CHAPTER 1

Evolution of the Nineteenth Century Newtonian Electrodynamics

The Birth of Electromagnetism

The concepts of electricity and magnetism have existed since the time of the ancient Greeks. Since then it has been believed that the attraction of iron by lodestone and of many kinds of matter by electrified amber had something in common. At least since the Middle Ages man had known that when lightning struck iron it could imbue this metal with magnetism. By the same token, the fire from heaven was capable of changing the polarity of a compass needle. Dibner [1.1] reported that in 1802, Romagnosi, a lawyer and physicist at the University of Parma in Italy, reversed the polarity of a compass by passing a galvanic current along the needle. This experiment came close to the discovery of electromagnetism which has been universally attributed to the Danish scientist Hans Christian Oersted (1777-1851), eighteen years later.

Oersted, a professor of natural philosophy in Copenhagen, determined the direction in which a compass needle would turn when a straight wire with electric current flowing along it was brought near to the needle without touching it. One might ask why this particular experiment was singled out as the beginning of electromagnetism?

Oersted felt so certain of the enthusiastic reception of his discovery that he had a paper printed for the occasion and sent to all scientists and journals of note. [1.2] The paper was dated July 21, 1820. It claimed that 'magnetic flux' encircled the current, but Oersted called this flux 'electric conflict'. Here was the missing link between electricity and magnetism.

It has to be remembered that Oersted's explanation of the magnetic influence of an electric current came at a time when effluvia, ether, and ether vortices were not in vogue because of the success of Newton's and Coulomb's action at a distance laws which avoided any reference to what was happening in the space between interacting bodies. Newton's own words serve best to describe the prevailing philosophy of contemporary natural philosophers. In the preface to the first edition of the 'Principia' he said:

> "Then from these forces, by other propositions which are also mathematical, I deduce the motion of the planets, the comets, the moon and the sea. I wish we could derive the rest of the phenomena of Nature by the

> same kind of reasoning from mechanical principles, for I am induced for many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards one another, and cohere in regular figures, or are repelled and recede from one another. These forces being unknown, philosophers have hitherto attempted the search of Nature in vain; but I hope the principles here laid down will afford some light either to this or some truer method of philosophy."

These words apply equally to the Ampère-Neumann electrodynamics which followed on the heels of Oersted's experiment. Neither Ampère nor F.E. Neumann took heed of the magnetic circles around electric currents. They wrote down the laws of electrodynamic force according to the far-action model of Newtonian gravitation. Newton was more precise and spoke of mutual simultaneous interaction between particles so as not to create the impression of something travelling at finite or infinite speed between the interacting elements of matter.

In 1820, as today, there lived many scientists who disliked action at a distance. In their hearts they had adhered to the Aristotelian principle that matter cannot act where it is not. Oersted felt many of them would welcome an electromagnetic theory in terms of an active field and field-contact action. Forty years later Maxwell placed this electromagnetic field on mathematical foundations, and within a few decades the world of physics had shed all of its remote action concepts.

Oersted's announcement [1.2] triggered a frenzy of activity in Paris which had been established by Napoleon as the world's capital of science. It caused the French Academy to stage a demonstration of the Copenhagen experiment. As Hammond [1.3] recounts the event, Ampère was present and went straight home after the demonstration to begin work on a new science which he called 'electrodynamics'. He even left before the discussion at the Academy, at which he was a regular contributor. This was September 11, 1820. Precisely one week later, Ampère read a paper before the Academy and reported that parallel wires carrying electric currents attract or repel each other, depending on whether the two currents flow in the same or opposite directions. This was as great a leap forward in electromagnetism as Oersted's. Ampère followed up with weekly presentations to the Members of the Academy of the progress he was making in his experimental investigation of the interaction of electric currents. In only a few months he had laid the foundations of the new science of electrodynamics.

Like Ampère, Jean-Baptiste Biot (1774-1862) was also a professor in Paris. He was an expert in the measurement of the strength of the earth's magnetic field. The frequency of oscillation of a compass needle had been found to be a measure of the field strength. Biot had accompanied Gay-Lussac on the first balloon flight in order to determine if the earth's magnetic field varied with height above ground level. Biot was another French scientist who was present at the Academy meeting of September 11 and, like Ampère, he too rushed back to his laboratory. With his assistant Felix Savart he set up a galvanic current in a long vertical wire. With due compensation for the terrestrial field, the two proceeded to survey the magnetic field strength around the wire using their well established method. Figure 1.1(a) illustrates the nature of the Biot-Savart result. The force H which would be exerted on a unit magnetic pole was found to be inversely proportional to the shortest distance r to the wire. These investigators had no means of measuring the strength of the current i, as the galvanometer was still to be invented by Ampère. It is not clear, therefore, if the proportionality of H to i was taken for granted or established by a later series of measurements. At any rate, the Biot-Savart experiments led to the following well-known formula for straight conductors

$$H = k \frac{i}{r} \tag{1.1}$$

where k is a dimensional constant. The units of the early electrodynamics were the fundamental electromagnetic units based on the centimeter, the gram and the second (e.m.u).

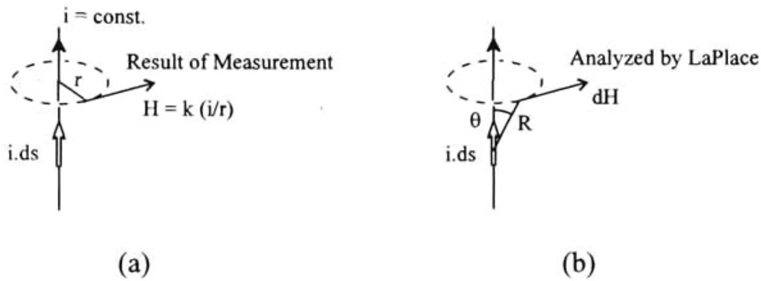

Figure 1.1 : The Biot-Savart Law

Biot and Savart spoke of their findings to the French Academy on October 30, 1820. They were obviously competing with Ampère in the unravelling of further electromagnetic mysteries. What today is understood to be the Biot-Savart law is not Eq.1.1, but the differential of it with respect to the current element i.ds, as shown in figure 1.1(b). The law may be written

$$dH = \frac{k}{R^2}\, i.ds \ \sin\ \theta \tag{1.2}$$

where k is again a dimensional constant. According to Biot and Ampère, Eq.1.2 was derived from Eq.1.1 by LaPlace, who never claimed credit for it. An excellent account of these happenings has been written by Tricker [1.4, 1.5].

The Biot-Savart law introduces the concept of the 'current-element' which has become the 'particle' of the Ampère-Neumann electrodynamics. Without subdividing the wire into small elements it would have been impossible to compute the magnetic field strength at a point due to a closed circuit. Ampère employed the same current element in his force law. Before considering Ampère's work, a little has to be said about the status of electrostatics at the beginning of the nineteenth century.

Whittaker in his 'History of the theories of aether and electricity' [1.6] maintains:

> "By Franklin's law of the conservation of electric charge, and Priestley's law of attraction between charged bodies, electricity was raised to the position of an exact science."

Benjamin Franklin working in isolation in America propounded the one-fluid theory of electricity which may be taken as the natural precursor of our present electronic theory of solids. He also came to the conclusions that certain substances, which we now describe as dielectrics, are impenetrable to electric effluvia. Therefore the two electrodes of a Leyden jar had to communicate by far-action. Through his friend Priestley in England, Franklin's work was published in Europe. Particularly his experiments of drawing lightning out of thunderclouds received much attention. During the 1750s the German physicist Aepinus took up Franklin's ideas and with them elucidated the phenomenon of electrostatic induction.

In 1766 Franklin wrote to Priestley asking him to repeat an experiment with cork balls in an electrically charged metal container. Franklin had found to his surprise that the cork balls did not respond to the charge. Subsequently Priestley established that the electric field strength, as it would be called today, was zero inside an electrically charged closed metallic vessel. He clearly recognized the analogy to gravitation and concluded [1.6]:

> "May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation, and is therefore according to the square of the distances; since it is easily demonstrated that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than to another?"

The inverse-square-law of the interaction of electric charges did, however, not become common property of the scientific community of the eighteenth century until Coulomb proved it directly in 1785 by measurements with a torsion balance. Coulomb's law may be written

$$F = k \frac{q_1 \, q_2}{r^2} \tag{1.3}$$

The charges q_1 and q_2 are separated by the distance r between their centers and k is a dimensional constant. If the charges are both positive or both negative, the force F is positive and this represents repulsion. For charges of unlike sign the force is negative which stands for attraction.

Ampère's Force Law

In the tradition of the great French mathematicians, who developed the science of mechanics from Newton's laws, Ampère set out to cast electromagnetism in a Newtonian mould. For this he required an appropriate fundamental law of the interaction of electrodynamic matter elements. He suspected this would turn out to be an inverse-square-law akin to those of Newton and Coulomb which, to use his own words in English translation [1.4]: ".... opened a new highway into the sciences which have natural phenomena as their object of study."

