give a detailed derivation of the energy loss in the two-capacitor problem. Section 4 is devoted to results and discussion.

2. Charging a capacitor by a power supply

Consider a parallel-plate capacitor with capacitance C, whose plates are circular of radius a and separated by a distance d. This capacitor is connected to a power supply of electromotive force V_0 using superconducting cylindrical wires of length L and radius R. In the charging process, the charge on the capacitor is instantaneously built to its final value Q, since the wires are assumed to be superconductors. The charge on the capacitor is thus represented by [4]:

$$Q(t) = Q\theta(t) \tag{1}$$

where θ is a step function given by

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases} \tag{2}$$

As a result, the electric current, I, is given by:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = Q\delta(t) \tag{3}$$

where $\delta(t)$ is the delta function.

We are now in a position to show that the energy stored in the capacitor if $\frac{1}{2}CV_0^2$, as expected, and the energy loss required to achieve a final equilibrium state is also $\frac{1}{2}CV_0^2$.

The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ inside the capacitor are

$$\boldsymbol{E} = \frac{q(t)}{A\varepsilon_0}\hat{Z} \qquad \boldsymbol{H} = \frac{\rho I}{2A}\hat{\varphi} \tag{4}$$

where A is the plate area of the capacitor.

The current I in equation (4) is the displacement current which is the same as the conventional current I [5]. Accordingly, the Poynting vector is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = -\frac{q(t)}{\varepsilon_0 A}\frac{\rho I}{2A}\hat{\rho}. \tag{5}$$

The minus sign shows that energy is being stored inside the capacitor. Using (1), one obtains the power that crosses the capacitor by integrating over the surface of a cylinder of radius a and length d, namely

$$-\oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = \frac{Q^2}{2\varepsilon_0 A^2}\theta(t)\delta(t)a\oint \mathrm{d}a.$$

The surface integration gives $2\pi ad$ and thus

$$-\oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = \frac{Q^2\theta(t)\delta(t)}{C} \tag{6}$$

where $C = \varepsilon_0 A/d$.

The total energy stored is thus

$$-\int \mathrm{d}t \oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = \frac{Q^2}{C}\int \theta(t)\delta(t)\mathrm{d}t = \frac{Q^2}{C}\theta(0) \tag{7}$$

where $\theta(0)$ can be thought of as an average value [6], so that

$$\theta(0) = \frac{1}{2}[\theta(0^-) + \theta(0^+)] = \frac{1}{2}. \tag{8}$$

Therefore, equation (7) becomes

$$-\int \mathrm{d}t \oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = \frac{Q^2}{2C} = \frac{1}{2}CV_0^2 \qquad (9)$$

which is the expected electrostatic energy stored in the capacitor when charged by a power supply of electromotive force V_0. At this point one must appreciate the importance of the Maxwell term [7] that introduces a displacement current which plays a crucial role in deriving the energy stored in the capacitor.

For the energy loss, the electromagnetic fields in the connecting wires are

$$\boldsymbol{H} = \frac{I}{2\pi R}\hat{\varphi} \qquad \boldsymbol{E} = \frac{V_0 - V_c}{L}\hat{Z} \qquad (10)$$

where V_c is the potential difference across the capacitor. The Poynting vector at the surface of the wire (radius R) is

$$\boldsymbol{S} = -\frac{V_0 - V_c}{2\pi RL}Q\delta(t)\hat{\rho}. \qquad (11)$$

The power passing in through the surface of the wire is therefore

$$-\oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} Q(V_0 - V_c)\delta(t) \qquad (12)$$

and thus the total energy transferred into the wire is

$$-\oint \mathrm{d}t \int \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = QV_0 - \frac{Q^2}{2C} = \frac{1}{2}CV_0^2 \qquad (13)$$

where we have used equation (8). Equations (9) and (13) show that the energy delivered by the power supply, which is equal to CV_0^2, is divided into two halves: one half is electrostatic energy stored in the capacitor and the other half is energy transferred into the wires which is required to achieve a final equilibrium state, i.e. the power supply and the capacitor have the same potential V_0.

3. The two-capacitor problem

Consider a charged capacitor (hereafter called the first capacitor) of capacitance C_1 and charge Q_0 that is being connected to an uncharged capacitor of capacitance C_2 (hereafter called the second capacitor). The final equilibrium state is determined by the conservation of charge and the requirement for a common final potential difference across the two capacitors. Straightforward calculations yield that the final common potential, V, is

$$V = \frac{C_1 V_0}{C_1 + C_2} \qquad (14)$$

while the final electrostatic energy stored inside the two capacitors is

$$U = U_0 \frac{C_1}{C_1 + C_2} \qquad (15)$$

so that the energy loss is

$$U' = U_0 - U = U_0 \frac{C_2}{C_1 + C_2} \qquad (16)$$

where $U_0 = \frac{1}{2}C_1 V_0^2$ is the initial energy of the system.

Our aim here is to derive the energy loss given in equation (16) without any reference to the nature of the mechanism responsible for the dissipated energy needed to establish a final equilibrium state.

