The self-resonance and self-capacitance of solenoid coils

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Abstract

Medhurst's semi-empirical formula for the self-capacitance of a single-layer solenoid inductor is derived from low and high-frequency limiting behaviour except for an approximately frequency-independent shape factor. The expression is further improved by the inclusion of a $1/\text{Cos}^2\psi$ factor (where ψ is the pitch angle), which accounts for a neglected second-order relationship between self-capacitance and the number of turns. The $1/\text{Cos}^2\psi$ dependence is consistent with experimental data, whereas formulae based on the inter-turn capacitance are radically in error.

Medhurst gathered data for coils wound on solid polystyrene rods and analysed it on the basis that the coil former dielectric makes no difference to the self-capacitance. We presume that he did so because data for long thin coils converges to an asymptotic formula which depends only on external permittivity. By consideration of boundary conditions relating to an electromagnetic wave propagating along the helix, we deduce an expression which reconciles Medhurst's data with supplementary data for air-cored coils.

The lumped-component model for an inductor only works insofar as the curve of apparent inductance vs. frequency given by:

$L' = (X_L // X_{CL}) / 2\pi f$

corresponds to the apparent inductance of a dispersive short-circuited transmission line given by:

$L' = (R_0/2\pi f) \text{Tan}(2\pi f\, \ell_{TL} / v_p)$

Where R_0 is the characteristic resistance, ℓ_{TL} is the equivalent line length, and v_p is a phase velocity. The dispersive property of the line is effected by allowing both v_p and R_0 to vary with frequency.

Solutions for R_0 and v_p, in Bessel functions, have been given by Schelkunoff for the Ollendorf sheath-helix model and are well known. This model however is best related to infinitely long filamentary helices, and provides only an approximate description for practical round-wire coils. We can however use it to deduce the circumstances under which the lumped-component theory breaks down. The situation is that, at very low frequencies, the superposition of an axial slow wave and a wave following the helix results in a helical phase velocity several times the speed of light. In this region however, the transmission-line equation asymptotes to the approximation $\text{Tan}\theta=\theta$, and the relationship between R_0 and v_p is fixed from purely magnetic considerations. At intermediate frequencies, v_p decreases smoothly in such a manner as to cause a reasonable match between the transmission-line model and the lumped component model. As the SRF is approached however, the scattering cross-section of the coil increases dramatically and the helix propagation mode comes to dominate the superposition. This forces v_p to become close to c, causing the SRF to occur when the length of the wire in the coil is about $\lambda_0/2$. The change in behaviour gives rise to a kink in the L' curve and a corresponding deviation from lumped-element theory; the effect being most noticeable in short coils. Ultimately, the apparent self-capacitance of an inductor is decided by the range over which the two expressions for L' agree. The kink, combined with lesser differences of curvature at lower frequencies, implies that self-capacitance is not an accurate predictor of the SRF.

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Glossary

// = parallel impedance operator, defined such that: **a**//**b**=**ab**/(**a**+**b**)

$\varepsilon = \varepsilon_0\ \varepsilon_r$ = electric permittivity
$\varepsilon_0 = 1/(\mu_0\ c^2)$ = 8.854187818 pF/m = permittivity of free space
ε_r = relative permittivity (dielectric constant in the lossless approximation).
ε_{rf} = dielectric constant of coil-former material.
ε_{ri} = relative permittivity inside the solenoid
ε_{rx} = relative permittivity outside the solenoid
$\lambda = v / f$ = wavelength in the surrounding medium
$\lambda_0 = c / f$ = wavelength in free space
$\mu = \mu_0\ \mu_r$ = magnetic permeability
$\mu_0 = 400\pi$ nH/m = permeability of free space
μ_r = relative permeability (neglecting losses)
χ^2/ν = reduced chi-squared = variance of an observation of unit weight.
$\psi = \mathrm{Arctan}[p/(2\pi r)]$ = pitch angle

$c = 1/\sqrt{(\mu_0\ \varepsilon_0)}$ = 299 792 458 m/s = Speed of light
C_{ee} = End-effect capacitance
C_L = Self-capacitance
D = Effective diameter of solenoid ($D<D_a$)
DAE = "Doubly asymptotic, empirically corrected"
d = wire diameter
D_a = Average diameter of solenoid (measured from middle of wire)
f = frequency

f_0 = resonant frequency
f_{0s} = self-resonant frequency (SRF)
GDO = Grid-dip oscillator
h = height
k_c = capacitance correction coefficient
k_E = electric correction factor
k_H = Generalised magnetic field inhomogeneity coefficient
k_L = Nagaoka's coefficient (current-sheet field inhomogeneity coeff.)
L = Low-frequency inductance
$L' = X_L'/(2\pi f)$ = Apparent inductance
$\ell = Np$ = Length (or height) of solenoid.
$\ell_w = 2\pi rN / \cos\psi$ = Wire (or conductor) length
N = Number of turns
n = refractive index
$1/n_{ax}$ = axial velocity factor
$1/n_{hx}$ = helical velocity factor
p = pitch distance
$Q = X_L' / R_{ac}$
r = Effective radius of solenoid ($r<r_a$)
R_0 = Characteristic resistance of transmission line
r_a = Average radius of solenoid (measured to middle of wire)
r_i = inside radius of tubular coil-former
r_o = outside radius of coil-former
R_{ac} = AC resistance
SRF = Self-resonance frequency
v = apparent propagation velocity (in medium. v=c in vacuo)
v_p = phase velocity
w = relative wall thickness
$X_{CL} = -1/(2\pi fC_L)$ = reactance attributable to self-capacitance
$X_L = 2\pi fL$ = Inductive reactance
$X_L' = 2\pi fL'$ = Measured reactance
$Z_0 = \sqrt{(\mu_0/\varepsilon_0)} = 376.7303134\ \Omega$ = Impedance of free space
$Z_0/2\pi = 59.9584916\ \Omega$, not 60.

1. Introduction

Of pure resistance, capacitance and inductance, the latter is the least amenable to realisation in practical devices. The reason is that the lumped-component theory depends on the assumption that every physical dimension is negligible in comparison to the wavelength of the incident radiation. In a wound inductor, the problematic dimension is the length of the piece of wire used to make it. That wire may be coiled-up in a tiny volume, but its electrical length at radio frequencies cannot be ignored.

When a solenoid coil is operated in the regime where wire length is negligible in comparison to wavelength, its inductance can be calculated with phenomenal accuracy from magnetic considerations alone. A straightforward basis for so doing is the hypothetical current-sheet inductor which, with corrections for realistic wire, allows inductance to be determined from physical parameters to better than one part per thousand[1]. Magnetic theory must, of course, be respected in the asymptotic behaviour of any hypothesis which purports to describe inductors at high frequencies, and this has been a weakness in some of the more influential studies. It is one thing to say that a theory of wave propagation in coils reverts to the lumped theory at low frequencies, but another to ensure that it actually does.

At audio and low-radio frequencies, the situation is complicated by the onset of the skin effect, and its companion, the proximity effect. The internal impedance of isolated wires is however a solved problem[2]; and the proximity effect, albeit more difficult to quantify, is nevertheless susceptible to attack[3] [4]. We may also note that the current redistribution which affects external inductance is constrained by boundary conditions on Maxwell's equations; i.e., the loop radius is always determined to a point which lies within the body of the conductor. Hence, with a little empiricism, we can push the inductor model into the radio-frequency range and still obtain respectable accuracy. More relevant to the present discussion however, we can envisage the existence of data corrected for minor non-idealities, which frees us from distractions when looking to higher frequencies still.

As the lumped component theory would have it: corrected for strays and losses (and better-still also corrected for the effects of non-uniform current distribution within the wire); the reactance of a coil looks like the reactance of a pure inductance in parallel with a capacitance. There is even a school of thought which says that the self-capacitance is due to the capacitance between adjacent turns; and although this is partly true for multi-layer coils, the hypothesis turns out to be a hopeless predictor of the reactance of single-layer coils. Experimentally, it transpires that self-capacitance *increases* as the spacing between turns increases, and we will shortly derive an expression which accounts for that phenomenon.

Attributing self-capacitance to the static turn-to-turn electric field is a fallacy akin to taking the coil apart and trying to find the capacitor. The lumped element approach also breaks down in the vicinity of the self-resonance frequency (SRF), a difficulty which gives rise to inaccuracies in circuit simulation. The solution, of course, lies in recognising that the coil is a transmission line; except that the line in question turns out to be a rather complicated one.

Even for resistors and capacitors, the lumped approximation is just a special limiting case of transmission line theory. It is just that those components can generally be made small in comparison to wavelength; and when they do become large, the distributed parameter models remain fairly simple. Not so for the inductor; where different regions of the line overlap, giving rise to strong interactions between competing propagation processes.

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1 See, for example: **Inductance Calculations: Working Formulas and Tables**, F. W. Grover, 1946, 1973. Dover Phoenix 2004, ISBN: 0 486 49577 9.

2 **Practical continuous functions for the internal impedance of solid cylindrical conductors**. D. W. Knight. Available from www.g3ynh.info.

3 **Practical Model and Calculation of AC resistance of Long Solenoids**. E. Fraga, C Prados, and D.-X Chen. IEEE Transactions on Magnetics, Vol 34, No. 1. Jan 1998.

4 **Solid State Tesla Coil**. Gary L Johnson, 2001. Chapter 6. http://eece.ksu.edu/~gjohnson/

If it were just a matter of replacing the expression for the reactance of a pure inductor and capacitor in parallel with the expression for a short-circuited conventional line, then inductor modelling would present no serious challenge. Actual measurements remain anomalous when compared against either model however; an issue which came to this author's attention while investigating and attempting to predict the behaviour or broadband current transformers.

The lumped component and the simple transmission line models agree over a fairly wide frequency range. Hence there is a correspondence between self-capacitance and time delay, which might be exploited in the modelling of phase-sensitive circuits. In the interests of predicting self-capacitance therefore, attention was given to the classic 1947 work of R. G. Medhurst[5], which offers a widely used empirical formula for solenoids which can, in principle, be adapted to deal with toroids and other shapes.

It quickly became apparent that Medhurst's formula was not good enough for the task in hand. It is clearly on the right track, because it gives results which are accurate within about -50 to +100%; but an accuracy of ±15% can easily be obtained by assuming that high-frequency waves propagate along the coil wire with a phase velocity equal to the speed of light (v_p=c). The latter approach incidentally, does not represent the general physical situation; but it works fairly well. Instead of abandoning Medhurst however; a number of supplementary measurements were collected in an endeavour to find the source of the errors. That investigation, outlined below, results in a new formula which takes pitch-angle and coil-former dielectric into account, and brings the uncertainty in estimating the self-capacitance of typical RF inductors to a more respectable ±2%.

It was the business of acquiring self-capacitance data, rather than that of predicting it, which produced anomalies. Measurements, for want of exotic test equipment, were made using the ancient but venerable resonance method of G W O Howe. This involves resonating the test coil against a series of known capacitances, then fitting the data to a regression line. If we take, for the purpose of illustration, the simple case where the Q is sufficiently high to permit the neglect of losses, the parallel resonance formula reduces to:

$(2\pi f_0)^2 = 1/[\ L\ (C_L+C_{ref})\ ]$

Where C_L is the self-capacitance, and C_{ref} is the added parallel capacitance including all strays. Rearranging gives:

$C_{ref} = -C_L + 1/\ [\ (2\pi f_0)^2\ L\ ]$

which, insofar as the lumped component theory is valid, is a straight-line graph of the form y=a+xb. Hence, if $1/(2\pi f_0)^2$ is chosen as x, a linear regression procedure returns the slope 1/L and the y-axis intercept $-C_L$.

Medhurst's measurements were made using a variant of Howe's method, but there is no functional difference. For a given test coil, he acquired data using reference capacitances considerably greater than C_L. This amounts to shooting at the graph intercept from a distance; and although the measurements were performed with great care, the limited perspective removes all knowledge of possible deviations from the model. To put the matter in its proper historical context however; in 1947 engineers were inclined to *believe* in the lumped component theory, rather than see it as a mathematical convenience, in which case the approach was perfectly reasonable.

For the measurements made by this author, the decision was made to take data over a wide frequency range: generally at least three and sometimes as many as six octaves for each test coil. Also, a method was developed using a set of pre-calibrated plug-in mica reference capacitors, and a jig with a stray capacitance of only 0.73pF. This meant that, for most of the coils, measurements at the high end of the frequency range involved external capacitances considerably less than C_L. The results turned out to have both explicable and, initially, inexplicable deviations from linearity.