However difficult it may appear today, the question of what constituted the elementary 'particle' of electrodynamics apparently posed no problem to Ampère, Biot and Savart. They all employed the metallic current-element. It is uncertain who may have thought of this concept first. Ampère clearly recognized that, unlike the elementary particles of gravitation and electrostatics, which were characterized by a simple scalar magnitude (of mass or charge), the current-element would in addition to its magnitude of current strength have to possess length and direction.

On the basis of his first electrodynamic experiments, showing the attraction and repulsion of straight and parallel current carrying wires, Ampère expected the law of mechanical force between two current elements to be of the general form

$$\Delta F_{m,n} = - i_m \, i_n \, \frac{dm \;\; dn}{r_{m,n}^2} \, f(\alpha, \beta, \varepsilon) \qquad (1.4)$$

The Δ in Eq.1.4 infers that we are dealing with an elemental force which cannot be measured directly because current elements of wires are not available in isolation. The forces that are measured in the laboratory are sums of many elemental forces. In Eq.1.4 the elements carry currents of i_m and i_n, and their lengths are dm and dn. The distance between the center points of the elements is $r_{m,n}$, and the angles of the function f are shown in figure 1.3.

If the angle function $f(\alpha, \beta, \varepsilon)$ is positive, then $\Delta F_{m,n}$ is negative, which indicates attraction between the elements. Ampère originally proposed the opposite sign convention, but this was subsequently dropped in order to coordinate Eq.1.4 with Coulomb's law, Eq.1.3. Both the current strengths and element lengths were taken to be positive scalar quantities, while the directional properties of the elements were given by the angle function f.

With respect to the proportionality of the elemental force to the lengths and currents of the two elements Ampère said [1.4]:

> "First of all, it is evident that the mutual action of two elements of electric current is proportional to their lengths; for assuming them to be divided into infinitesimal equal parts along their length, all attractions and repulsions of these parts can be regarded as directed along one and the same straight line, so that they necessarily add up. This action must also be proportional to the intensities of the two currents."

In his early papers on electrodynamics Ampère also assumed the proportionality of the elemental force to the inverse square of the distance of separation, because he believed all fundamental forces of nature concurred with this distance dependence. Later he proved the validity of this early assumption with the three-circle experiment of figure 1.2. For the sake of clarity the figure does not show the current leads to the three parallel and coaxial current circles in vertical planes. All three circles were connected in series to ensure equal current intensities in all of them. Ampère's method of compensating for the effect of the earth's magnetic field has also been omitted in figure 1.2. The radii of, and the distances between the three current circles were chosen such that the geometrical relationship of circle 1 to circle 2 was similar to the relationship of circle 2 to circle 3. In other words, the only difference between the 1-2 and 2-3 combinations was a linear scale factor. Circles 1 and 3 were fixed to the laboratory frame, while circle 2 was held coaxial with the other two, but with its insulator arm free to rotate about the vertical line YY. The purpose of this experiment was to show that, if the currents in 1 and 3 encircled the common axis in the same direction, circle 2 would remain stationary, for it was either attracted or repelled equally strongly by the two adjacent circles. When the current in circle 2 flowed in the direction shown on figure.1.2, the force was repulsion.

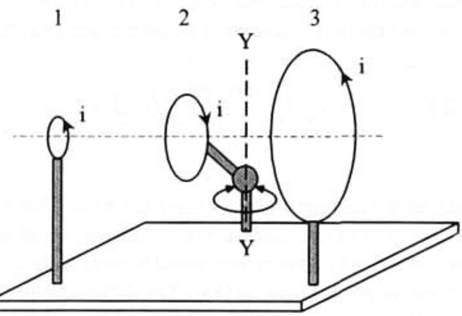

Figure 1.2 : Ampère's three-circle experiment

This experiment proved that the reciprocal electrodynamic forces between two current loops were independent of the linear scale factor and therefore independent of the size of the electrodynamic system of conductors. The same had to be true for all elemental forces into which the total force could be divided. Hence the geometrical factor $dm\ dn / r_{m,n}^2$ of Eq.1.4 had to be a dimensionless number, a condition which could only be fulfilled by the inverse square law. In addition to this proof by the three-circle experiment, the widely known fact that geometrically similar small and large conductor arrangements are subject to the same mechanical forces for the same currents, further corroborates Ampère's conclusion.

Not surprisingly, Ampère's most challenging task proved to be the determination of the angle function $f(\alpha, \beta, \varepsilon)$. A recurrent difficulty for those who have tried to understand Ampère's force law, and others who have used it for engineering calculations, has been the visualization of the three angles, particularly when the two elements do not lie in the same plane. Expressing the law in vector form does not eliminate the problem. Figure 1.3 attempts to make the visualization as easy as possible. M and N are the center points of two unequal current elements. The distance between M and N, that is $r_{m,n}$, must be treated as a vector. The polarity of this vector is arbitrary. It may be chosen to point from M to N, or from N to M. The current elements must also be treated as vectors and have to point in the direction of current flow. The angle through which the element $i_m dm$ has to be turned about M to make it point in the same direction as $r_{m,n}$ is α. Similarly, the angle through which the element $i_n dn$ has to be turned to point in the same direction as $r_{m,n}$ is ß. Since both of these angles appear in cosines, and since $\cos\alpha=\cos(2\pi-\alpha)$, it does not matter in which direction the element vectors are turned to make them coincide with the distance vector. Each element and the distance vector lie in a plane of their own. The two planes intersect along the distance vector. Of the two complimentary angles between the planes, γ is that angle through which the plane containing i_m dm would have to be turned in the direction indicated, in order to make the components of the current elements, which are perpendicular to the distance vector, point in the same direction.

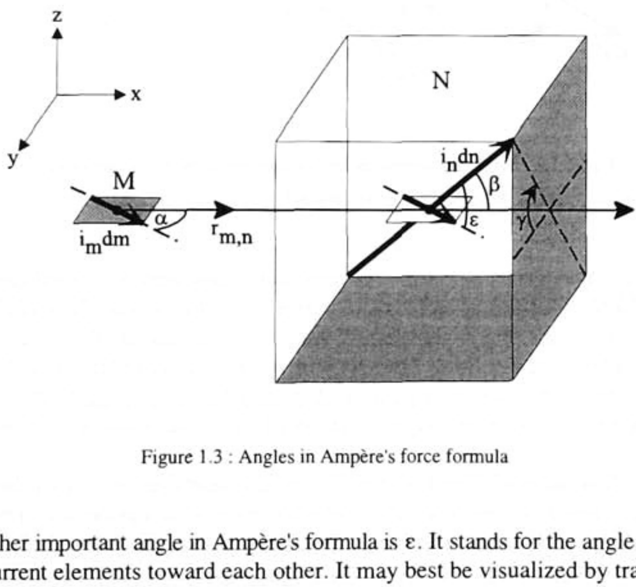

Figure 1.3 : Angles in Ampère's force formula

Another important angle in Ampère's formula is ε. It stands for the angle of inclination of the two current elements toward each other. It may best be visualized by transferring one of the elements parallel to itself along MN until its center coincides with the center of the other element. In figure 1.3 the dm element has been transferred from M to N and ε is the angle through which the transferred element has to be turned about N to make it point in the same direction as dn. Since ε also appears in a cosine, its direction of rotation is as arbitrary as that of α and ß.

To see how Ampère determined $f(\alpha, ß, \varepsilon)$ we resolve the two current elements of figure.1.3 into their cartesian components shown in figure 1.4. The elements i_m dm and i_n dn are there represented as vectors m and n, pivoted at the centers of the elements. The resolved components of the two current elements along the x, y, and z axes are given by

$$m(x) = i_m dm \cos\alpha \quad ; \quad m(y) = i_m dm \sin\alpha$$
$$n(x) = i_n dn \cos\beta \quad ; \quad n(y) = i_n dn \sin\beta \cos\gamma \quad ;$$
$$n(z) = i_n dn \sin\beta \sin\gamma \tag{1.5}$$

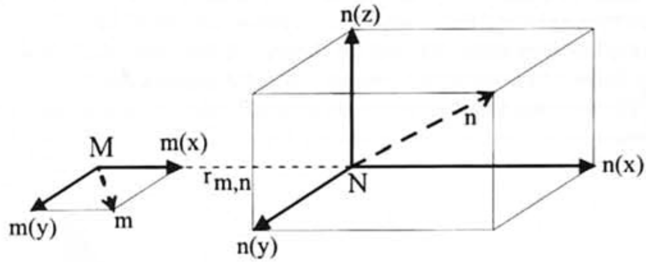

Figure 1.4 : Resolved component vector representation of the two general current elements of figure 1.3

Now each component of m interacts with each component of n, resulting in a total of six contributions to the elemental force between two current elements. Four of them are zero according to a theorem first enunciated by Ampère. Generations of physicists have been uncertain how this theorem follows from Ampère's experiments or how it could be deduced from his postulates. We will refer to it as 'Ampère's Rule', leaving the question open of whether it is a theorem or an assumption. Ampère [1.7] wrote about it as follows:

> "An infinitely small portion of current exerts no action on another infinitely small portion of a current which is situated in a plane which passes through the midpoint and which is perpendicular to its direction. In fact, the two halves of the first element produce equal actions on the second, the one attractive and the other repellent, because the current tends to approach the common perpendicular in one of those halves and to move away from it in the other. These two equal forces form an angle which tends to two right angles according as the element tends to zero. Their resultant is therefore infinitesimal in relation to these forces and in consequence it can be neglected in the calculations."