As in section 2, we assume that the connecting wires between the two capacitors are superconductors with cylindrical symmetry, of length L and radius R. After connecting the capacitors and during the transient state, let the charge on the second capacitor be $q_2(t)$ and the charge on the first capacitor be $q_1(t)$, so that

$$q_2(t) = Q_2\theta(t) \tag{17}$$

$$q_1(t) = Q_0 - Q_2\theta(t) \tag{18}$$

and thus the current is

$$I = \frac{\mathrm{d}q_2}{\mathrm{d}t} = -\frac{\mathrm{d}q_1}{\mathrm{d}t} = Q_2\delta(t). \tag{19}$$

The electric field parallel to the wire is

$$\boldsymbol{E} = \frac{1}{L}\left(\frac{q_1}{C_1} - \frac{q_2}{C_2}\right)\hat{Z}. \tag{20}$$

Using equations (17) and (18), we obtain

$$\boldsymbol{E} = \frac{1}{L}\left[\frac{Q_2 - Q_2\theta(t)}{C_1} - \frac{Q_2\theta(t)}{C_2}\right]\hat{Z}. \tag{21}$$

The $\boldsymbol{H}$ field at the surface (radius R) is

$$\boldsymbol{H} = \frac{I}{2\pi R}\hat{\varphi}$$

which, using equation (19), becomes

$$\boldsymbol{H} = \frac{Q_2\delta(t)}{2\pi R}\hat{\varphi}. \tag{22}$$

Accordingly, the Poynting vector is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = -E\frac{Q_2\delta(t)}{2\pi R}\hat{\rho}. \tag{23}$$

The power passing in through the surface of the wire is therefore

$$-\oint \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = Q_2\delta(t)\left[\frac{Q_0 - Q_2\theta(t)}{C_1} - \frac{Q_2\theta(t)}{C_2}\right]. \tag{24}$$

Hence, the energy passing in through the surface of the wire is

$$-\int \mathrm{d}t \oint \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = \frac{Q_0 Q_2}{C_1} - \frac{Q_2^2}{2C_1} - \frac{Q_2^2}{2C_2} = V_0 Q_2 - \frac{Q_2^2}{2}\left(\frac{C_1 + C_2}{C_1 C_2}\right). \tag{25}$$

Using equation (14) we obtain the final charge on the second capacitor which is

$$Q_2 = \frac{C_1 C_2 V_0}{C_1 C_2} \tag{26}$$

and thus equation (25) becomes

$$-\int \mathrm{d}t \oint \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = U_0 \frac{C_2}{C_1 + C_2} \tag{27}$$

which is the electrostatic energy loss given in equation (16). Again, this derived energy loss is the price that one must pay for obtaining a final equilibrium state, i.e. final common potential difference across the two capacitors.

4. Results and discussion

The electrostatic energy loss in the two-capacitor problem has been derived., This energy loss is a consequence of the requirement that a common final potential difference across the two capacitors is imposed at the final equilibrium state,. Our derivation for the energy loss is independent of the nature of the damping mechanism involved in the process but based purely on the Poynting vector in electrodynamics. The method adopted for the derivation of the energy loss is interesting because no use is made of the resistance in the circuit and thus it is not the Joule heating that is responsible for this energy loss, as is widely believed. The reason for this wide belief is that one usually starts by assuming a resistance in the circuit as a damping mechanism and then the integrated power loss yields the energy loss even in the limit as the resistance is chosen arbitrarily close to zero.

Here are some examples of other systems that have similar behaviour to that which we have discussed [3]. The first one is the one-dimensional totally inelastic collision between two 'point' masses. Here, the imposed condition is the common final velocity, which is responsible for the dissipation of the kinetic energy. The second system consists of two interacting (massless) Hooke's law springs. Here, we have two horizontal springs whose far ends are attached to fixed walls. In the initial state, one spring is held in a stretched position by an applied force (analogous to an emf charging a capacitor), while the other spring is relaxed (analogous to an uncharged capacitor). In this initial state the two springs just fill the space between the fixed walls. The springs are then allowed to interact by connecting their adjacent ends and removing the force on the first spring. The final state is achieved by conservation of elongation and the equality of the two spring forces. Again, the price that one must pay is the dissipation of a specific fraction of the system energy.

References

[1] Halliday D and Resnick R 1988 *Fundamentals of Physics* (New York: Wiley)
[2] Serway R 1996 *Physics for Scientists and Engineers with Modern Physics* (London: Saunders)
[3] Sciamanda R 1996 *Am. J. Phys.* **64** 1291
[4] Arfken G 1970 *Mathematical Methods for Physicists* (New York: Academic)
[5] Wangsness R K 1979 *Electromagnetic Fields* (New York: Wiley)
[6] Griffiths D J 1993 *J. Phys. A: Math. Gen.* **26** 2265
[7] Griffiths D J 1999 *Introduction to Electrodynamics* (Englewood Cliffs, NJ: Prentice-Hall)