One obvious feature of all of the datasets covering sufficient range was the dispersion due to skin and proximity effects. This however could be corrected-out using techniques described in

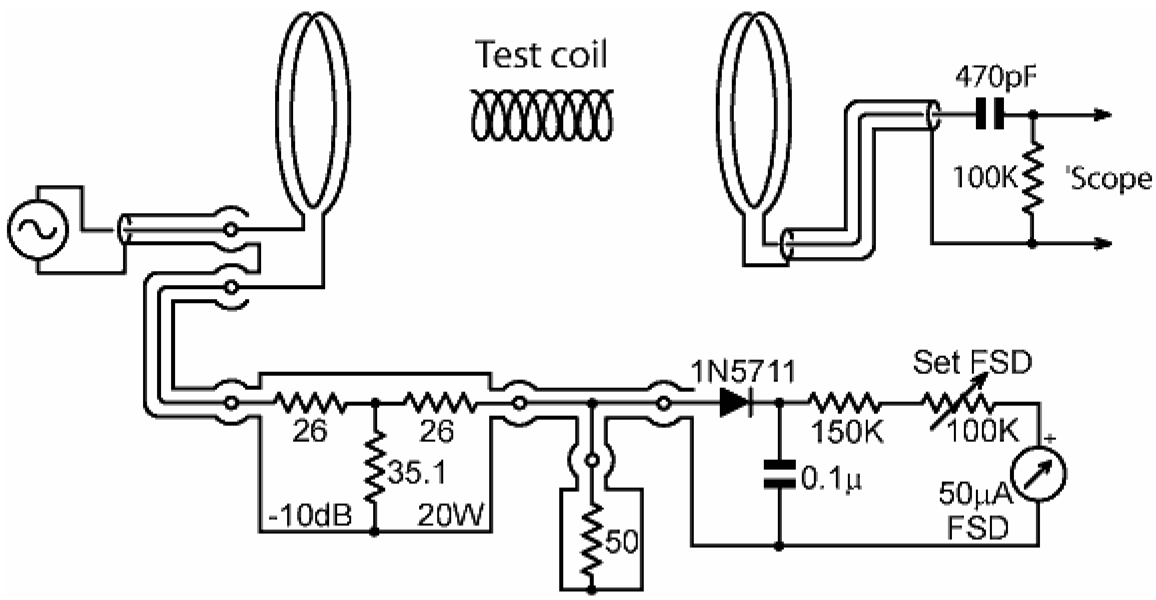
5 **H. F. Resistance and Self-Capacitance of Single-Layer Solenoids.** R G Medhurst (GEC Research Labs.). Wireless Engineer, Feb. 1947 p35-43, Mar. 1947 p80-92. Corresp. June 1947 p185, Sept. 1947 p281.

another article[6]. Hence, in the discussion to follow, we will concern ourselves only with what remains after that correction.

If the coil behaves as a conventional short-circuited transmission line, we should see a low frequency agreement between the data and the lumped component model, a disagreement at intermediate frequencies, and a reconvening as the frequency approaches the SRF. In experiments however, what was seen was an agreement over a surprisingly wide range, with a sudden deviation on approaching the SRF. If, for the sake of argument, we say that the inductance is constant, then it appears that the self-capacitance starts to decline when the external capacitance is reduced to the point where it becomes comparable to the self-capacitance.

One difficulty with this behaviour is that it looks like an artefact. Specifically, it looks like a calibration problem associated with the jig strays and the smaller reference capacitors. No amount of careful recalibration would make it go away however; and so for some time the only viable solution was to exclude the deviant data from the analysis by setting their fitting weights to zero. Still thinking that it was a systematic error, this led the author to devise a method for measuring the SRF of coils without using a test jig.

The solution to the troublesome 'calibration' issue (which never existed) was to scatter radiation from isolated coils and detect the resultant field using a small loop or an electrometer dipole connected to an oscilloscope. A giant version of the experiment devised for public demonstration is shown in the photograph below. In that case the generator is an HF radio transmitter modified to give continuous coverage, and the induction and pick-up coils are 2-turn loops of about 90mm diameter made from stiff wire. Note that the generator passes current through the induction loop to a coaxial load resistor; the point being to maintain a reasonable impedance match and thereby avoid provoking the transmitter's protection circuitry. Thus, although the power delivered to the load is in the 2 to 10W range, the actual radiated power is somewhat less than 100µW. The primitive high-pass filter before the oscilloscope serves to reject mains hum.



6 Reference required for self-C measurements by DWK.

For the demonstration shown, a large self-supporting coil of copper tubing is suspended from a plastic metre rule. As we will see later, a small amount of dielectric material on the *inside* of the coil makes very little difference to the resonant behaviour; but an exclusion zone of at least 5 coil diameters is necessary on the outside if the experiment is to give accurate results.

It takes several hundred microvolts at the oscilloscope input to produce a usable display, and so for the antenna separation shown (about 0.8m) very little is seen in the absence of a test coil. With a coil in situ however; numerous scattering resonances can be detected (depending on the coil geometry and the available frequency coverage), some visible with the pick-up loop on axis and some requiring it to be moved around to the side. By far the strongest resonance however is the fundamental SRF; which gives rise to an enormous increase in scattering cross-section, and a consequent sharp increase of about four orders of magnitude in the received signal. Note that the detected signal is the superposition of incident and scattered waves, giving rise to interference phenomena when investigating weak resonances; but the scattering signal at the SRF is so great that there is no ambiguity in finding the centre of the peak.

Which brings us to the main observation, which is that the SRF occurs when the total length of wire is very close to half the free-space wavelength, just as it does with a wire antenna. The data for the big demonstration coil serve to illustrate the point:

Solenoid length:	$\ell = 152$mm
Average diameter :	$D_a = 96$mm (= inside diameter + wire diameter)
Number of turns:	$N = 18.09$
Wire (tubing) diameter:	$d = 5.7$mm
Winding pitch:	$p = \ell / N = 8.4$mm
Pitch to wire diam. ratio:	$p / d = 1.47$
Conductor length:	$\ell_w = \sqrt{[(\pi D_a N)^2 + \ell^2]} = 5.458$m
SRF:	$f_{0s} = 26.92$MHz
Free-space half-wavelength:	$\lambda_0/2 = c/(2f_{0s}) = 5.631$m
Helical velocity factor:	$1/n_{hx} = 5.458 / 5.631 = 0.97$

From this, the principal resonance mechanism is clear. Waves travel along the helix and reflect from the impedance discontinuities which occur at the ends of the wire. The strong scattering resonance at the fundamental SRF corresponds to the standing-wave pattern which builds up when a single round trip brings the wave back to its starting point in phase with itself. The phase velocity moreover, despite interaction with the fields from adjacent turns, is not greatly different from c ; which implies that the wave is intimately associated with the guiding wire, at least at the SRF. Note however that the test coil has a pitch to wire diameter (p/d) ratio of about 1.5. In coils with more closely spaced turns ($p/d<1.1$) the helical phase velocity is somewhat greater than c at the SRF; a phenomenon which is explained by theory to be discussed later.

More information about the nature of the travelling wave can be had by replacing the loops with small dipoles; with cables brought straight-out at the back and choked-off with ferrite sleeve baluns. With the test coil removed, and an uncluttered working area, it can be shown that there is a minimum in the direct signal when the two dipoles are at right-angles. When the coil is introduced however, with its axis along the path between the antennas, there is no-longer a minimum when the dipoles are crossed, particularly on approaching and going above the SRF. Hence an observation which will come as no surprise to those who work with helical antennas; which is that the coil converts linearly polarised radiation into circularly polarised radiation. By sampling the electric field while looking along the axis, we see an advancing wave with a rotating electric vector; i.e., the wave in the dominant propagation mode travels along the helix with its electric vector substantially perpendicular to the axis.

In truth, there is nothing particularly novel or controversial about the scattering experiment as described so far. The association between self-resonance and wire length has probably been known since shortly after the invention of the grid-dip oscillator (GDO) in the 1920s; and there is theory which can account for the observed velocity factors, at least approximately. What is of interest is the way in which the information sheds light on the relationship between self-capacitance and the transmission-line resonance.

The scattering experiment shows, to a fair approximation, that the SRF of an air-cored solenoid occurs when the wire length is $\lambda_0/2$. What Medhurst's formula says however, is that the $\lambda/2$ rule only applies when the coil is long and thin. Specifically, for the squat solenoids typically used in radio applications ($\ell/D_a \approx 1$), the self-capacitance deduced from regression analysis predicts the SRF at a frequency which is too low in comparison to that of the disconnected coil. This does not imply, incidentally, that self capacitance is a useless conception. It is still the appropriate modelling parameter for conventional parallel LC resonators, because it correctly describes the circuit behaviour whenever the minimum padding capacitance is greater than C_L (which is usually the case). When parallel resonance data are acquired over a wide frequency range however; the points are seen to veer away from the regression line at high frequencies as the coil 'locks-on' to the $\lambda/2$ wire-length resonance. One clue to the origin of this attraction behaviour lies in the huge increase in scattering cross-section which occurs at the SRF.

The theoretical analysis which comes closest to describing the high-frequency behaviour of solenoids is that given by Schelkunoff, for the Ollendorf sheath-helix. This is reproduced in J. R. Pierce's classic paper on the travelling-wave tube[7] and elsewhere[8 9 10 11]. The case considered is that where the circumference of the coil is small in comparison to wavelength, so that the field patterns are radially symmetric; i.e., for coils of many turns, it is still appropriate at frequencies well above the fundamental SRF. The story is essentially that of a system which has two principal modes for electromagnetic propagation: one being associated with a plane wave, the 'slow-wave', travelling along the coil axis; the other being associated with a wave travelling along the helical conductor. Note that the two waves are not physically distinguishable. The overall field pattern is given by their superposition. They represent two ways in which radiation can traverse the coil, and since they must always remain in lock-step, the ratio of their phase velocities is given by the ratio of the

7 **Theory of the Beam-Type Traveling-Wave Tube**. J R Pierce. Proc. IRE. Feb. 1947. p111-123. See Appendix B, p121-123, "**Propagation of a wave along a helix**", which gives Schelkunoff's derivation of propagation parameters for the Ollendorf sheath-helix.

8 **RF Coils, Helical Resonators and Voltage Magnification by Coherent Spatial Modes**, K L and J F Corum, Microwave Review, Sept 2001 p36-45.

9 **Multiple Resonances in RF Coils and the Failure of Lumped Inductance Models**. K L Corum, P V Pesavento, J F Corum. 6th International Tesla Symposium 2006.

10 **Fields and Waves in Communication Electronics**. S Ramo, J R Whinnery and T Van Duzer. Wiley 1994. ISBN 0-471-58551-3. Section 9.8: The idealixed helix and other slow-wave structures.

11 **Coaxial Line with Helical Inner Conductor**. W Sichak. Proc. IRE. Aug. 1954. p1315-1319. Correction Feb. 1955, p148.

coil length to the conductor length.

Ramo et al. liken the propagation environment for the axial slow-wave to the situation within a disk-loaded waveguide[12]. The wave passes a series of slits, each of which leads to a short-circuited transmission-line stub, the resulting inductive loading causing a reduction in phase velocity. The series of Hertzian loops let into the walls of the mis-named 'Cavity' Magnetron have an analogous effect[13]. The additional complication in the solenoid lies in the varying relative dominance of the axial and helical processes.

It was said by Michael Faraday, that for every highly mathematical theory, we should seek an equivalent explanation in plain words. Be warned however, that some interpretational license is to be expected in the following view. It seems that when a plane wave of *low-frequency* moves along the coil axis, is is slowed by the cylinder-wall discontinuities, but the interaction is relatively weak. In effect, the scattering cross-section of the coil is small, and so the retardation is not at its greatest. Maximum retardation only occurs at high frequencies, when the scattering cross-section is large; in which case the helical mode is strongly excited and, according to both theory and experiment, attains a limiting phase velocity of c.

Now, bearing in mind that the axial and helical phase velocities are linked; it follows that, at low frequencies, if the axial wave is not slowed by the maximum amount, then the helical phase velocity must be considerably greater than c. Indeed, the sheath-helix theory predicts a limiting low-frequency helical velocity factor of about 2.7 for widely-spaced helices. For those unfamiliar with advanced optical concepts incidentally, note that a phase velocity is the apparent velocity of a superposition of waves; the theory does not imply the transmission of information at greater than the speed of light.