In compliance with Ampère's Rule, the four vanishing force contributions of the element components drawn in figure 1.4 are

$$\Delta F_{m(x),n(y)} = \Delta F_{m(x),n(z)} = \Delta F_{m(y),n(x)} = \Delta F_{m(y),n(z)} = 0 \tag{1.6}$$

A corollary of Ampère's Rule is that the mechanical interaction of two current elements arises from two sets of parallel element components, one of them being the set which lies along the line connecting the two elements, and the other is the set which is perpendicular to that line. Ampère then assumed that the two non-vanishing force contributions may be expressed by

$$\Delta F_{m(y),n(y)} = - \frac{m(y) \cdot n(y)}{r_{m,n}^2} \tag{1.7}$$

$$\Delta F_{m(x),n(x)} = - k \frac{m(x) \cdot n(x)}{r_{m,n}^2} \tag{1.8}$$

where k is a constant and the element components are defined by Eq.1.5. At this stage Eq.1.4 may be expressed as

$$\Delta F_{m,n} = - i_m i_n \frac{dm \;\, dn}{r_{m,n}^2} (\sin\alpha \;\sin\beta \;\cos\gamma + k \;\cos\alpha \;\cos\beta) \tag{1.9}$$

Ampère then introduced the trigonometrical equation

$$\cos\varepsilon = \cos\alpha \;\cos\beta + \sin\alpha \;\sin\beta \;\cos\gamma \tag{1.10}$$

For proof of Eq.1.10 he referred to a spherical triangle, but it may also be derived with the help of figure 1.5 from the direction cosines of the two general current elements. It is known that the cosine of the angle of inclination between two vectors is equal to the sum of the three products of corresponding direction cosines of the two vectors. With regard to figure 1.5 the direction cosines along the x, y and z axes of the dm-element are

$$\cos\alpha \quad ; \quad \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \quad ; \quad \cos\delta = \cos\left(\frac{\pi}{2}\right) = 0$$

and those of the dn-element are

$$\cos\beta \ ; \ \cos\left(\frac{\pi}{2}-\beta\right)\cos\gamma = \sin\beta\cos\gamma \ ; \ \cos\left(\frac{\pi}{2}-\beta\right)\cos\left(\frac{\pi}{2}-\gamma\right) = \sin\beta\sin\gamma$$

Hence

$$\cos\varepsilon = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta\ \cos\gamma + 0\times\sin\beta\ \sin\gamma$$

which confirms Eq.1.10.

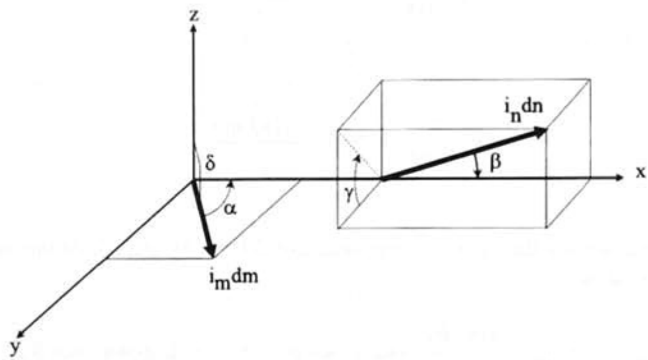

Figure 1.5 : Angles for determining the direction cosines of two general current elements

Attempting to deduce a force law for two co-planar elements, he used Eq.1.10, and the force formula Eq.1.9, to arrive at

$$\Delta F_{m,n} = -\ i_m\ i_n \frac{dm\ \ dn}{r_{m,n}^2}\ (\cos\varepsilon + (k-1)\ \cos\alpha\ \cos\beta) \tag{1.11}$$

After this step Ampère converted the cosines to partial differentials of $r_{m,n}$ with respect to small displacements of the centers of the elements, M and N, along the line of action. These partial differentials are further defined by figure 1.6. In the limit, as the displacements of M and N tend to zero, and writing r for the distance between the elements, we find that

$$\cos\alpha = \frac{\partial r}{\partial m} \quad ; \quad \cos\beta = -\frac{\partial r}{\partial n} \tag{1.12}$$

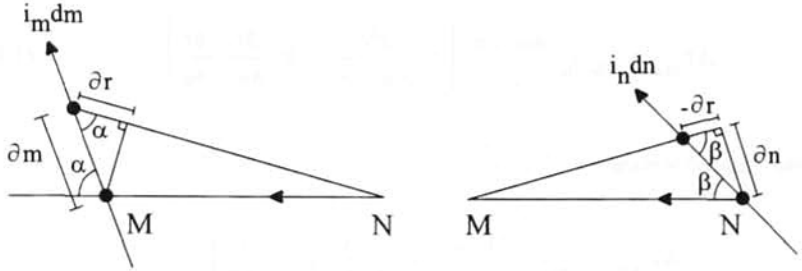

Figure 1.6 : Partial differentials of the distance vector with respect to displacements of current elements

Furthermore, if M and N have the coordinates x_m, y_m, z_m, and x_n, y_n, z_n, we have

$$r^2 = (x_m - x_n)^2 + (y_m - y_n)^2 + (z_m - z_n)^2 \tag{1.13}$$

Differentiating Eq.1.13 with respect to m results in

$$r \frac{\partial r}{\partial m} = (x_m - x_n) \frac{\partial x_m}{\partial m} + (y_m - y_n) \frac{\partial y_m}{\partial m} + (z_m - z_n) \frac{\partial z_m}{\partial m} \tag{1.14}$$

and a second differentiation with respect to n gives

$$r \frac{\partial^2 r}{\partial m \, \partial n} + \frac{\partial r}{\partial m} \frac{\partial r}{\partial n} = - \frac{\partial x_m}{\partial m} \frac{\partial x_n}{\partial n} - \frac{\partial y_m}{\partial m} \frac{\partial y_n}{\partial n} - \frac{\partial z_m}{\partial m} \frac{\partial z_n}{\partial n} \tag{1.15}$$

The right-hand side of Eq.1.15 contains the negative products of the direction cosines of the two current elements. Therefore

$$\cos \varepsilon = - r \frac{\partial^2 r}{\partial m \, \partial n} - \frac{\partial r}{\partial m} \frac{\partial r}{\partial n} \tag{1.16}$$

Substituting Eq.1.12 and Eq.1.16 into the force equation, Eq.1.11, yields

$$\Delta F_{m,n} = i_m \, i_n \, \frac{dm \;\, dn}{r^2} \left(r \, \frac{\partial^2 r}{\partial m \, \partial n} + k \, \frac{\partial r}{\partial m} \, \frac{\partial r}{\partial n} \right) \tag{1.17}$$

This may also be written

$$\Delta F_{m,n} = i_m \, i_n \, \frac{dm \;\, dn}{r^2} \, \frac{1}{r^{k-1}} \, \frac{\partial}{\partial n} \left(r^k \, \frac{\partial r}{\partial m} \right)$$

$$= i_m \, i_n \, r^{-(k+1)} \, \frac{\partial}{\partial n} \left(r^k \, \frac{\partial r}{\partial m} \right) dm \;\, dn \tag{1.18}$$

Ampère then invoked the result of another of his null-experiments to determine the value of k. The experiment to which he referred is sketched in figure 1.7. To distinguish it from the other null-experiments it will be called the wire-arc experiment. It proved that the mechanical force on a circular arc section of a current carrying circuit 1, due to current in a separate closed circuit 2 of any shape and disposition, was entirely perpendicular to the arc. As shown in figure 1.7, Ampère floated the arc section on two mercury troughs and left it free to rotate the insulator arm OX about the pivot O. During the experiment the arc remained stationary as circuit 2 was brought up to it and moved around. From this behaviour Ampère concluded that the net tangential force on the wire arc portion was zero.