All of which brings us at least to a qualitative picture of what goes on with coils: At low frequencies the helical phase velocity is several times greater than c (i.e., the axial wave is dominant). It declines steadily as the frequency increases, but then the rate of change diminishes as the limiting value is approached. It is this cross-over from declining to constant velocity which gives rise to the anomalous impedance characteristic in the run-up to the SRF. Hence the high-frequency breakdown of the regression analysis indicates the end of a long dispersion region. Indeed, as demonstrated by Rhea[14] [15] a simple non-dispersive transmission-line model gives a fair description of a coil operated above its SRF.

The difficulty which remains is that of finding an overall quantitative theory of coils. It is often assumed that the sheath-helix theory is definitive because it is so successful at explaining peculiar phenomena. There are a few serious caveats however, as some have found to their cost. Here springs to mind the innocent phrase: "it passes to the lumped-component theory at low frequencies". Well sadly, no, it doesn't; not if you want numbers which are actually right.

The sheath-helix theory is a theory of what happens in the middle region of infinitely long solenoids. It lacks fringing field corrections (cf. Nagaoka's coefficient) and so can only be argued to pass to the lumped theory for very long coils. Even then however, its accuracy for predicting inductance is only about ±15%, a limitation which must be attributed artificial constraints inherent in the model. Particularly, Maxwell's equations are solved on the basis that current can only flow strictly in the helical direction. The nearest physical realisation is a coil of very fine wire, or several fine wires in parallel laid side-by-side on a coil former[16]. Hence corrections for the difference between the sheath helix and actual wire are needed, and these are not the same as the corrections

12 **Fields and Waves in Communication Electronics**. [cited earlier] Section 9.9: Surface guiding.

13 **Technical and Military Imperatives: A Radar History of World War II.** Louis Brown. 1999. Taylor and Francis. ISBN13: 978-0-7503-0659-1. See Ch. 4. Resonant Magnetron: p153, p409.

14 **Filters and an Oscillator Using a New Solenoid Model**, Randy Rhea, Applied Microwave & Wireless, Nov 2000, p30-42.

15 **A Multimode High-Frequency Inductor Model**, Randall W Rhea. Applied Microwave & Wireless, Nov/Dec 1997, p70-80.

16 **Fields and Waves in Communication Electronics**. [cited earlier] Section 9.8.

used with the current-sheet model.

A theory of infinitely long coils moreover, deals only with travelling waves. Adapting it to describe resonant behaviour involves truncating the cylinder and (in the state of the art so far) ignoring the fringing fields. This approach has been highly successful in predicting the voltage magnification of Tesla coils[17] but it does not permit exact prediction of the SRF. A limitation of the sheath-helix theory is that the turn-over point for the transition from declining to nearly-constant phase velocity is not specifically linked to a resonant process. In practice however, since scattering cross-section is involved, we must expect that it should be.

Intriguingly, we can devise coils where the helical velocity has not reached its limiting value at the SRF (closely spaced turns), and coils where it has (widely-spaced turns). This is as allowed by the sheath-helix theory, but not predicted because it is strictly a phenomenon associated with wires of finite width. We can explain by saying that narrowing the slits in the cylinder wall reduces the braking applied to the axial slow-wave, and so forces-up the helical phase velocity. It may therefore be possible to derive a correction, involving the pitch to wire-diameter ratio (p/d), by combining the sheath-helix and the disk-loaded waveguide models.

So, overall, we have a theory of coils which is not-quite fully fledged; but as we will see, by evoking the occasional empirical parameter, we can still devise transmission-line equations which account for all of the data. That then is the story in plain-text. Now, to take the matter further (with no apologies for the pun); we must engage in a little light mathematics.

2. Self Capacitance (asymptotic form)

The reactance of a coil is only strictly defined by a transmission-line model. For purposes of circuit design however, we can assume that the coil behaves approximately as a lumped inductance in parallel with a fixed capacitance on some interval between very low frequencies and the fundamental SRF. Hence the logical starting point for a general expression for self-capacitance is to use the electrical resonance formula as an asymptotically-correct bridge, linking the low-frequency inductance (derived from magnetics) to the SRF (derived from the conductor length).

Self-resonance of the disconnected coil occurs when the length of wire used to wind it is one half-wavelength. This relationship is not exact, but it is in the nature of the derivatives of the resonance formula for the case where the L/C ratio is very large, that a small change in capacitance gives a large change in f_0. Hence the inverse corollary, which is that even a large uncertainty in f_0 does not greatly affect the self-capacitance; i.e., since we take it that:

$2\pi f_{0s} = 1/\sqrt{(L\, C_L)}$ . . . . . (**2.1**)

then:

$C_L = 1/[\, (2\pi f_{0s})^2\, L\,]$

and

$\partial C_L \,/\, \partial f_{0s} = -1/\, [\, 2\pi^2\, L\, f_{0s}^3\,]$

If the uncertainty in C_L is δC_L, and the uncertainty in f_{0s} is δf_{0s}, then:

$\delta C_L = -\delta f_{0s} \,/\, [\, 2\pi^2\, L\, f_{0s}^3\,]$

This inverse-cubic relationship makes it difficult to predict the SRF accurately from the self-capacitance, but (in principle) easy to predict the self-capacitance from the SRF. It allows us to regard the effects of turn spacing and the fringing fields on wave propagation as second-order effects, and thereby produce first-order formulae for self capacitance.

The length of wire in a helical coil is given by:

$\ell_w = 2\pi r N \,/\, \mathrm{Cos}\psi$ . . . . . (**2.2**)

17 See papers by the Corum brothers, cited earlier.

where r is the effective solenoid radius, N is the number of turns, and ψ is the pitch angle. At the SRF (to a first-order approximation):
$\ell_w = \lambda/2$
where λ=v/f ; v being the apparent velocity of light in the surrounding medium. v is given by:
$v = 1/\sqrt{(\mu\,\varepsilon)} = 1/\sqrt{(\mu_0\,\mu_r\,\varepsilon_0\,\varepsilon_r)}$
and for a self-supporting coil in air or vacuum, $\lambda=\lambda_0$ and:
$v = c = 1/\sqrt{(\mu_0\,\varepsilon_0)}$
Hence, at the SRF:
$\ell_w = 1/[\,2\,f_{0s}\,\sqrt{(\mu\,\varepsilon)}]$
Using this in (**2.2**) gives:
$2rN / \cos\psi = 1/[\,2\pi f_{0s}\,\sqrt{(\mu\,\varepsilon)}]$ . . . . . (**2.3**)

At this point it might seem logical to substitute for $2\pi f_{0s}$ using (**2.1**), but to do so will not result in an equality. This problem arises because of a hidden generalisation in eliminating f_{0s}. Equation (**2.1**) defines the relationship between L and C_L only at the SRF, whereas we want the relationship to hold from DC to the SRF, and we want it to capture any changes in wave propagation which might result from connecting the coil to a circuit. This implies that the self-capacitance is not strictly a constant, it may even change in the presence of a shunt impedance, and so it cannot be defined as a constant. Hence we need an additional parameter, which is as yet unquantified, but which may prove to be a function of coil geometry, or frequency, or load impedance, or perhaps all of those. It is analogous to a relative permittivity, and so we will include it with the permittivity and permeability factors. It is also analogous to Nagaoka's coefficient (the fringing magnetic field correction for the current-sheet solenoid), and so we will call it k_E (electric correction factor). Hence, substituting (**2.1**) into (**2.3**) and restoring equality by including k_E :
$2\,r\,N / \cos\psi = \sqrt{[L\,C_L /(\mu\,\varepsilon\,k_E)]}$
Squaring both sides then gives:
$4\,r^2\,N^2 / \cos^2\psi = L\,C_L /(\mu\,\varepsilon\,k_E)$
Hence:
$C_L = \mu\,\varepsilon\,4\,r^2\,N^2\,k_E / (L \cos^2\psi)$ . . . . . (**2.4**)

The equivalent lumped inductance of a solenoid can be written[18]:
$L = \mu\,\pi\,r^2\,N^2\,k_H / \ell$ . . . . . (**2.5**)
where k_H is an aggregation of correction factors which is generally within 1% of Nagaoka's coefficient (k_L), and is assumed to be equal to k_L in the current sheet approximation. Substituting (**2.5**) into (**2.4**) gives:

$$C_L = \frac{4\,\mu\,\varepsilon\,r^2\,N^2\,\ell\,k_E}{\mu\,\pi\,r^2\,N^2\,k_H \cos^2\psi}$$

i.e.;

$C_L = (4\,\varepsilon / \pi)\,\ell\,(k_E / k_H) / \cos^2\psi$	**2.6**

For a self-supporting coil in air, this becomes:
$C_L = (4\,\varepsilon_0 / \pi)\,\ell\,(k_E / k_H) / \cos^2\psi$
where
$4\varepsilon_0/\pi$=11.27350207 pF/m
The coefficient k_E , of course, remains to be determined.

18 Solenoids. D W Knight. Still in HTML form at time of writing.

3. Medhurst's formula

The most widely cited study of solenoid self-capacitance is that reported by R G Medhurst in 1947 [reference given earlier]. The intended remit of that work however, was probably not as wide as is generally assumed. Medhurst engaged in a study of the AC resistance of solenoids with a view to producing formulae and tables for the prediction of Q. In order to do that, he needed to correct his measurements for the effect of self-capacitance, and it was his original intention to use the theory of A J Palermo (to be discussed in section **7**) for that purpose. Palermo gives a formula based on the hypothesis that the self-capacitance can be deduced by considering the capacitance between adjacent turns. Medhurst, being a meticulous experimenter, soon ran into difficulties with that approach; and so was forced to "find out whether Palermo's formula did in fact agree with experiment". He concluded that the data supporting Palermo's theory were suspect; and fell only a little short of accusing Palermo of scientific fraud.

Medhurst's solution to the dearth of believable theory was to make self-capacitance measurements on a large number of test coils, all of which were wound on solid polystyrene rods. He then corrected the data for strays and fitted them to the following regression formula:

$C_L / D = 0.1126(\ell/D) + 0.08 + 0.27\sqrt{(D/\ell)}$ [pF / cm]	Medhurst's Formula

The first thing to notice here is that the coefficient 0.1126 is $4\varepsilon_0/\pi$ in pF/cm with an error in the last digit as occurs when the value of c is taken to be 300M m/s (instead of 299 792 458 m/s, which is now the standard value). As Medhurst stated in his paper: "The first numerical factor follows from Nagaoka's inductance formula for long coils and the experimental fact that the self-resonant wavelength for long coils equals twice the length of the winding". In other words; he employed a derivation somewhat akin to that which led us to equation (**2.6**), but using the long current-sheet approximation (k_L=1), and without encountering the $1/\mathrm{Cos}^2\psi$ factor. The latter omission is understandable however, because Medhurst was aware that self-capacitance is substantially independent of turn-spacing provided that the coil has plenty of turns. He therefore chose to keep the number of turns per unit length high to eliminate pitch effects, and thus worked in the regime where $\mathrm{Cos}^2\psi \approx 1$.

Medhurst's formula can, of course, be put into the form of equation (**2.6**), and it is instructive to do so. We start by multiplying throughout by D, restoring natural constants to their proper identities, and converting to SI units by multiplying by 10^{-12} to get rid of the p in pF, and multiplying it by 100 to get rid of the c in /cm. This gives:

$C_L = (4\varepsilon_0\, \ell/\pi) + 8\times10^{-12} D + 27\times10^{-12} D\sqrt{(D/\ell)}$ [Farads]

Now, factoring $4\varepsilon_0\ell/\pi$ from each of the terms we get:

$C_L = (4\varepsilon_0\ell/\pi)\, [\, 1 + \{8\times10^{-12}\pi/(4\varepsilon_0)\}D/\ell + \{27\times10^{-12}\pi/(4\varepsilon_0)\}(D/\ell)^{3/2}\,]$

Which, after re-enumerating the empirical constants (and avoiding the introduction of rounding error by retaining more significant figures than is justified), gives:

$C_L = (4\varepsilon_0/\pi)\, \ell\, [\, 1 + 0.7096(D/\ell) + 2.395(D/\ell)^{3/2}\,]$ [Farads] . . . . (**3.1**)

Comparing this with equation (**2.6**), shows that Medhurst has given us the coefficient k_E/k_H as:

$k_E / k_H = [\, 1 + 0.7096(D/\ell) + 2.395(D/\ell)^{3/2}\,]$

We will not however accept Medhurst's formula as it stands, there being several shortcomings which need to be addressed.