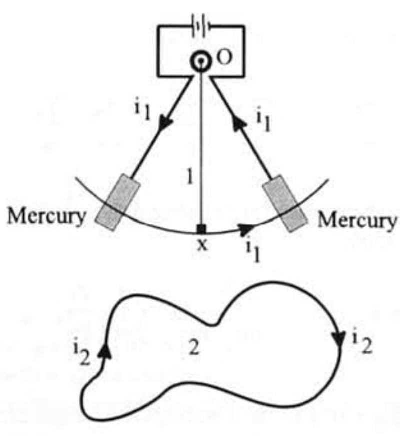

Figure 1.7 : Ampère's wire-arc experiment

Take Eq.1.18 and substitute for $\partial r/\partial m$ from Eq.1.12, leaving

$$\Delta F_{m,n} = i_m \, i_n \, dm \; r^{-(k+1)} \frac{\partial}{\partial n} (r^k \cos\alpha) \; dn \tag{1.19}$$

The component of the mutual force which acts tangentially on the dm-element is obtained by multiplying Eq.1.19 by $\cos\alpha$. If it is to agree with experiment, this tangential force, when integrated over all dn-elements in circuit 2, must come to zero. Hence we may write

$$\oint_2 \Delta F_{m,n} \cos\alpha = i_m \, i_n \, dm \oint_2 r^{-(2k+1)} \, r^k \cos\alpha \, \frac{\partial}{\partial n} (r^k \cos\alpha) \; dn = 0 \tag{1.20}$$

For integration by parts according to $\int u \, dv = uv - \int v \, du$, we let

$$u = r^{-(2k+1)} \quad ; \quad \frac{\partial u}{\partial n} = -\,(2k+1) \, r^{-2(k+1)} \frac{\partial r}{\partial n}$$

$$v = \frac{1}{2} r^{2k} \cos^2\alpha \quad ; \quad dv = r^k \cos\alpha \, \frac{\partial}{\partial n} (r^k \cos\alpha) \; dn$$

Therefore

$$\oint_2 \Delta F_{m,n} \cos\alpha = \frac{1}{2} i_m \, i_n \, dm \left[\left(\frac{\cos^2\alpha}{r} \right) \Bigg|_{n'}^{n} + (2k+1) \oint_2 \frac{\cos^2\alpha}{r^2} \, dr \right] = 0 \tag{1.21}$$

The limits n and n' of the first term are actually adjacent infinitely short elements on circuit 2, and therefore the first term of Eq.1.21 vanishes. As Ampère pointed out, however, many closed circuits can be imagined for which the integral in the second term of Eq.1.21 will not vanish. Hence we are left with

$$k = -\frac{1}{2} \tag{1.22}$$

as the only possibility of reducing Eq.1.20 to zero, whatever the shape or disposition of circuit 2. As can be seen from Eq.1.7 and Eq.1.8, k determines the difference in the mutual interaction of equal parallel current elements between (a) elements lying along the line connecting their centers, and (b) elements set perpendicular to that line. With Eq.1.22 substituted into Eq.1.8, it is evident that two elements of unit strength and separated by unit distance repel each other half as strongly when lying along the distance vector, than they would attract each other when arranged transverse to this vector.

Using Eq.1.11 and Eq.1.22 Ampère wrote his force law as follows

$$\Delta F_{m,n} = - i_m \, i_n \, \frac{dm \;\; dn}{r_{m,n}^2} \left(\cos \varepsilon - \frac{3}{2} \cos \alpha \;\; \cos \beta \right) \qquad (1.23)$$

It was Ampère who first clarified what was meant by voltage and current, and his force law indicated that the square of current must have the same dimension as mechanical force. With this knowledge he defined an electrodynamic unit of current which was smaller than the electromagnetic unit of current called the absolute-ampere. (1 ab-amp = 10 amps). To obtain these units of current, the electrodynamic measures have to be multiplied by $\sqrt{2}$. After this change of units we arrive at Ampère's force law for co-planar current elements, in its modern form

$$\Delta F_{m,n} = - i_m \, i_n \, \frac{dm \;\; dn}{r_{m,n}^2} \left(2 \;\cos \varepsilon - 3 \;\cos \alpha \;\; \cos \beta \right) \qquad (1.24)$$

This gives the elemental force in dynes provided the currents i_m and i_n are inserted in absolute-amperes. If Eq.1.24 is further multiplied by $(\mu_o/4\pi)$, then the force is in Newtons when the current is in Amps.

A most important aspect of Ampère's theory is that the individual current element does not interact with itself. There is little or no discussion of this point in Ampère's papers because he took it for granted that in Newtonian science every elemental force is a mutual interaction of two elements of matter. Ampère considered his elements to be particles of the conductor metal and not travelling charges. This was in harmony with the ideas of the 1820s when electricity was still considered to be a subtile fluid or continuum. The conductor metal was considered to be infinitely divisible, and forces could therefore be determined with differential and integral calculus. Today the Ampère electrodynamics is being applied to the ions of the metal lattice or plasma, which imposes a lower limit on the size of the current element. Finite-size current elements are easily handled with computer assisted finite element analysis.

A common error made in modern treatments of Ampère's law is to assume that it represents the force between moving conduction electrons. One of the foremost scholars of Ampère's work was Maxwell. He stressed the fact that the Ampèrian current element is a stationary piece of metal and said [1.8]:

> "It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. .... The only force which acts on electric currents is electromotive force, which must be

distinguished from the mechanical force."

An often voiced contemporary criticism of current elements is that they produce discontinuities at straight element junctions. This fact was also of concern to Ampère. He went to great length to demonstrate to his own satisfaction that a smooth wire curve could be adequately represented with his discontinuous elements. One of his null-experiments was specifically designed to prove this argument. The essential features of the experiment are shown in figure 1.8. It will be referred to as the bent-wire experiment. AA'DE is a rectangular current loop in a vertical plane and suspended so that it is free to rotate about its vertical center line. As before, for the sake of clarity, Ampère's additional circuit to offset the terrestrial magnetic field is not shown in figure 1.8.

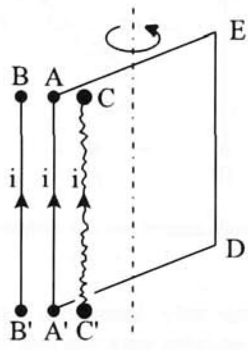

Figure 1.8 : Ampère's bent-wire experiment

BB' is a straight wire parallel to AA' and placed close to it. CC' is the bent wire arranged with its axis parallel to and at the same level as BB' and AA'. Ampère fitted the bent wire into a narrow slot of a wooden post and described it as being twisted over its entire length in a plane perpendicular to BCC'B' and such that the wire at no point departed more than a very short distance from the center line of the slot. The experiment proved that if the wires BB' and CC' carried currents in the same direction and were equidistant from AA', as indicated in figure 1.8, no turning moment was exerted on the loop AA'DE. Hence, depending on the direction of the current in AA', the bent wire CC' exerted the same force of attraction or repulsion on the AA' wire as did the straight wire BB'.

Based on this experimental result, Ampère argued that a curved wire section, as for example AA' of figure 1.9, was equivalent to the straight section BB' provided B coincided with A and B' with A'. For an explanation he offered the vectorial cancellation of the transverse sub-elements. Ampère believed the argument also applied to a three-dimensional curve.

The published record for the few years in which Ampère concerned himself with electrodynamics shows little discussion of the current distribution over the wire cross-section and to what extent this may have been compatible with the single filament representation of a conductor. At that time conductors were usually thin wires, and the three-dimensional nature of the current stream was not a pressing issue. As will be shown later, Ampère's theory can be adapted to large conductors by subdividing them into current elements of finite volume and filaments of finite cross-section.

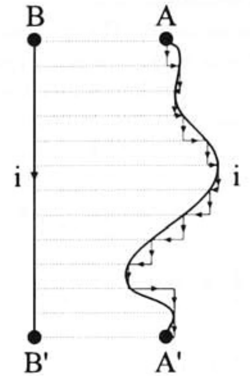

Figure 1.9 : Equivalence of bent and straight wire sections

Energy conservation was only introduced after Ampère's death, however electrodynamic forces are capable of doing work and they must therefore be associated with stored energy. Following Ampère's successes, this issue was taken up by F.E. Neumann (1798-1895).

Eq.1.24 is Ampère's empirical force law which he formulated to agree with the large body of his experimentation [1.7]. The law is not known to have failed during more than 170 years of its existence, so long as it is applied to metallic conductors for which it was created. The method by which Ampère deduced his law from experiments is only of academic interest. It has no bearing on the validity of the law. Like the scaffolding used when erecting a building, the method of deducing an empirical law may be discarded as soon as the law has been found.

Neumann's Electrodynamic Potential

Twenty years elapsed between the conclusion of Ampère's study of electrodynamics and F.E. Neumann's first memoir, in 1845, giving the original theory of electromagnetic induction. The author, Franz Neumann was the father of Carl Neumann, who also became a famous 'electrician' of the last century. Much had happened in these twenty years. Faraday (1791-1867) had discovered electromagnetic induction in 1831, and there was a general awakening to the atomicity of electricity.