The first issue is that, in 1947, carrying out a least-squares fitting procedure on any sizeable dataset could amount to several days of work. Consequently, Medhurst's statistical investigation is minimal. He does however report his data; which, although not in raw form, are nevertheless as adjusted prior to fitting. Hence we can repeat the analysis.

Writing equation (**3.1**) with undefined empirical coefficients we have:

$C_L = (4\varepsilon_0/\pi)\, \ell\, [\, 1 + k_1 (D/\ell) + k_2 (D/\ell)^{3/2}\,]$

Dividing both sides by $(4\varepsilon_0/\pi)D$ gives:

$(C_L/D) / (4\varepsilon_0/\pi) = (\ell/D) [1 + k_1 (D/\ell) + k_2 (D/\ell)^{3/2}]$
Multiplying (ℓ/D) into the right-most bracket gives:
$(C_L/D) / (4\varepsilon_0/\pi) = (\ell/D) + k_1 + k_2 (\ell/D)^{-1/2}$
and subtracting (ℓ/D) from each side gives:
$(C_L/D)/(4\varepsilon_0/\pi) - (\ell/D) = k_1 + k_2 (\ell/D)^{-1/2}$
This is a straight-line graph of the form y=a+bx, with:
$y = [(C_L/D)/(4\varepsilon_0/\pi)] - (\ell/D)$
Notice here that the derivative $\partial y/\partial(C_L/D)$ is a constant. Hence there is no non-linear scaling of uncertainties to contend with in this case.

Medhurst reports his adjusted data as a table of C_L/D vs. ℓ/D. What he actually measured in each case however, was capacitance in the range of about 1 to 10pF. Hence we should note that the uncertainty of a C_L/D value is probably best expressed as a percentage common to the whole dataset. It transpires however, that the regression line for Medhurst's function does not lie particularly close to the data for low C_L/D values when realistically weighted. The problem is that the choice of polynomial is not optimal, and needs artificial weighting in order to force it on to the lower asymptote. Thus, assuming for purely pragmatic reasons that all of Medhurst's data have equal uncertainties, a simple least-squares fit[19] (spreadsheet: **Medhurst.ods**, sheet 1) returns the following information :
$k_1 = 0.824903 \pm 0.089$
$k_2 = 2.328995 \pm 0.073$
$\sigma_{CL} = 3.6$ D pF
the latter statistic meaning that a value for C_L computed from the formula below has a standard deviation in pF of 3.6 times the coil diameter in metres. Hence, Medhurst's formula, in its best state of optimisation is:

$C_L = (4\varepsilon_0/\pi)\, \ell\, [1 + 0.8249(D/\ell) + 2.329(D/\ell)^{3/2}]$ [Farads]	r >> p	**3.2** Medhurst refitted

Our best estimate for the coefficient k_E/k_H using Medhurst's data and choice of fitting function is therefore:

$k_E / k_H = 1 + 0.8249(D/\ell) + 2.329(D/\ell)^{3/2}$	Solid polystyrene former	**3.3**

Equation (**3.2**) describes the data fairly well, but the pattern of residuals (observed minus calculated) is not random (see: **Medhurst.ods**, sheet 1) and, as mentioned above, the weighting required in order to obtain it is not realistic. This indicates that a different choice of fitting function will give a better result (i.e., a smaller standard-deviation of fit). There is however no point in pursuing this matter until we have solved the riddle of the missing dielectric constant.

19 For explanation of the statistical methods used in this article; see **Scientific Data Analysis**. D W Knight. Available from www.g3ynh.info

4. Coil-former dielectric

All of Medhurst's coils were wound on solid polystyrene rods. No matter whether we subscribe to the 'capacitance between adjacent turns' hypothesis, or to a transmission line theory; even the most cursory consideration of the fields involved will tell us that the dielectric constant of the coil-former must appear in any expression for self capacitance. In fact, it is odd that Medhurst did not raise this matter, especially since he did briefly consider the issue of dielectric losses in determining R_{ac}. If he accepted that the dielectric was penetrated by the electric field, then why did he assume that it would not affect the capacitance?

A clue to the conundrum can be had by plotting Medhurst's formula and comparing it against two lines: one being the long-coil asymptote for the case where the effective permittivity is the same as that of air, i.e.;

$C_L/D = (4\varepsilon_0/\pi)(\ell/D)$

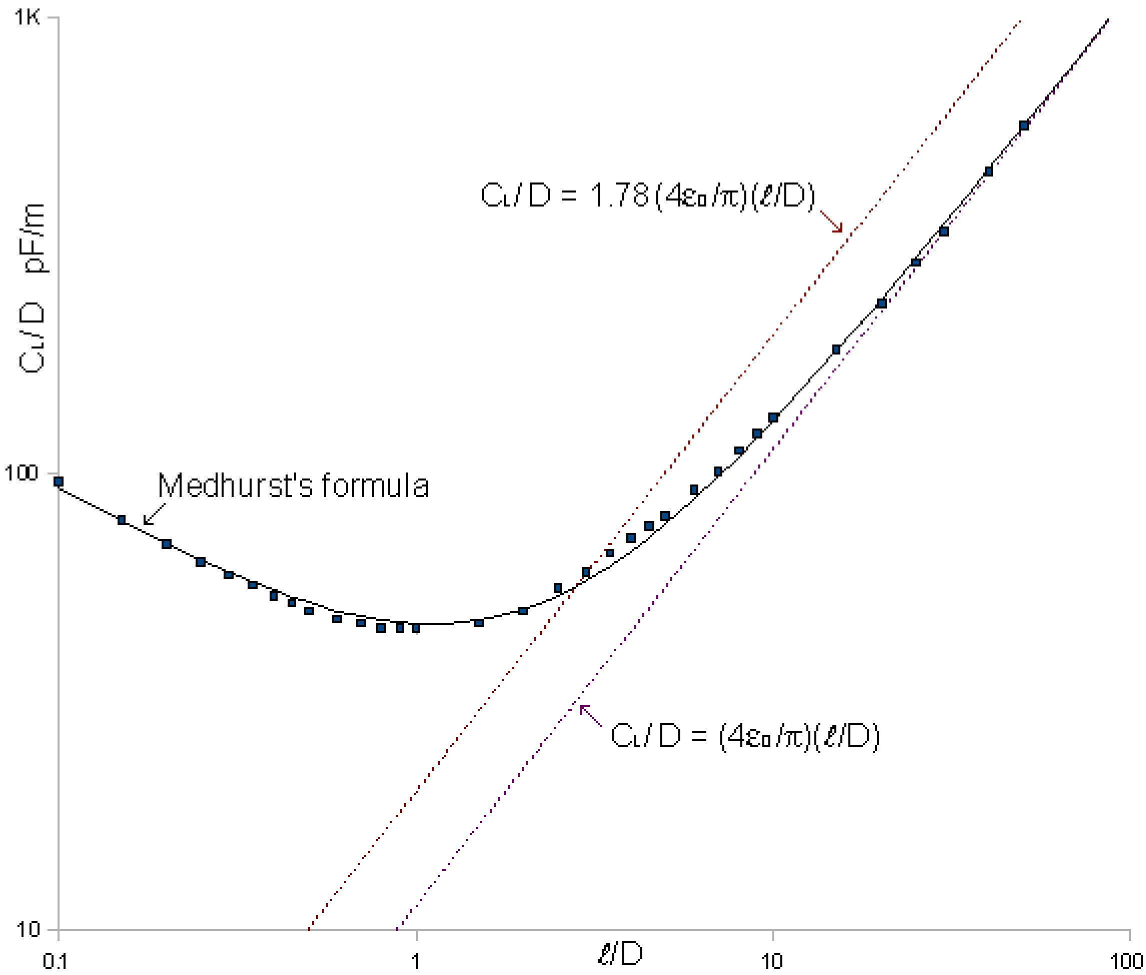
the other being the the long-coil asymptote for the case where the effective permittivity is average of that of the coil former and the surrounding air. Medhurst made all of his measurements at frequencies in the range 460KHz to 25MHz. The dielectric constant of typical polystyrene in this region of the spectrum is 2.56. Hence the average relative permittivity is:

$(2.56 + 1)/2 = 1.78$

and the asymptotic formula in that case is:

$C_L/D = 1.78\ (4\varepsilon_0/\pi)(\ell/D)$

The comparison is given in the graph below, with Medhurst's adjusted data superimposed (spreadsheet: **Medhurst.ods**, sheet 2).



It is obvious that when the coils become long and thin, the data converge with the line corresponding to an effective relative permittivity of 1. This behaviour can be interpreted in one of two ways: *either* the coil former dielectric plays no part in determining the self-capacitance; *or* the effect of the coil former material diminishes as ℓ/D increases.

Medhurst obviously chose the first interpretation. He appears to have done so (despite having used the relationship between conductor length and SRF in deducing the asymptotic form) because he was attempting to interpret his findings according to the then prevailing wisdom: which is that self capacitance is composed of internal and external components. The internal component is the capacitance between adjacent turns, which he considered to be the minor contributor. The external component is the capacitance from the coil body to the ground plane, which he considered to be dominant. This gave him qualitative grounds for rejecting Palermo's theory, and presumably a reason for ignoring the coil former dielectric; but he seems to have decided to terminate the investigation without pursuing the matter further. We can perhaps understand his lack of curiosity at this point, by observing that the shortcomings of Palermo's work had practically doubled the amount of work he needed to do in order to complete his study of AC resistance. What is less excusable is the assumption, by several generations of engineers since then, that Medhurst's formula can be applied to all coils, rather than just to coils with solid polystyrene cores.

Here, of course, we take the view that the SRF is a transmission-line resonance, in which case the disappearance of the coil-former permittivity from the long-coil asymptotic behaviour is not difficult to explain. Consider a wave travelling along the helix. In the middle region of the solenoid at least, the electric vector will be substantially perpendicular to the axis (actually, tilted by an amount equal to the pitch angle). The boundary conditions on Maxwell's equations for this problem also allow that the electric field is continuous across the conducting wall[20]. Hence the relative permittivity on the inside of the solenoid will affect wave propagation. When the circumference is small in comparison to the wavelength however, the electric field will be cylindrically symmetric. This means that the electric fields penetrating into the interior of the solenoid from opposite sides will be almost equal and opposite. Therefore the fields on the inside of the solenoid will tend to cancel; the degree of cancellation being minimal when $\ell/D \ll 1$ and almost complete when $\ell/D \gg 1$.

It follows that self-capacitance can be conceived as the sum of two parts, although not as envisaged by Medhurst. If we define the relative permittivity external to the solenoid as ε_{rx} and the relative permittivity on the inside as ε_{ri}, then we need to re-derive the formula in such a way that:

$\varepsilon_r = \varepsilon_{rx}$ when $\ell/D \gg 1$

and

$\varepsilon_r = (\varepsilon_{ri} + \varepsilon_{rx})/2$ when $\ell/D \ll 1$

i.e., for the latter case; when the diameter of the coil is large in comparison to the length, the effective permittivity tends to the average of the internal and external permittivities. It is worth noting here incidentally, that Sichak[21], in his study of helically-loaded coaxial lines, comes to effectively the same conclusion in deriving velocity factors.

The general first-order expression for coil self-capacitance was derived earlier as:

$$C_L = (4\, \varepsilon / \pi)\, \ell\, (k_E / k_H) / \mathrm{Cos}^2\psi$$

We can still retain this general form, and satisfy the required asymptotic behaviour, by re-writing the equation as follows:

$$C_L = (4\, \varepsilon_0 / \pi)\, \ell\, [\, \varepsilon_{rx} + k_c (\varepsilon_{rx}+\varepsilon_{ri})/2\,] / \mathrm{Cos}^2\psi$$

Here the external relative permittivity ε_{rx} has been separated from the original permittivity factor ε, and k_c is a coefficient which goes to zero when $\ell/D \gg 1$. Now factoring ε_{rx} from the square bracket we have:

20 See, for example, Corum & Corum, cited earlier.

21 **Coaxial Line with Helical Inner Conductor**. W Sichak. Proc. IRE. Aug. 1954. p1315-1319. Correction Feb. 1955, p148. See equations (5) and (6).

$C_L = (4\, \varepsilon_0\, \varepsilon_{rx} / \pi)\, \ell\, [\, 1 + k_c (1+\varepsilon_{ri}/\varepsilon_{rx})/2\,] / \cos^2\psi$	**4.1**

Which is in the same form as equation (**3.2**) (Medhurst's formula optimised) when $\varepsilon_{rx}=1$ and $\cos^2\psi=1$. Also noting that $\varepsilon_{ri}=2.56$ for Medhurst's data, we have an initial estimate for k_c from equation (**3.3**)
$1.78\, k_c = 0.8249(D/\ell) + 2.329(D/\ell)^{3/2}$
i.e.,
$k_c = 0.4634(D/\ell) + 1.3084(D/\ell)^{3/2}$

5. Empirically corrected formula for self-capacitance

Now having the complete form for an expression for self-capacitance (at least, in as far as Medhurst's approach is valid), we can use Medhurst's data, or indeed good data from any source, to find an expression for the coefficient k_c. We start by dividing both sides of equation (**4.1**) by D and rearranging:

$$\left[\frac{(C_L/D) \cos^2\psi}{(\ell/D)\,(4\varepsilon_0/\pi)\,\varepsilon_{rx}} - 1 \right] \frac{2}{(\, 1+\varepsilon_{ri}/\varepsilon_{rx}\,)} = k_c \qquad (\mathbf{5.1})$$

If Medhurst's data are used to evaluate k_c using this expression, it will be found that a roughly-straight line of negative gradient is obtained when $\log(k_c)$ is plotted against $\log(\ell/D)$. Hence, to a first approximation, there is a regression line having the form:
$\ln(k_c) = k_1 - k_2 \ln(\ell/D)$
We can, of course, make the gradient positive by inverting the argument of the logarithm on the right, i.e.:
$\ln(k_c) = k_1 + k_2 \ln(D/\ell)$
Fitting the data on this basis yields:
$\ln(k_c) = 0.604 + 1.363 \ln(D/\ell)$
(see spreadsheet: **Medhurst.ods**, sheet 3). Although the existence of a logarithmic relationship may be analytically significant however; the fit, corresponding to a standard deviation of 4.7% for unit variance of an observation of unit weight ($\chi^2/\nu=1$), is not as good as that obtained in the process of optimising Medhurst's formula (section **3**). Hence a more complicated function with a greater number of adjustable parameters is needed.

Taking the exponent of the expression above gives a first approximation for k_c as:
$k_c \approx \exp[\, \ln(1.83) + 1.363 \ln(D/\ell)\,]$
where $\ln(1.83) = 0.604$. Hence:
$k_c \approx 1.83\, (D/\ell)^{1.363}$
The fit can obviously be improved by replacing this with a polynomial in D/ℓ. Notice also that k_c must go to zero when ℓ/D is very large, i.e., when $D/\ell \to 0$. Hence the required polynomial has no zero-order (i.e., constant) terms. If the polynomial has a finite first-order term however, we can create a new expression with a finite zero-order term by multiplying throughout by ℓ/D; i.e., if we have a starting expression:
$k_c = k_1\, (D/\ell) + k_2\, (D/\ell)^{p2} + k_3\, (D/\ell)^{p3} + \ldots$
then
$(\ell/D)\, k_c = k_1 + k_2\, (D/\ell)^{p2-1} + k_3\, (D/\ell)^{p3-1} + \ldots$
Fitting the data to an expression of this type is easily accomplished using a modified linear regression procedure, where the third and higher terms (if needed) are manually adjusted. Hence

we rearrange equation (**5.1**) so that $(\ell/D)k_c$ is the coefficient to be represented as a polynomial:

$$\left[\frac{(C_L/D)\,\mathrm{Cos}^2\psi}{(4\varepsilon_0/\pi)\,\varepsilon_{rx}} - (\ell/D) \right] \frac{2}{(1+\varepsilon_{ri}/\varepsilon_{rx})} = (\ell/D)\,k_c \qquad (5.2)$$

This expression is now taken to represent a simple polynomial of the form:
$y = k_0 + k_1 x + k_2 x^2 + \ldots$
Note that the relationship between x and D/ℓ is yet to be decided; but by inspection of Medhurst's formula it should come as no surprise that something approaching an optimal fit is obtained when:
$x = \sqrt{(D/\ell)}$
In performing a least-squares fit, we must, of course, weight the data according to their relative uncertainties. Medhurst does not report the actual capacitances or diameters used in obtaining his table of C_L/D values, but by fitting the data initially with equal weights, it can be seen from the pattern of residuals that the largest C_L/D values have the greatest scatter. Also, the largest C_L/D values are those for long coils, and the data for long coils are least important because the long coil asymptotic behaviour is analytically defined. Hence it is reasonable to fit the data on the basis that the uncertainty in an observation is proportional to the absolute value of the observation, i.e.:
$\delta(C_L/D) = u\,(C_L/D)$
where u is a proportionate uncertainty (and 100u is a percentage uncertainty) common to the whole dataset. From this we can determine the uncertainty of a y value as:
$\delta y = [\partial y/\partial C_L/D)]\,\delta(C_L/D) = [\partial y/\partial(C_L/D)]\,u\,(C_L/D)$
The derivative $\partial y/\partial(C_L/D)$ is obtained by differentiating (**5.2**):
$\partial y/\partial(C_L/D) = [\,\mathrm{Cos}^2\psi\,/\,\{\,(4\varepsilon_0/\pi)\,\varepsilon_{rx}\,\}][\,2\,/\,(1+\varepsilon_{ri}/\varepsilon_{rx})\,]$
Hence:
$\delta y = u\,(C_L/D)\,[\,\mathrm{Cos}^2\psi\,/\,\{\,(4\varepsilon_0/\pi)\,\varepsilon_{rx}\,\}][\,2\,/\,(1+\varepsilon_{ri}/\varepsilon_{rx})\,]$
Which, for coils with external air dielectric and small pitch angle, reduces to:
$\delta y = u\,[\,(C_L/D)\,/\,(4\varepsilon_0/\pi)][\,2\,/\,(1+\varepsilon_{ri})\,]$
The statistical weight of an observation is given by:
$w_i = 1/\delta y_i^2$
and u is adjusted until the standard deviation of fit, $\chi^2/\nu=1$.

Details of the final fitting procedure can be had by examining the spreadsheet file: **Medhurst.ods**, sheet 4. A good fit (u = 0.021) is obtained using the polynomial:
$(\ell/D)\,k_c = k_0 + k_1\sqrt{(D/\ell)} + k_2\,(D/\ell)$
where:
$k_0 = 0.717439 \pm 0.027$
$k_1 = 0.933048 \pm 0.021$
$k_2 = 0.106$
Thus:

$C_L = (\,4\,\varepsilon_0\,\varepsilon_{rx}\,/\,\pi\,)\,\ell\,[\,1 + k_c(1+\varepsilon_{ri}/\varepsilon_{rx})/2\,]\,/\,\mathrm{Cos}^2\psi \quad \pm 2.1\%$ where $k_c = 0.717439(D/\ell) + 0.933048(D/\ell)^{3/2} + 0.106\,(D/\ell)^2$	**5.3** **C_L-DAE**

This will be referred to as the DAE (doubly-asymptotic, empirically corrected) formula for solenoid self-capacitance (generally applicable when $\ell/D>>1$, or when the externally connected capacitance is > C_L). Note that there is little point in extending the polynomial; a standard deviation of 2.1% being about right for the data available (and the theory as it stands), and the absence of high-order terms making it reasonably safe to extrapolate.

6. Tubular coil formers

Medhurst was either wise or fortunate in his decision to wind his test coils on solid rods all made from the same material. Had he chosen to use tubes, or a variety of dielectrics, he would not have been able to fit his short-coil data successfully. From this it seems likely that he *was* expecting to see a dielectric effect; but that the long-coil asymptotic behaviour he discovered allowed him to neglect it.

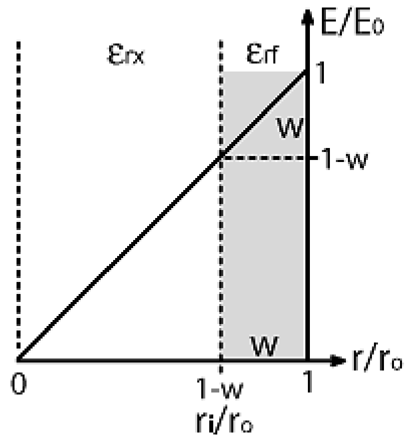
Most coil-formers used in practice are, of course, tubular. This gives us the problem of how to determine ε_{ri}, which will be some weighted average of the permittivities of the solid dielectric and the air inside. There are two principal issues here: one being that of determining the rate of decay of the electric field on moving from the conducting wall to the coil axis; the other (since this is a correction for short coils) being that of determining the fringing field corrections for turns close to the two ends of the coil.

By setting up the scattering experiment described in section **1**, it is easy to demonstrate that the introduction of any substantial amount of dielectric material to the inside of a coil causes a reduction in the SRF. Apart from that however, there is a dearth of good quantitative information on the effect of inhomogeneous internal dielectrics. The best we will do here therefore, is to posit a first-order correction which is reasonably realistic.

It seems likely that ignoring end effects and pitch angle will not cause too much error. In that case we only have to consider fields perpendicular to the solenoid axis. We will also assume that the circumference of the cylinder is short in comparison to the wavelength, in which case, there will be no phase shift from one turn to the next. Thus we assume that the electric field is radially symmetric, and that its strength is at a maximum at the conducting wall and zero at the axis. This arrangement precludes exponential decay (which only gives zero at infinite distance), and likewise precludes an inverse square law (which is associated with radiation). The conclusion is that, when the material on the inside of the coil is homogeneous, the decay will be roughly linear.

We now need to define the problem in such a way that dielectric close to the conducting wall has more effect than dielectric close to the coil axis. This can be done crudely by considering the area under the curve of relative field strength vs. relative radius divided into regions having different dielectric constants. We then assume that the average dielectric constant is weighted according to the relative areas in the two regions. This is a simple integration problem, so simple in fact that calculus is not needed.

In most practical situations, the medium in the hollow part of the cylinder will be the same as that outside the coil (i.e., air usually). Hence the inner medium is taken to have a dielectric constant ε_{rx}. The coil-former tube is made from a material having a dielectric constant ε_{rf}, and the tube has a relative wall thickness w.



The total area under the curve is ½. The area in the region having a dielectric constant ε_{rx} is $(1-w)^2/2$. The area in the region having a dielectric constant ε_{rf} is:

$[1 - (1-w)^2]/2$.

Hence:

$\varepsilon_{ri} = \varepsilon_{rx}\,(1-w)^2 + \varepsilon_{rf}\,[1 - (1-w)^2]$

If r_o is the outside radius of the coil former, and r_i is the inside radius of the coil former, then:

$(1-w) = r_i/r_o$

Hence:

$\varepsilon_{ri} = \varepsilon_{rx}\,(r_i/r_o)^2 + \varepsilon_{rf}\,[1 - (r_i/r_o)^2]$	**6.1**

Note that in determining the internal permittivity in this way, it is assumed that there is no difference between r_o and the effective radius of the coil. Since the effective radius tends towards the inner solenoid radius (average radius - wire radius) at high-frequencies, this approximation is unlikely to cause a large error. In the current-sheet approximation, of course, the effective radius is the same as r_o by definition.

When using equation (5.3) to calculate self-capacitance, the internal permittivity parameter required is $\varepsilon_{ri}/\varepsilon_{rx}$. Hence, equation (6.1) can be conveniently rewritten:

$\varepsilon_{ri} / \varepsilon_{rx} = (r_i/r_o)^2 + (\varepsilon_{rf} / \varepsilon_{rx})[1 - (r_i/r_o)^2]$	6.2

It must be stressed that this is a very crude correction. A detailed theoretical analysis will no doubt produce a different formula, and this one is offered strictly on the basis that any correction is better than no correction at all.

7. Inter-turn capacitance

A trivial investigation involving a Grid-Dip Oscillator and a set of engineer's callipers will confirm that the various resonances exhibited by a disconnected coil are associated with the total conductor length. It is therefore extraordinary that the self-capacitance of single-layer coils is still routinely attributed to the static capacitance which is presumed exist between adjacent turns. This warrants consideration of whether of not the inter-turn capacitance hypothesis is plausible, and whether or not it can it provide insights into the properties of inductors.

The basic idea is that if we inspect a small region of a solenoid wall we see a set of wires lying parallel to each other. It is then supposedly logical that there will be a capacitance between any chosen pair of wires; and that this capacitance can be calculated from physical dimensions. There is a small paradox inherent in the fact that every infinitesimal element of capacitance is shorted-out by a loop of wire, but we must put such details aside in order to proceed.