By theoretical reasoning Neumann [1.9, 1.10] arrived at the concept of the electrodynamic potential. With the notation of Ampère's theory, this may be expressed as

$$P_{m,n} = \pm \frac{1}{2} i_m i_n \int_m \int_n \frac{\cos \varepsilon}{r_{m,n}} dm \; dn \tag{1.25}$$

It is the mutual potential of two closed circuits, composed of Ampèrian current elements. The double integration involves each pair of current elements twice, but the energy of the pair is only stored once. Neumann took account of this by the factor ½. This has to be born in mind when summing the potential contributions by finite element analysis. Neumann felt uncertain about the sign of the electrodynamic potential until, with the help of the principle of 'virtual work', he was able to derive Ampère's force law from the potential.

Neumann is best remembered for his mutual inductance formula

$$M_{m,n} = \pm \oint_m \oint_n \frac{\cos \varepsilon}{r_{m,n}} dm \; dn \tag{1.26}$$

which arose directly from the electrodynamic potential. The sign of the mutual inductance is also determined by virtual work considerations. In Eq.1.26 the factor of ½ has been dropped because the integrals are now around the two closed circuits. Many precise inductance calculations continue to be based on Neumann's mutual inductance formula. Maxwell incorporated it into field theory. In the process however, he changed its meaning to magnetic flux linkage per unit current.

Comparing the electrodynamic potential, Eq.1.25, with Ampère's force law, Eq.1.24, it will be seen that the dimension of the potential is force times distance, that is energy. Today this potential is called magnetic energy. Any change in the current intensities or the relative distances of the current elements, which increases the potential, requires work to be done on the two circuits. Conversely, if the mutual potential is reduced, energy stored by or in the circuits will be transformed to mechanical work or Joule heat or both. Free energy divorced from matter does not exist in Neumann's theory. The Newtonian electrodynamics treats P as potential energy, depending only on the positions and orientations of the matter elements which enter Ampère's formula. Though both Ampère and Neumann used the term 'current', neither of them ascribed to it kinetic energy, as Maxwell would do later.

Neumann changed his mind about the sign which should be given to his electrodynamic potential. In his first paper [1.9] he defined it as follows:

> "The potential of two closed currents of unit intensity, relative to each other, is the sum of the products of the elements of one current with the elements of the other, each product of the two elements being multiplied with the cosine of the angle of their inclination and divided by their distance."

Following this definition, he used Eq.1.25 with the positive sign. In his second paper [1.10], presented two years later in 1847, he repeated the definition but inserted "the negative half-sum". From then onward he used Eq.1.25 with the negative sign.

In potential theories there have always existed difficulties in agreeing on a universal sign convention. Kellog [1.11] pointed out that the most popular rule was to assign negative potential energy to elements of like sign which attracted each other, and positive potential energy to elements of like sign which repelled each other. Gravitating particles were an example of the former class, and electric charges were an example of the latter class. If matter elements have signs attached to them, they normally represent scalar quantities. Current elements are not of this nature. They have definite directions and therefore they are vectors. Depending on their directions, two current elements sometimes attract and sometimes repel each other. Hence it would seem the potential energy of current elements, and circuits made up of these elements, may sometimes be positive and at other times negative, leaving us to ponder what could be meant by negative energy? We cannot conceive of less than no energy. Consequently, positive and negative potential energy must be two kinds of energy, like positive and negative charge are two kinds of electricity. One kind of potential energy would be associated with attraction and the other with repulsion. Unlike charge, however, the two kinds of energy cannot be neutralized by putting them together.

To see this more clearly, let us now examine two very long straight and parallel wires m and n, as sketched in figure 1.10. In case (a) of that figure they carry currents in the same direction. From experience we know that they will attract each other. By the rules of potential energy they are therefore associated with negative potential energy. Assume an externally applied force F_x tends to increase the separation x and brings about the displacement ∂x by moving n to n'. This external force has to do work and expend an amount of energy equal to $F_x\,\partial x$. At first it may be thought that this energy is being added to the stored potential energy. This cannot be so, however, because the magnitude of the electrodynamic potential given by Eq.1.25 decreases as a result of the lengthening of the distances between current elements. Not only does the mechanical source sustaining the external force supply energy to the system of conductors, but the potential energy store also gives up energy. What absorbs these two streams of energy? As the currents are assumed to remain constant, no additional Joule heat will be dissipated. Later we will show that, according to Neumann's theory, all of this energy flows to the two electrical sources which maintain the currents. Some of the Joule heat normally furnished by these sources will, during the displacement of one conductor from n to n', be supplied by the potential energy store and the mechanical energy source.

In the case of figure 1.10(b), where the currents flow in opposite directions and the conductors repel each other, the displacement x from n to n' again requires the supply of energy by the mechanical source exerting the external force, but now the magnitude of the stored energy increases because of the shortening of element distances. This opens the possibility of all the energy provided by the mechanical source being stored as potential energy, and the electrical sources maintaining the currents are either not involved in the transaction or they exchange energy with each other.

To appreciate that in a system of conductors positive potential energy does not cancel negative potential energy, we consider three parallel and equidistant wires A, B, and C, in accordance with the model of figure 1.10. If A and B carry current in the same direction, the associated stored energy will be negative, say -P. Let the current in the wire C flow in the opposite direction to the current in B. Then the energy stored between B and C will be positive, that is +P. The forces inside the two sets of conductors do not eliminate each other, and therefore +P-P≠0! The fact that the interaction between A and C adds further positive energy to the system does not change the argument. We have to conclude that positive and negative potential energy are two different kinds of energy which might as well have been named "red" and "green" energy.

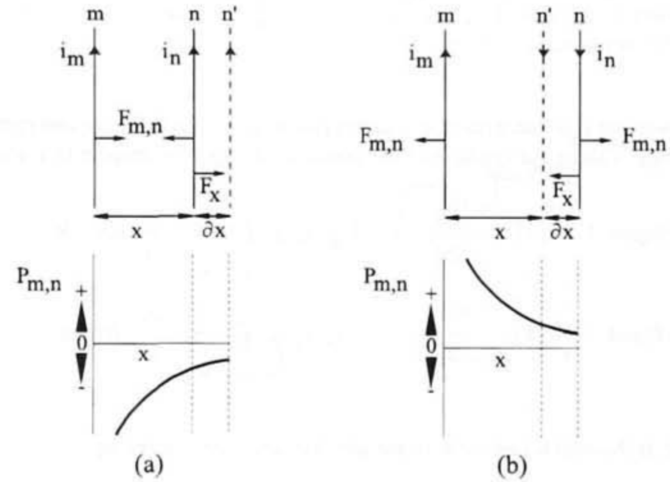

Figure 1.10 : Electrodynamic potential energy of straight and parallel currents

Neumann related the reciprocal force of repulsion or attraction between two circuits m and n to the mutual potential of the circuits and the principle of virtual work by

$$(F_{m,n})_x = - \frac{\partial P_{m,n}}{\partial x} \tag{1.27}$$

where x denotes a particular direction in which the virtual displacement ∂x takes place. At the same time he chose the negative sign for the potential given by Eq.1.25. The cosine of ε then decides whether, in any particular circuit arrangement, the mutual potential energy turns out to be positive or negative.

When applying Neumann's sign convention to the two conductor arrangements of figure 1.10, it is found that

$$\text{for figure 1.10(a)} \qquad \cos\varepsilon = +1 \tag{1.28}$$

$$\text{for figure 1.10(b)} \qquad \cos\varepsilon = -1 \tag{1.29}$$

Therefore

$$\text{for figure 1.10(a)} \quad P_{m,n} = - i_m i_n \oint_m \oint_n \frac{1}{r_{m,n}} \, dm \, dn \tag{1.30}$$

$$\text{for figure 1.10(b)} \quad P_{m,n} = + i_m i_n \oint_m \oint_n \frac{1}{r_{m,n}} \, dm \, dn \tag{1.31}$$

This agrees with the rule that attraction is associated with negative energy and repulsion with positive energy. Taking the gradient of the potential energy with respect to x we find

$$\text{for figure 1.10(a)} \quad \frac{\partial P_{m,n}}{\partial x} = + i_m i_n \oint_m \oint_n \frac{1}{r^2_{m,n}} \frac{\partial r}{\partial x} \, dm \, dn \tag{1.32}$$

$$\text{for figure 1.10(b)} \quad \frac{\partial P_{m,n}}{\partial x} = - i_m i_n \oint_m \oint_n \frac{1}{r^2_{m,n}} \frac{\partial r}{\partial x} \, dm \, dn \tag{1.33}$$

So we arrive at the interaction force in the specific direction x given by

$$\text{for figure 1.10} \quad (F_{m,n})_x = - \frac{\partial P_{m,n}}{\partial x} = \pm i_m i_n \oint_m \oint_n \frac{1}{r^2_{m,n}} \frac{\partial r}{\partial x} \, dm \, dn \tag{1.34}$$

In the case of figure 1.10(a) the force defined by Eq.1.34 is negative, signifying attraction, in agreement with experience. Similarly, for figure 1.10(b), the force becomes positive or repulsion. Hence Neumann's sign convention gives the correct direction of the forces. He extended this proof to the general case of two circuits of any shape. The potential function ultimately adopted by Neumann therefore was

$$P_{m,n} = - i_m i_n \oint_m \oint_n \frac{\cos \varepsilon}{r_{m,n}} \, dm \, dn \tag{1.35}$$

Eq.1.35 will henceforth be used in preference to Eq.1.25.