The capacitance between a parallel pair of conducting cylinders is given by Russell's formula[22]:

$$C = \frac{\varepsilon \pi h}{\ln\{ (p/d) + \sqrt{[(p/d)^2 - 1]} \}} \quad \text{[Farads]} \qquad \text{Russell's formula } h \gg p,\ h \gg d \qquad (7.1)$$

Where h is the length of the cylinders, p is the distance from axis to axis, and d is the cylinder diameter. Note that, if the surrounding medium is air, $\varepsilon = \varepsilon_0$; and in old publications, $\varepsilon_0 \times \pi$ is sometimes given approximately as 1/3.6 pF/cm. Also:
$\ln[x + \sqrt{(x^2 - 1)}] = \text{Arccosh}(x)$
which gives rise to the compact form:

$$C = \frac{\varepsilon \pi h}{\text{Arccosh}(p/d)} \quad \text{[Farads]} \qquad (7.1a)$$

Since Arccosh (inverse hyperbolic cosine) is a built-in function of spreadsheets and some programming languages, the latter formula is usually most convenient. Note that Arccosh(1) = 0 , which means that the capacitance goes to infinity when the cylinders are just touching without making electrical contact.

22 See, for example: **Radio-Frequency Measurements by Bridge and Resonance Methods**, L. Hartshorn, Chapman & Hall, 1940 (Vol. X of "Monographs on Electrical Engineering", ed. H P Young). 3rd imp. 1942. Ch VI, section 3: Calculation of capacitance. (Russell's formula for wires on p104).

The 1934 paper of A J Palermo[23] was mentioned earlier as the source of Medhurst's frustrations. Palermo, using a somewhat dubious argument involving charge, asserts that, since the voltage between adjacent turns is 1/N times the voltage across the whole coil, we should take the self-capacitance of the coil to be 1/N times the capacitance calculated using equation (**7.1a**). To calculate that capacitance he integrates over the whole coil (ignoring pitch angle), effectively choosing to identify $h=\pi DN$, and arrives at the formula:

$$C_L = \frac{\varepsilon_0 \pi^2 D}{\mathrm{Arccosh}(p/d)} \qquad \text{Palermo's formula} \qquad (7.2)$$

Relative permittivity is ignored, and there is no mention of coil-former dielectric anywhere in the paper.

Notwithstanding the painful logic involved in getting this far, there is a straightforward mathematical error in Palermo's formula. The point of objection lies in the assumption that the turns overlap for the entire length of the wire; whereas there is no adjacent turn on the outside for the two turns at the ends of the coil. Hence he should have taken h to be $\pi D(N-1)$, in which case he would have obtained the expression:

$$C_L = \frac{\varepsilon_0 \pi^2 D\,(N-1)}{N\,\mathrm{Arccosh}(p/d)} \qquad \text{CT2T} \qquad (7.3)$$

For the sake of working nomenclature, we will refer to this corrected version of Palermo's formula as "CT2T" (capacitance from turn to turn).

We cannot know whether Palermo started by making measurements and then derived his formula, or vice versa. It seems likely that he had at least one measurement available initially however, that of his coil No. 1. The coil was made from 2 turns of 6.24mm diameter wire, with an average diameter of 74.7mm and a pitch of 16.7mm. Palermo measured the self-capacitance as 3.2pF, and calculated 3.9pF using an approximate version of equation (**7.2**). He considered this to be a "very severe test" of his formula; whereas, in view of the mathematical error, it is actually just a coincidence. Subsequent coils had turns numbers in the range of 5 to 112 however, in which case the difference between (**7.2**) and (**7.3**) is less significant.

Palermo reported a total of 19 self-capacitance measurements, 12 of which he carried out himself, and 7 of which were communicated to him by F W Grover of the National Bureau of Standards. It was in the group of measurements performed by Palermo himself that Medhurst found some of the numbers to be unreproducibly large. Later we will compare the measurements against the DAE formula and show that Medhurst was right to cry foul; but, in fact, the extent of the tampering was even greater than Medhurst had suspected.

Palermo's calculations are repeated in the spreadsheet **CL_theor_test.ods** (sheet 3). His formula often produces values which are much too large. In such cases, he appears to have adopted the habit of adjusting the calculated value downwards and the measured value upwards in order to obtain plausible agreement. Since he acknowledges the help of F W Grover however, he was evidently not in a position to tamper with the NBS data; and so in that case he confined himself to writing down false calculation results. In the worst instance, his formula gives 27pF, but he reports 12.9pF to confer with an NBS measurement of 12.8pF. There are other sleights of hand for those who wish to pursue the issue, but overall the paper is a travesty.

That then is the insalubrious basis on which the inter-turn capacitance hypothesis became part of electromagnetic folklore. What Palermo hoped to gain by promoting his defective theory is difficult

23 **Distributed Capacity of Single-Layer Coils**, A J Palermo. Proc. IRE. Vol 22, No. 7, July 1934. p897-905.

to guess; but he may have been motivated by inability to accept failure after an early success. His formula was subsequently turned into tables and abacs to 'assist' the radio engineer; and his dogma diffused naturally into the textbooks to lie in wait for the unwary.

The 'capacitance between adjacent turns' hypothesis re-emerged in a new guise in 1999, in a paper by Grandi, Kazimierczuk, Massarini and Reggiani (GKMR)[24]. These authors cite Medhurst, only to dismiss his work for being empirical; and make no mention of Palermo despite the strong parallel and Medhurst's barbed discussion.

In the GKMR approach, the coil is considered to be equivalent to a set of wire rings. In that case, since there are N-1 gaps between N turns, the capacitance of an isolated coil is given by the capacitance between any pair of rings divided by N-1. The length of a ring is πD ; and so, using Russell's formula (**7.1**), and presuming the use of un-insulated wire, we have:

$$C_L = \frac{\varepsilon_0 \pi^2 D}{(N-1) \ln\{ (p/d) + \sqrt{[(p/d)^2 - 1]} \}} \qquad \text{GKMR} \qquad (7.4)$$

Grandi et al. support this derivation by measuring the capacitance of actual sets of wire rings. In doing so they demonstrate that the neglect of capacitance between non-adjacent turns is not important, and that the curvature of the wires does not significantly affect the validity of Russell's formula. The issue which must concern us here however, is that the wire-ring model does not have the magnetic field of the actual solenoid, and it does not consider the electromagnetic propagation which dictates the relationship between the electric and magnetic vectors at all points in the field.

A fair test of all of the theories discussed so far can be had by using data from a source with no theoretical axe to grind. Such data appear in the documentation for the Coilcraft Maxi Spring™ (132-xxSM) series of surface-mount air-core inductors[25], the guaranteed lower limit of SRF having been recorded to 3 decimal places. In fact these "SRF" data are extrapolations, made from jig measurements with finite stray capacitance, and so correspond to the pseudo-SRF calculated from the parallel combination of inductance and self-capacitance. Hence, using the published nominal inductance (±2%), they can be converted back to the self-capacitances they cryptically represent. What is particularly useful about this dataset is that the solenoid length ℓ and diameter D, and the wire diameter d, are constants. The only variables are the number of turns N and the pitch to wire diameter ratio (p/d). Hence there is no ambiguity in deciding between theories which predict no or minimal variation in self capacitance with turn-spacing, and those which make predictions to the contrary.

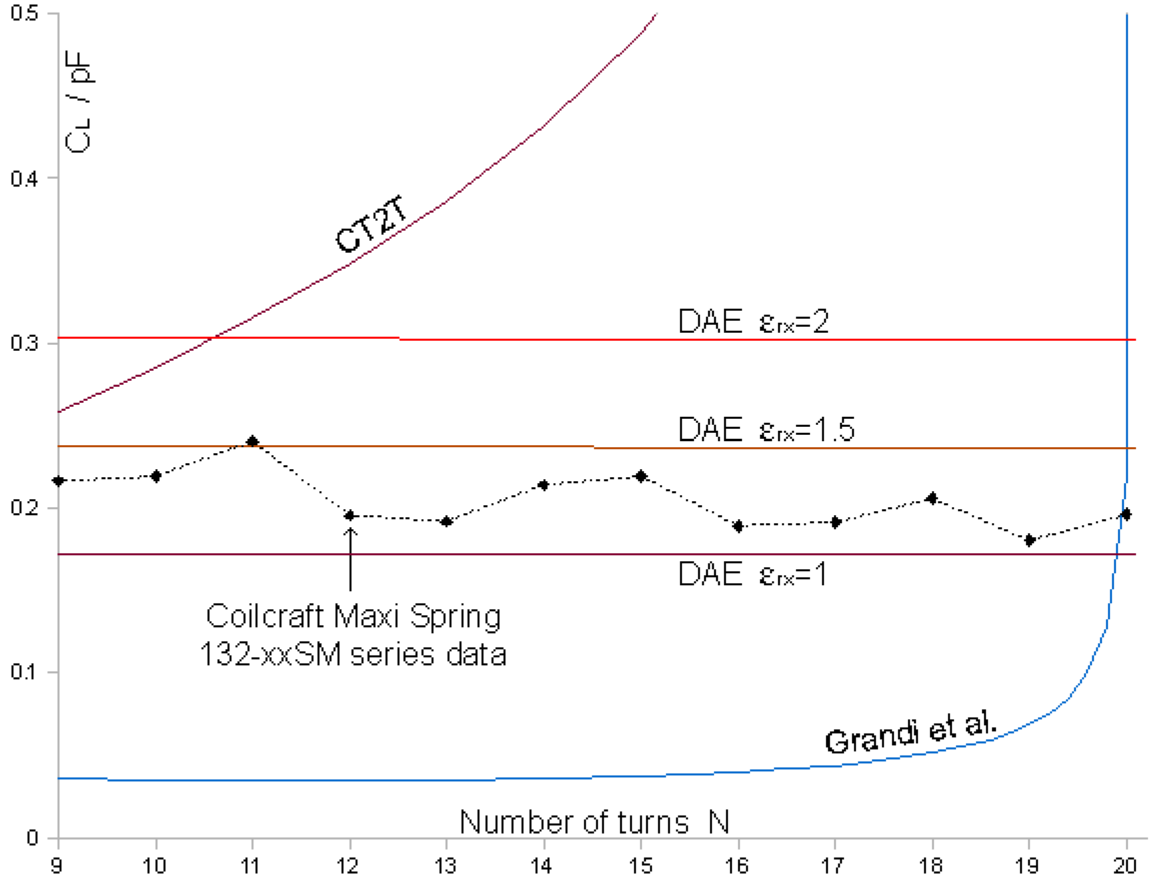
In order to test the various theories, the Coilcraft solenoid parameters were extracted from the published mechanical data as follows:

$\ell = 7.98 \pm 0.51$mm
D = 4.8mm (estimated)
d = 0.397mm (estimated).

The calculations are given in the spreadsheet: **CL_theor_test.ods** (sheet 2), and the results are shown graphically below:

24 **Stray Capacitances of Single-Layer Solenoid Air-Core Inductors"**, G. Grandi, M K Kazimierczuk, A Massarini, U Reggiani. IEEE Transactions on Industry Applications, Vol 35, No. 5, Sept/Oct 1999, p1162-1168.

25 **Coilcraft Maxi Spring Air Core Inductors.** Document 185-1, 2003. www.coilcraft.com



The data are noisy, but there is obviously minimal correlation between the capacitance and the p/d ratio. Instead, the points lie slightly above the ε_{rx}=1 external-permittivity contour from the DAE formula (5.3). The fact that the DAE prediction is a little low is explicable on two counts: Firstly, the experimental capacitances are calculated from the guaranteed minimum pseudo-SRF, and so correspond to a guaranteed 'no-greater-than' value. Secondly, each of the coils has a tight-fitting rectangular plastic cover, which touches the cylinder at three points around the circumference (see illustration in datasheet).

The curve labelled "CT2T" is produced by the corrected version of Palermo's formula (7.3). The uncorrected version (7.2) is even worse. The curve labelled "Grandi et al." is produced by the GKMR formula (7.4). None of these theories bears any resemblance to the data series, but all are capable of matching a single measurement by deliberate or accidental choice of turn-spacing. In the case of the GKMR formula, the turns need to be very close together in order for the coincidence to occur.