Neumann did not set out to derive the electrodynamic potential. He discovered it while developing a theory of electromagnetic induction which he based on Ampère's force law. This meant that the potential equation, Eq.1.35, had to be compatible with the force equation, Eq.1.24. The connection lead Neumann to the discovery of the principle of virtual work, as expressed by Eq.1.27. The formal mathematical proof of these facts is very long. It has been fully documented in reference [1.12] and will not be repeated here.

A long forgotten aspect of Neumann's theory is the derivation of electrodynamic turning moments, or mechanical torques, from the electrodynamic potential. Consider two rigid closed circuits carrying currents i_m and i_n, respectively. If circuit N is fixed to the laboratory frame, circuit M will feel torques $(T_{m,n})_x$, $(T_{m,n})_y$, and $(T_{m,n})_z$ on it about arbitrarily chosen cartesian coordinates x, y, and z. Alternatively, if M is fixed, N will feel torques of the same magnitudes, but in opposite directions. Figure 1.11 shows the various torque parameters. The angular displacement about the z-axis is denoted by Ψ_z.

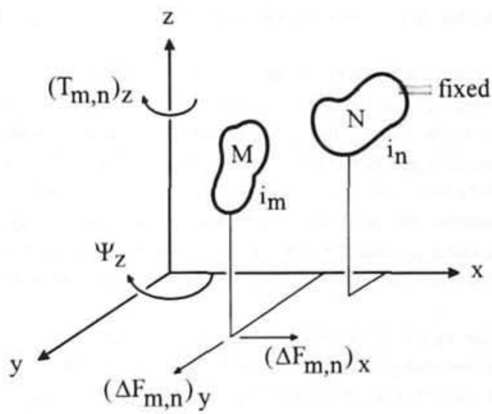

Figure 1.11 : Parameters used for electrodynamic torque calculations

The mutual inductance, and therefore the mutually stored potential energy of Eq.1.35 will change when one circuit is turned about an arbitrary axis with respect to the other circuit. This change in stored energy can only be brought about with the aid of a mechanical interaction via mutual torques. With the principle of virtual work, embodied in Eq.1.27 for linear displacements, the corresponding equations for angular displacements must be

$$(T_{m,n})_x = -\frac{\partial P_{m,n}}{\partial \Psi_x} \tag{1.36}$$

$$(T_{m,n})_y = -\frac{\partial P_{m,n}}{\partial \Psi_y} \tag{1.37}$$

$$(T_{m,n})_z = -\frac{\partial P_{m,n}}{\partial \Psi_z} \tag{1.38}$$

Neumann [1.13] did not take the virtual work principle for granted and actually proved Eqs.1.36 - 1.38 from first principles, starting from Ampère's law. This proof has been reproduced in reference [1.12]. In conclusion, Neumann's torque theorem is:

> "The mutual torque between two rigid current-carrying circuits, with respect to any arbitrary axis, is the negative angular gradient of the mutual electrodynamic potential."

In modern field theory Neumann's electrodynamic potential is called stored magnetic energy. Maxwell assumed that this was kinetic energy and therefore it had to be always positive. This has caused difficulties which were absent in Neumann's potential energy treatment.

The potential of Eq.1.35 refers to mutually stored energy between two circuits. The total stored energy should contain additional contributions from the interactions of current element pairs residing in the same circuit. This self-inductance potential and the resulting internal reaction forces were not considered by Neumann. They have become a major point of disagreement between modern relativistic electromagnetism and Newtonian electrodynamics. Neumann was probably held back by the same integration singularities in isolated circuits which have caused considerable controversy in recent years. It will later be shown how these difficulties can be overcome with computer aided finite current element analysis.

In Neumann's theory the stored energy of metallic circuits is associated with the forces of attraction and repulsion between current elements. These elements consist of the substance of the conductor metal, and not the contained electric fluid (conduction electrons). Hence the energy never becomes detached from matter. This is a characteristic of Newtonian mechanics. In contrast to this, modern field theory is based on energy stored in the field and in vacuum. For the magnetic field energy to change, some has to be emitted or absorbed by metal conductors. It requires flying energy transport at the velocity of light and gives rise to many philosophical difficulties which are absent in the Newtonian electrodynamics. For example, there seems to be no satisfactory mechanism that can explain how magnetic field energy is recalled from the far reaches of space when a current is switched off.

Neumann's virtual work method of calculating ponderomotive forces and torques from the change of magnetic energy has survived to this day. It is often preferred to calculations using the Lorentz force. The relevant equations involve Neumann's mutual inductance formula. In many practical arrangements the mutual inductance can be measured with small AC currents. This avoids the most difficult part of the calculations.

It has become common practice to calculate the reaction forces between two parts of the same circuit with Neumann's virtual work concept, but in this case, the mutual inductance is replaced by the self-inductance of the isolated circuit. This procedure gives correct answers, but it cannot be traced back to Neumann.

He shied away from defining the mutual inductance and electrodynamic potential of an isolated pair of current elements. Most of his critical formulae refer to the mutual inductance and stored energy of a pair of complete circuits. The lack of a formula for the mutual inductance between two conductor elements leaves Neumann's theory strangely incomplete. The later chapters of this book fill the gap. The consequences of the mutual inductance formula for two current elements have not been contradicted by experiment.

Neumann's Laws of Electromagnetic Induction

As mentioned previously, Neumann discovered the electrodynamic potential while working on his theory of electromagnetic induction. He started by setting up an elemental law of induction for relative motion between two current elements. For this purpose he assumed that the electromotive force, abbreviated e.m.f., induced in one of the elements, was a function of the current intensity in the other and the Ampèrian force between the elements, provided the element experiencing the induction carried unit current. The latter assumption was made more precise by speaking of the Ampèrian force per unit current in the second element, that is $\Delta F_{m,n}/i_n$, where i_n is the current in the element which experiences the induction. This condition makes the induced e.m.f. in element n, that is Δe_n, independent of the current in this element. Hence the induced e.m.f. in n is finite even when $i_n = 0$. Neumann's induction mechanism is seen to be a one-way process in which the element which carries current is the cause of the induction and the second element experiences the effect. Whether or not there is a back-e.m.f. induced in element m depends on the current i_n. This latter process, however is completely separate from the induction of forward-e.m.f 's.

There is a clear distinction between Neumann's one-way induction forces and the mechanical forces described by Ampère's law, which are reciprocal forces always involving a two-way process. This difference is reflected by the fact that electromotive forces are measured in volts, while ponderomotive forces are measured in dynes or newtons.

With these postulates, Neumann's elemental law of induction due to relative motion between a current element $i_m dm$ and a conductor element dn can be expressed as

$$\Delta e_n = - v_x \frac{\Delta F_{m,n}}{i_n} \cos\theta_{r,x} \qquad (1.39)$$

where Δe_n is the induced e.m.f. in the conductor element dn shown in figure 1.12. The element dn is taken to be moving with velocity v_x along the arbitrary x-direction relative to the orientation of the inducing element $i_m dm$. $\Delta F_{m,n}$ is Ampère's mechanical force given by Eq.1.24. The angle $\theta_{r,x}$ lies between the distance vector $r_{m,n}$ and the positive x-direction. The negative sign in Eq.1.39 arises from Lenz's law which Neumann [1.9] quotes as follows:

> "If a metallic conductor moves relative to, and in the vicinity of, a galvanic current or magnet, the current induced in the conductor will flow in such a direction that, were the conductor at rest, it would be set in motion in the opposite direction, it being understood that the line of relative motion is fixed."

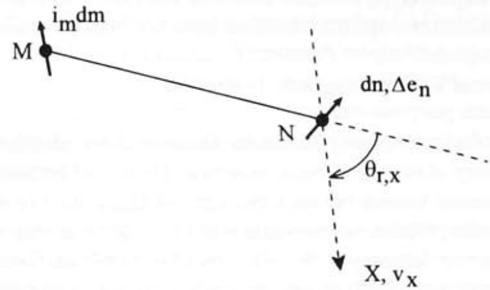

Figure 1.12 : Diagram for Eq.1.39

Neumann treated the proportionality of the induced e.m.f. to the relative velocity v_x and to the inducing current i_m (inside $\Delta F_{m,n}$) as experimentally established facts following Faraday's work and that of others. The elemental law of induction Eq.1.39 is in accord with the Newton-Ampère philosophy of simultaneous mutual matter far-actions. Therefore Δe_n should at all times be proportional to v_x, irrespective of relative acceleration. Neumann felt a little uncertain on this issue. On a number of occasions he referred to the 'stationary state' in which changes in current intensity and relative position of the elements progressed at a rate which was slow compared to the 'velocity of electricity'. As an example of non-stationary phenomena he quoted discharges of capacitors. In order to allow for delays between the cause of induction and its effect, Neumann made a certain dimensional constant a function of time. But then he did not proceed with the analysis of non-stationary phenomena. As can be seen from Eq.1.39, the time-dependent constant was dropped in the later years of Neumann's life. Since then it has been assumed that his law of induction is a law of simultaneous matter interaction.

That the simultaneity of inductive interactions is actually a requirement of the Ampère-Neumann electrodynamics can be seen from the following example. Consider a simple wire circuit connected to the terminals of a battery via a switch. Experience has shown that, when the switch is closed, the current increases smoothly from zero at a rate dictated by the total self-inductance of the circuit. Regardless of how large the circuit may be, the inductive interactions of all wire element pairs appear to spring into action instantaneously when the switch is closed. If the inductive interactions were delayed, the current would, in the first instance, jump to an infinite value, and then decrease to the level dictated by the increasing inductance. However, experiments have never shown a discontinuous jump in the current when the switch is closed, and furthermore, the initial smooth current rise is precisely as predicted from the total self-inductance. Therefore Neumann's original fear that inductive interactions may be delayed was unfounded.

Neumann noticed that the induced e.m.f. was related to the rate of change of the electrodynamic potential. This can be seen by using figure 1.12, and observing that

$$v_x \cos\theta_{r,x} = \frac{dr}{dt} \tag{1.40}$$

If we assume that the mutual potential of two closed currents as described by Eq.1.35 is the sum of the elemental potential contributions $\Delta P_{m,n}$ from all current element pairs, with one element in either circuit, then Eq.1.39 may be given the form

$$i_n \,\Delta e_n = \frac{d}{dt}\,\Delta P_{m,n} \tag{1.41}$$

The left side of this equation represents power or energy flow to element dn, and the right side gives the rate of change of mutually stored potential energy of the two elements.

Eq.1.39 will be called Neumann's first law of induction. It is an empirical law because it was derived from the experimental facts discovered by Faraday. Even though, as Eq.1.41 indicates, this law could have been derived from the principle of virtual work, the empirical basis is necessary to place it squarely on the foundation of nineteenth century Newtonian electrodynamics.

Interestingly, one experiment has come to light in recent years which can only be explained with Neumann's first law of induction, and not with Maxwell's equations. It is concerned with the operation of railguns and will be discussed in detail in Chapter 5. Maxwell claimed that he had incorporated Neumann's theory into his equations, and particularly in what has become known as Faraday's law of induction. It is now evident that this marriage of the far-action theory with field-contact action was less than perfect.

To establish his second law of electromagnetic induction, Neumann went to great length to show that an equation like Eq.1.41 also applies to two complete circuits m and n. This can be written

$$i_n \, e_n = \frac{dP_{m,n}}{dt} \tag{1.42}$$

Substituting for the electrodynamic potential from Eq.1.35, the total e.m.f. induced in the circuit n becomes

$$e_n = -\frac{d}{dt}\oint_m \oint_n \frac{i_m \cos\varepsilon}{r_{m,n}}\, dm\; dn \tag{1.43}$$

or

$$e_n = -\frac{d}{dt}\, i_m\, M_{m,n} \tag{1.44}$$

where $M_{m,n}$ is the mutual inductance of the two circuits given by Eq.1.26. The step from Eq.1.43 to Eq.1.44 may be taken as the definition of Neumann's mutual inductance in terms of matter interactions rather than magnetic flux linkage. Modern textbooks on electromagnetism call Eq.1.44 Faraday's law because the product of current and mutual inductance is the mutual flux linkage between the two circuits. Without wishing to take away anything from the important experimental achievements of Faraday, we will describe Eq.1.44 as Neumann's second law of electromagnetic induction. This corrects the historical record.

When computing mutual inductances with Neumann's formula, Eq.1.26, it is necessary to assign directions of current flow to the two closed circuits, as only this will make the angle ε unique for every element pair. The $\pm$ sign of Eq.1.26 acknowledges this uncertainty with respect to the current directions. Reversing the direction of current in one of the circuits will not change the magnitude of the mutual inductance but reverses its sign.

Provided the conductor elements belong to two closed circuits, it follows from Eq.1.42 that the elemental e.m.f. may be expressed as

$$\frac{\Delta e_n}{dn} = - \frac{d}{dt}\left(\frac{i_m \, dm \, \cos\varepsilon}{r_{m,n}}\right) \tag{1.45}$$

The quantity inside the bracket turns out to be the magnetic vector potential of the current element $i_m dm$ at point N, the center of the conductor element dn. Neumann wrote his papers before vector analysis was invented and he did not mention the magnetic vector potential. Denoting the vector potential by $\vec{A}$, Neumann's second law of induction may be stated as

$$\frac{\Delta \vec{e}_n}{dn} = - \frac{d}{dt} \Delta \vec{A}_{m,n} \tag{1.46}$$

where the vector potential is given by

$$\Delta \vec{A}_{m,n} = \frac{i_m \, d\vec{m}}{r_{m,n}} \quad ; \quad \Delta \vec{A}_{n,m} = \frac{i_n \, d\vec{n}}{r_{m,n}} \tag{1.47}$$

The vector potential is not a reciprocal interaction parameter because it involves only one current element at a time. Therefore

$$\Delta \vec{A}_{m,n} \neq \Delta \vec{A}_{n,m} \tag{1.48}$$

As a consequence of Eq.1.35, the mutual electrodynamic potential of a pair of current elements, belonging to separate closed circuits, is

$$\Delta P_{m,n} = - i_m \, i_n \frac{\cos\varepsilon}{r_{m,n}} \, dm \, dn \tag{1.49}$$

and using Eq.1.47 this is equal to the scalar products

$$\Delta P_{m,n} = i_m d\vec{m} \cdot \Delta \vec{A}_{n,m} = i_n d\vec{n} \cdot \Delta \vec{A}_{m,n} \tag{1.50}$$

Eq.1.50 reveals just how closely the magnetic vector potential is related to Neumann's electrodynamic potential.

Closed conduction currents may induce e.m.f 's in open-circuited conductor sections. An example is the combination of a loop antenna with a dipole antenna. This problem was examined by Neumann. Consider a conductor n consisting of just a single element dn. In order to obtain the action of the closed current i_m on dn, using Eq.1.41 and Eq.1.49 it follows that

$$\Delta e_n = - dn \frac{d}{dt} \left(i_m \oint_m \frac{\cos \varepsilon}{r_{m,n}} dm \right) \tag{1.51}$$

The angle function in Ampère's force law, Eq.1.24, is written as

$$f(\alpha, \beta, \varepsilon) = (2 \cos \varepsilon - 3 \cos \alpha \cos \beta) \tag{1.52}$$

But Neumann had proved that, when one of the circuits is closed

$$dn \oint_m \frac{2 \cos \varepsilon - 3 \cos \alpha \cos \beta}{r_{m,n}} dm = dn \oint_m \frac{\cos \varepsilon}{r_{m,n}} dm \tag{1.53}$$

and thus in this case

$$f(\alpha, \beta, \varepsilon) = \cos \varepsilon \tag{1.54}$$

It is this restricted angle function which is being used in Eq.1.51.

If the conductor n consists of more than one element and extends from n_1 to n_2 the e.m.f. induced in this length of conductor is

$$e_n = \frac{d}{dt} \left[i_m \int_{n_1}^{n_2} \oint_m \frac{\cos \varepsilon}{r_{m,n}} dm \; dn \right] \tag{1.55}$$

Neumann's theory of electromagnetic induction, pertaining to metallic conductors, has survived in modern field theory, however the words around the formulae have changed. Where Neumann spoke of interacting conductor elements and complete circuits modern physics now talks of magnetic flux linkage. The e.m.f. per unit length has become the electric field intensity, and so on. The flux linkage idea breaks down when one of the circuits is unclosed. Neumann's method, on the other hand, can deal with the e.m.f. in an unclosed conductor, as has been shown with Eq.1.55.