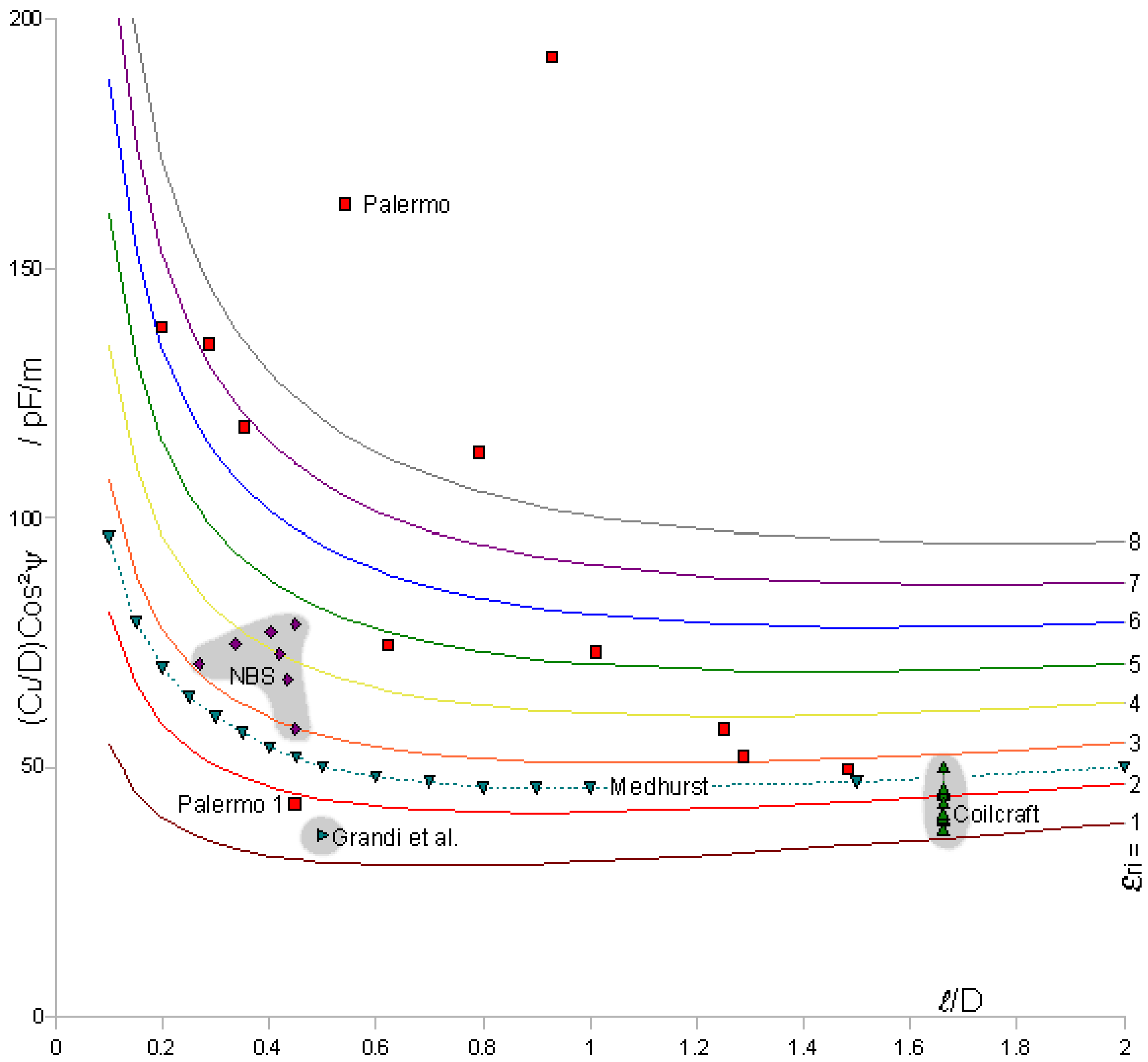
Interestingly, the theoretical work of Grandi et al. was supported by a single measurement on an actual coil, this having the very low p/d ratio of 1.02. The coil was wound using 16 turns of 10mm diameter wire (presumably tubing), with a pitch of 10.2mm maintained by using "plastic spacers" present for about 10% of the turn length. The type of plastic and its dielectric constant were not reported. The mean coil diameter D was 326mm, and its length (Np) was 163.2 mm. The inductance of the coil was measured to be 82.3µH at 10KHz, and its (pseudo) SRF was 5.1MHz, giving the self capacitance as:

$C_L = 1 / [(2\pi f_{0s})^2 L] = 11.83$ pF

Equation (7.4) (GKMR) predicts the capacitance to be 9.51pF, and Grandi et al. argue that the higher value in practice is due primarily to the effect of the plastic spacers, and to a lesser extent to approximations used in the derivation of their formula. It seems curious however, given that the coil was robust enough to be self-supporting and that the spacers affected the capacitance, that the helix had to be compressed to the point where spacers were needed before the sole reported measurement was made.

The DAE formula (5.3) predicts the capacitance of the GKMR test coil to be 10.1pF in the absence of dielectric materials. The DAE prediction, of course, is not greatly affected by turn spacing.

Of the self-capacitance prediction methods considered, all except the transmission-line related DAE approach have now failed. Hence it is interesting to compare the data discussed so far against a set of coil-former permittivity contours generated by the DAE formula. The data are shown plotted below as $(C_L/D)\text{Cos}^2\psi$ vs. ℓ/D, and individual coil details can be had by inspecting the spreadsheet: **CL_theor_test.ods** (sheet 1).



The most arresting feature of the graph is the enormous scatter in the 'measurements' performed by Palermo. The six points above the curve for ε_{ri}=6 are those which Medhurst suspected to be fake. Presuming that Palermo was not in the habit of winding his coils on ferroelectric formers, it seems that Medhurst's suspicions were well grounded. The Palermo measurements falling below ε_{ri}=4 are probably genuine however, especially that of his coil No. 1. The latter lies a little above the ε_{ri}=1 contour, even though the wire was thick enough to be self-supporting; but there may have been dielectric supporting material present during the measurement, and no details of the stray capacitance corrections were given.

The measurements supplied to Palermo by the NBS all fall in a tight group, despite the coils having turns numbers between 5 and 112. The situation, between the contours for ε_{ri}=3 and ε_{ri}=5, is consistent with the then (1934) standard practice of winding reference inductors on wooden cylinders.

The remaining data have been discussed previously, and all lie above the ε_{ri}=1 contour due (arguably) to the presence of dielectric material, or due to the measurement being an upper limit.

In summary, it is fair to say that theories which attempt to attribute the self-capacitance of single-layer solenoids to the inter-turn capacitance are wrong. In Palermo's case, the problem lies firstly in the assumption that a single wire can behave like two wires lying parallel, and secondly that the resulting capacitance should be divided by N. Logically, his theory is no better than a guess; which happens to work roughly for some coils, but has no actual predictive power.

The GKMR theory however is more plausible and challenges us to explain why it fails. A coil of wire is not a short-circuit at high frequencies. It can sustain a voltage across its terminals, and the current which flows can be resolved theoretically into three components; these being the resistive, inductive and capacitive contributions. If we cut the solenoid wall parallel to the axis and flatten it out, we will have a set of N parallel wires with N-1 gaps, and the capacitance of this structure is easily calculated. Why then is it not the self-capacitance of the coil?

The fallacy lies in the assumption that, since the coil can be modelled electrically as a set of lumped components, then the lumped components must have an independent existence within the coil. In fact, it is bad enough to assume that the resistance is independent of the reactance, but at least the error in that case is not serious provided that the Q is high. The GKMR theory fails because the reactive elements are primarily aspects a single energy storage mechanism, and the failure is clear and positive evidence in favour of that view. Cutting the coil open destroys the inductance, and thereby disrupts the all-important relationship between the electric and magnetic fields.

From the relationship between conductor length and self resonance, and from the ability of the coil to emit circularly polarised radiation when excited by linearly polarised radiation; we infer the existence of an electromagnetic wave propagating along the helix, and presume it to be the principal reservoir of stored energy giving rise to the reactance. The overall field surrounding the coil will be the superposition of the fields from the individual turns. In this, the overall electric field will be at a maximum in a direction perpendicular to the wires and parallel to the pitch direction, i.e., it will be tilted away from being perpendicular to the coil axis by an amount equal to the pitch angle. If we cut the coil open lengthwise, the helical waveguide will cease to function, the structure will turn into a capacitor, and the principal direction of the electric field will switch to point parallel to the axis. It is this difference in the principal field directions between the static capacitance model and the actual coil which causes the GKMR theory to fail.

It is conceivable however, that although most of the energy is stored in a propagating wave, the inter-turn capacitance might still exist as a parasitic component. In that case, we would need to include the GKMR capacitance in the total self-capacitance, and in correcting the DAE formula using actual data we might have missed a systematic offset. Fortunately, no such offset is evident, as may be seen by re-examining the graph comparing Coilcraft data and theory which was given earlier. When the spacing between turns is large, the GKMR formula predicts a small capacitance which is nearly independent of N. It is only when the gap between turns starts to close that the GKMR capacitance suddenly shoots up, but there is no corresponding trend in the data.

It was suggested by Medhurst, that the fact that the capacitance does not increase asymptotically when the gap between turns closes is due to the *proximity effect*; i.e., due to the tendency for the current streams in adjacent turns to repel each other when they are very nearly in phase. The proximity effect would indeed modify the asymptotic behaviour, but it does not explain the complete absence of observable effect.

The overall electric field of the coil ($\boldsymbol{E}_0$ say) can be resolved into two components; $E_0\text{Cos}\psi$ (perpendicular to the axis) and $E_0\text{Sin}\psi$ (parallel to the axis). Hence we should examine the possibility that the GKMR capacitance might be associated with the $E_0\text{Sin}\psi$ component. The counter-indication here is that this component is typically very small, and if we allow the GKMR capacitance to set the ratio of the radial and axial fields, the energy stored in the radial field becomes unrealistically large.

We are drawn to the conclusion that the most prominent feature of the inter-turn capacitance hypothesis is that it consistently fails to explain any aspect of self-capacitance. This takes us back to an observation made at the beginning of this section, which is that the idea is paradoxical. If we draw a line between two points situated on adjacent turns, there will always be a loop of conductor connecting those points; and in the regime in which the concept of self-capacitance is valid (i.e., well below the SRF) the length of that loop will be small in comparison to the wavelength. Hence the inter-turn capacitance is shorted-out until the frequency becomes relatively high, an awkward fact which does not bode well for the derivation of a constant static capacitance by that method. This does not mean that there will be no 'adjacent turns' effect however, it is just that we are looking for it in the wrong way.

It was mentioned in section **2** that, although variations in self-capacitance make a large difference to the SRF, small variations in the SRF do not make much difference to the self-capacitance. It is also obvious that the distributed capacitance from turn to turn will become significant at high frequencies. Hence we expect the turn spacing to affect the SRF, even if we cannot detect its effect on the self capacitance. Specifically, the SRF will increase as the gap between turns closes; there being a limit where the solenoid becomes a continuous conducting tube, helical propagation ceases, and the axial 'slow wave' attains a phase velocity close to c.

8. Interfacial capacitance

Now that we have rejected the inter-turn capacitance hypothesis; it needs to be stated that the so-far preferred DAE formula, while being perhaps the best we can do with Medhurst's approach to self-capacitance, is by no means definitive. There are ways in which a static capacitance might be evoked to account, at least in part, for the self-capacitance of a single-layer solenoid.

It is, of course, essential that corrections for jig and lead capacitance are made whenever self-capacitance is measured, but there is the problem of deciding where the connecting wires end and the coil begins. Normally we assume that the leads end at the cylinder wall, but this is merely a pragmatic solution in aid of an approximate analysis. Hence it is legitimate to ask whether (say) the two turns at the ends of the coil might act like capacitor electrodes. This is also equivalent to asking whether there should be a static component in the fringing-field corrections.

It is also arguable that a notionally static capacitance might be invoked in order to account for self-capacitance in full. If we treat the end turns as electrodes, what we have is a capacitor stuffed with a bizarre type of dielectric which, according to recent parlance, can be classified as a "metamaterial". An ordinary dielectric acquires its properties by virtue of the scattering of radiation from the atoms and molecules within; the emergent radiation being the superposition of the incident and scattered waves. A metamaterial, on the other hand, acquires its properties by the scattering of radiation from engineered structures. In the present case, we have a wire helix, which sustains an electric field and therefore has some dielectric quality. This medium is, of course, highly dispersive; but its permittivity will not change rapidly with frequency provided that we keep well away from the SRF. Thus, at low frequencies, we might usefully assume it to have a dielectric 'constant'. To derive an expression for that parameter is not straightforward, but we can at least see what the experimental data have to say about it.

The model in this case is a capacitor with parallel wire-ring electrodes; the wire diameter being d, the electrode separation being ℓ (i.e., the overall length of the coil), and the electrode circumference being πD. To the relative permittivity of the intervening metamaterial, averaged over all space, we will assign the symbol ε_{rh} (h for helix). Thus, using equation (**7.1a**) we have:

$$C_{ee} = \frac{\varepsilon_0\, \varepsilon_{rh}\, \pi^2\, D}{\mathrm{Arccosh}(\ell/d)} \quad \text{[Farads]} \qquad (\mathbf{8.1})$$

Now, since we do not know ε_{rh}, we can simply set it to 1 and see how the result compares against some actual measurements. This is done in the spreadsheet: **CL_dynamic.ods** (sheet 1), and the outcome is remarkable. As can be seen from the table below, the formula produces plausible but somewhat low estimates of the self-capacitance of air-cored coils.

Coil	C_L Measured / pF	C_{ee} / pF, ε_{rh} = 1
Palermo No. 1	3.2	2.76
GKMR	11.83	8.18
Colicraft 132-xxSM series (average)	0.205	0.114

It is surprising that advocates of the static origin of coil self-capacitance have never discovered this simple formula. Had they done so it would have given good service, being more accurate than any of the inter-turn capacitance theories, and actually better then Medhurst's formula when the coil has no core.

Which brings us to the issue of whether or not we might have missed something in the doubly asymptotic approach of section **2**. Mercifully however, the answer is 'no', or at least 'nothing serious'. The clue lies in the metamaterial permittivity factor ε_{rh}; which is effectively defined as: 'that factor which makes the self-capacitance come out correctly'. If we want to get equation (**8.1**) to work perfectly, we will need to put all of the helical transmission-line theory into ε_{rh}. Hence the static capacitance approach does not invalidate the DA approach, because it is just another way of looking at the same problem.

Finally however, we need to return to the question of whether or not there is a static inter-electrode capacitance component hidden in the DAE formula. To clarify this issue a little, we might also refer to it as the 'circuit interaction capacitance' or the 'interfacial capacitance'; the point being that it will be present when wires are connected to the coil, but not otherwise. Equation (**8.1**) tells us that, depending on where the wires end and the coil begins, it might be large enough to make a significant contribution to the self-capacitance. It is also consistent with what we know so far, which is that the apparent SRF of a coil connected to a jig is generally lower than that obtained from scattering measurements; unless the coil is very long and thin, in which case the two converge (arguably) because the hypothetical inter-electrode capacitance goes to zero when the two ends of the coil are a long way apart.

Ultimately, it is difficult to deny the existence of an interfacial capacitance. It amounts to saying that the connecting leads effectively penetrate into the coil to some extent; or that the end turns are special because they correspond to regions where helical propagation is not fully established. It can also be understood in optical terms, it being related to the difference between having a sharp refractive-index boundary at the ends of the coil (no external wires) or a diffuse boundary (parallel impedance connected). To some extent; it is already absorbed into the empirical coefficient of the DAE formula, except for a second-order (i.e., reasonably small) wire diameter effect. The concern is raised however, that the DAE formula could become seriously inaccurate for short coils (i.e.,

ℓ/D<<1) because there might be a need for terms involving ℓ/d as well as ℓ/D .

9. Dynamic model for self-capacitance

(provisional)

We cannot deny the existence of the line-length resonance. It is easily demonstrated and logically obvious; and if we assume that the coil has a definable inductance, it can be converted into an equivalent capacitance. There is however, no obvious basis on which to assume that the capacitance so calculated will remain unchanged at frequencies below the SRF, and so we overcame that problem partially in section **2** by introducing the parameter k_E . Now however, it is interesting to conjecture that the equivalent transmission-line resonance capacitance (C_{TL} say) is roughly constant from DC to the SRF, and look for end-effects to explain why the in-circuit capacitance is greater than that of an isolated coil. How this might work is envisaged as follows.

Imagine a coil connected to the end of a transmission line, exchanging energy with a capacitor at the other end of the line. A wave propagating along the line has its maximum E-field pointing across the plane in which the wires lie. As it enters the coil however, it has to tilt over, so that it is in-line with the pitch direction by the time it reaches the middle of the coil. Hence, at the ends of the coil, there is a substantial E-field component parallel to the coil axis, but this component largely disappears if the coil is disconnected from the line. Hence a capacitance which evaporates when the external reactance heads for infinity.

An expression which appears to be capable of analysing all of the author's multi-octave self-capacitance measurements is:

$C_L = C_{TL} + C_{FP} [1 - C_{FP} / (C_{FP} + C_{ref})]$ . . . . (**9.1**)

Where C_{FP} is a hypothetical 'field-perturbation' capacitance, which is evoked to account for the change which must occur when the coil is attached to a circuit. C_{ref} is the external circuit capacitance, including strays. When C_{ref} is greater than C_{FP} (but not hugely so), the expression tends towards the limit:

$C_L \rightarrow C_{TL} + C_{FP}$

This limit, of course, corresponds to the regime in which in-circuit self-capacitance measurements (including Medhurst's) are made.

An expression for C_{TL} is given by setting k_E to 1 in equation (**2.6**). It lacks any provision to account for coil-former dielectric however, and so we will start by only considering only the air-cored case. Also, we will presume that the magnetic-field inhomogeneity parameter is well-approximated by Nagaoka's coefficient (k_L), in which case we have:

$C_{TL} = (4\, \varepsilon_0 / \pi)\, \ell / (k_L \operatorname{Cos}^2\psi)$

and for coils with closely-spaced turns:

$C_{TL} = (4\, \varepsilon_0 / \pi)\, \ell / k_L$ . . . . (**9.2**)

Now observe that, if we put $\varepsilon_{ri} = \varepsilon_{rx} = 1$ into the DAE formula (**5.3**), we have a curve which corresponds to Medhurst's empirical data corrected back to air core.

$C_L = (4\, \varepsilon_0 / \pi)\, \ell\, [1 + k_c]$

Certainly there is an assumption inherent in the way in which the effect of the polystyrene former was removed, but it is not unrealistic. Thus we have tentative a way of isolating C_{FP} .

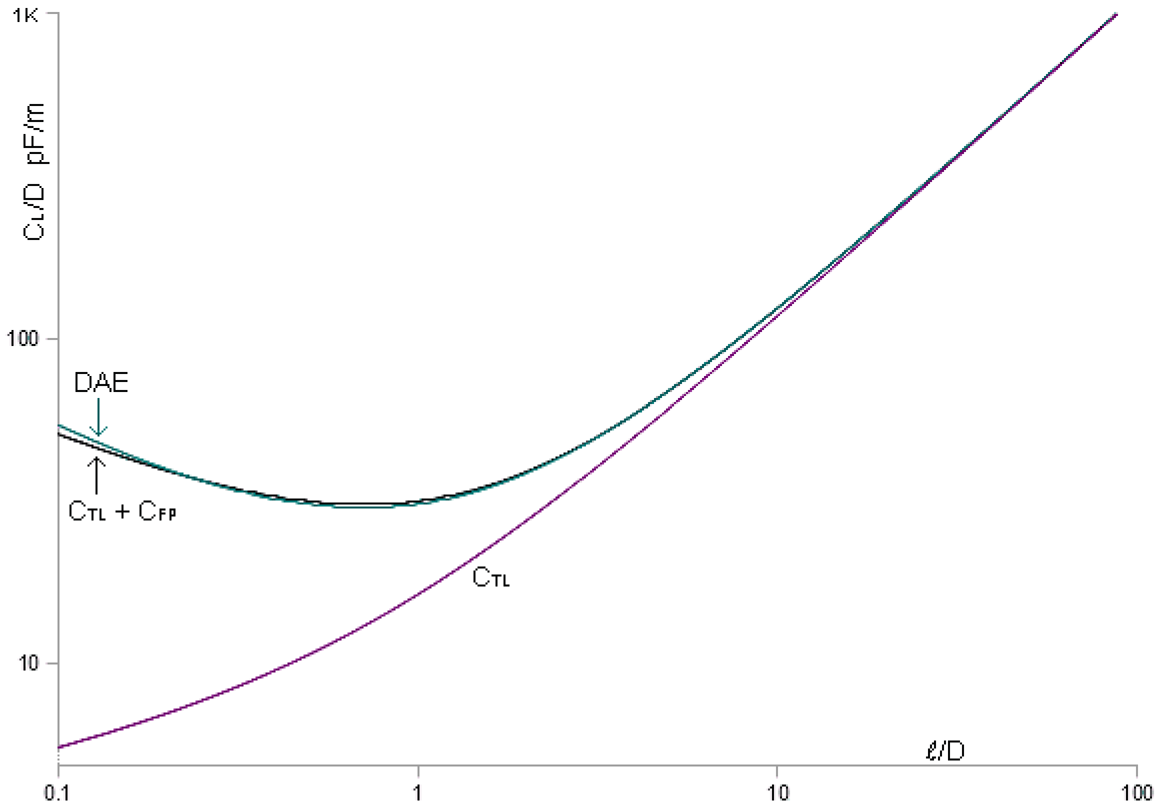
The first concern must be that C_{FP} might be proportional to the end-to-end capacitance (C_{ee}) discussed in the previous section. That would make a nonsense of the DAE formula, because it would imply that the self capacitance of short coils is strongly related to the ratio of coil length to wire diameter (ℓ/d). There is however, insufficient scatter in the data to suggest that such might be the case in first order, and then there is the matter of boundary conditions. When the length to diameter ratio (ℓ/D) of a solenoid goes to zero, the first turn is superimposed upon the last, and the self-capacitance due to end-effects must go to infinity. That certainly is where the data for short coils are heading, and we should look for a first order dependence on ℓ/D which satisfies that condition.

So now we might imagine a virtual electrode structure which will allow us to model the difference between the perturbed and the un-perturbed field as a static capacitance. This is just something to account for the distribution of electric field lines extending from one end of the coil to the other when the un-perturbed field is taken away. Such an origin suggests that it will not look like the capacitance between the two ends when the middle of the coil is removed; because it is doubtful that phenomena related to external impedance (and concerning fields which extend some distance from the coil) will be greatly affected by the diameter of the wire.

It turns out that if we try to model the capacitance perturbation as being due to a set of plates, then the resulting curve does not fit the data. Beyond that however, there are various possibilities, one of which is:

$C_L = C_{TL} + k_{FP}\, \varepsilon_0\, \pi^2 D / \mathrm{Arccosh}(1 + \ell/D)$ . . . . (**9.3**)

This curve is shown below for $k_{FP} = 0.2308$; with C_{TL} (**9.2**) on its own and the DAE formula for comparison (all quantities in units of coil diameter).



The fit is not perfect, but bear in mind that the DAE formula involves assumptions and has a standard deviation of fit of 2.1%. The weighted square error sum comparison between the formula above and DAE indicates that the the two curves are statistically indistinguishable; i.e., given some physically reasonable way of removing the effect of the coil-former dielectric, (**9.3**) should fit Medhurst's data just as well as the DAE formula (see spreadsheet, **CL_dynamic.ods**, sheet 2).

Another formula, which give an even better fit is:

$C_L = C_{TL} + 4\, \varepsilon_0\, D / \ln(1 + \pi^2\, \ell/D)$ . . . . (**9.4**)

Author's note

Despite having known about the difference between jig measurements and scattering measurements for some time, it had nevertheless not occurred to me to try to analyse my own measurement results in a way which deviates strongly from standard practice. Equation (**9.1**) is a recent invention which results from much thinking about the anomaly and the assumptions which might hold-up when trying to account for the fringing electric field. Now I am prompted to re-analyse a lot of old data and see if the parameters required to reconcile jig and scattering measurements are realistic. That will take some time.

>>>> To be continued . . . .

10. Transmission line theory

$$Z = \frac{R_0[\, Z_{\mathbf{a}} + \mathbf{j}R_0 \,\mathrm{Tan}(2\pi\, \ell_{\mathrm{TL}} / \lambda) \,]}{[R_0 + \mathbf{j}Z_{\mathbf{a}} \,\mathrm{Tan}(2\pi\, \ell_{\mathrm{TL}} / \lambda) \,]}$$

99. Discussion

Remaining problems:
1) To predict the exact high-frequency limiting phase velocity for helical propagation.
2) To quantify those cases when the velocity factor has not quite reached the limiting value at the SRF, i.e., when the turns are closely spaced.

■