Since the electrodynamic potential was derived from Ampère's force law, and since this potential largely survives in field theory, one might expect Maxwell's equations to contain Ampère's force law, but in fact they do not. Maxwell [1.8] himself was aware that field theory does not contain a force law. He strongly endorsed Ampère's law but thought the Grassmann formula, to be discussed in the next section, would do equally well. The Grassmann law has become the magnetic component of the Lorentz force acting between two moving charges. In modern electromagnetism this has taken over the function of Ampère's law.

Classical Newtonian physics was based on the pillars of three empirical force laws, those of Newton, Coulomb, and Ampère. They were all simultaneous far-action laws and ushered in the first two centuries of quantitative science. Modern physics has made a complete break with far-actions. The first step in this direction was taken by Maxwell (1831-1879), however before then, the far-action electrodynamics had been developed in other directions, as discussed in the following section.

Grassmann's Force Law

Both Ampère and Neumann had concerned themselves not only with the interaction of linear currents in metallic conductors, but also with the mechanical and electromotive forces between magnets and in mixed systems containing magnets and current circuits. Ampère's concept of the magnetic molecule has had lasting value. Neumann showed the equivalence of a current carrying circuit and a magnetic shell bounded by the circuit. Our book does not deal with the behaviour of magnetic materials and concerns itself solely with the interaction of electric currents in non-magnetic metals. In this restricted sense, electrodynamics is the science of metallic current elements.

The force law for two current elements which has been used almost to the exclusion of all others during the past eighty years was first proposed by Grassmann (1809-1877) in 1845 [1.14], the same year Neumann published his theory of induction. Grassmann's is an unsymmetrical law and therefore has to be stated by two equations. One is for the force on element dm, to be written ΔF_m, and the other for the force ΔF_n acting on element dn. In vector form, but otherwise with the previously employed notation, these two equations are

$$\Delta \vec{F}_m = \frac{i_m\, i_n}{r_{m,n}^2}\, d\vec{m} \times (d\vec{n} \times \vec{a}_{r,m})$$

$$\Delta \vec{F}_n = \frac{i_m\, i_n}{r_{m,n}^2}\, d\vec{n} \times (d\vec{m} \times \vec{a}_{r,n}) \tag{1.56}$$

where the direction of the unit distance vectors $\vec{a}_{r,m}$ and $\vec{a}_{r,n}$ is along the line connecting the elements and pointing toward the element at which the force is being determined. Grassmann considered Eq.1.56 to be a far-action law between two elements of matter, strictly in the Newtonian sense.

To see the relationship to the Ampère-Neumann electrodynamics it is best to resolve the triple vector products of Eq.1.56 according to

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\ (\vec{A}\cdot\vec{C}) - \vec{C}\ (\vec{A}\cdot\vec{B}) \tag{1.57}$$

Applying Eq.1.57 to Eq.1.56 results in

$$\Delta\vec{F}_m = \vec{dn}\ \frac{i_m\ i_n\ dm}{r_{m,n}^2}\cos\alpha_m - \vec{a}_{r,m}\ \frac{i_m\ i_n\ dm\ dn}{r_{m,n}^2}\cos\varepsilon$$

$$\Delta\vec{F}_n = \vec{dm}\ \frac{i_m\ i_n\ dn}{r_{m,n}^2}\cos\alpha_n - \vec{a}_{r,n}\ \frac{i_m\ i_n\ dm\ dn}{r_{m,n}^2}\cos\varepsilon \tag{1.58}$$

The angles α_m and α_n must not be confused with α and β of Ampère's force law, Eq.1.24, but ε is the same in both laws. Figure 1.13 should help to illustrate the respective angle conventions.

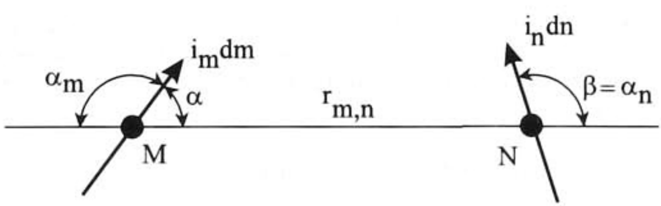

Figure 1.13 : Angle conventions for Ampère's and Grassmann's force laws, Eqs.1.24 and 1.58

According to Eq.1.56 a pair of current elements do neither attract nor repel each other. Each experiences a force perpendicular to itself which has its cause in the existence of the other. The transverse force lies in the plane containing the element in question and the line connecting both elements.

Grassmann also pointed out that, as a result of Ampère's rule, the interaction of two current elements always reduces to a two-dimensional problem. With respect to figure 1.13 this means that if we wish to determine the force on element dm we need only consider that component of the other element which lies in the plane of dm and $r_{m,n}$. In compliance with Ampère's rule, the component of the other element which is perpendicular to this plane produces no force on the element dm. Therefore the interacting components of the two current elements and the Grassmann forces all lie in the same plane.

The abandonment of mutual attraction and repulsion between matter elements of electric conductors, and the violation of Newton's third law which this entailed, signalled the end of Newtonian physics. The Grassmann and Lorentz force laws required a new mechanics which was to become that of the theory of special relativity. Regardless of velocity of light issues, special relativity was already inherent in the electrodynamics of Grassmann and Lorentz.

In the expanded form of Grassmann's law, Eq.1.58, the second term is a Newtonian term of repulsion or attraction. The remaining term is a force acting in the direction of the other current element. It is this term which violates Newton's third law. It will be called the relativistic term of the Grassmann or Lorentz force. Whittaker [1.6] and others have shown that when the Grassmann force on, say, dm is summed over a closed circuit n, all the relativistic contributions add up to zero. Only the force contributions made by the Newtonian term survive, and they agree with Ampère's law. Hence if in the Grassmann electrodynamics forces are calculated which involve one or more closed metallic circuits, we automatically, and in most cases unknowingly, slip back into the mathematics of the Newtonian electrodynamics. This mathematical deception has confused many field theoreticians.

Grassmann gave the magnitude of the perpendicular force acting on dn as

$$\Delta F_n = i_m \, i_n \, \frac{dm \;\; dn}{r_{m,n}^2} \, \sin\theta \qquad (1.59)$$

where θ is the angle of the Biot-Savart law, Eq.1.2, but with the ds-element replaced by the dm-element. Figure 1.14 depicts the connection between the Grassmann and Biot-Savart laws. The Biot-Savart law gives the magnetic field strength $d\vec{H}$ at N due to the dm current element. Therefore

$$\Delta \vec{F}_n = i_n \, d\vec{n} \times d\vec{H} \qquad (1.60)$$

This last equation clearly reveals that the Grassmann force is actually the magnetic component of the Lorentz force of modern field theory.

It is rather surprising to find that Grassmann had enunciated his law twenty years before Maxwell wrote his field equations, but at the time only Faraday was speaking of magnetic flux. Grassmann was a mathematics teacher at a German high school. Grassmann's great achievement as a mathematician was the introduction of vector calculus. There is some suspicion that he proposed his new electrodynamics [1.14] mainly in order to have a good application for vectors. He certainly achieved this with Eq.1.56. The Lorentz force expression made its appearance in the 1890s, fifty years after Grassmann's paper. Lorentz was led to Eq.1.60 by his theory of electrons [1.15].

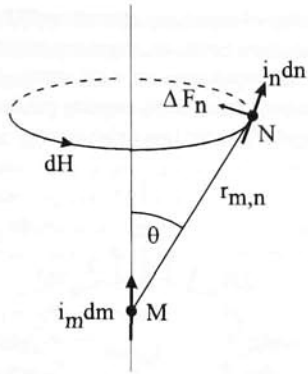

Figure 1.14 : Derivation of Grassmann's force law from the Biot-Savart law

On Grassmann's own authority, his investigation was prompted by two objections to Ampère's force law. He considered attraction and repulsion to be an arbitrary assumption, and also could see no reason why current elements should behave like gravitating and charged particles which were scalar quantities while current elements were vectors.

As far as his second objection to Ampère's law was concerned he said [1.4]:

> "The complicated form of this formula arouses suspicion, and the suspicion is heightened when an attempt is made to apply it. If, for example, the simplest case is considered, in which the circuit elements are parallel, so that $\varepsilon=0$ and $\alpha=\beta$, the Ampère expression becomes
>
> $$\frac{i_m\, i_n\, dm\, dn}{r_{m,n}^2}\,(2 - 3\cos^2\alpha)$$
>
> from which it appears that, when $\cos^2\alpha$ is equal to 2/3 or, which comes to the same thing, $\cos 2\alpha$ is equal to 1/3, that is the position of the midpoint of the attracted element lies on the surface of a cone whose apex is at the attracting element, and whose apex angle is arccos(1/3), there is no interaction; while for smaller angles there is repulsion, and for larger ones attraction. This is such an unlikely result, that the principle from which it is derived must come under the gravest suspicion and with it the supposition that the force in question must show an analogy with all other forces."

Ampère's force reversal which takes place when a current element, held parallel to itself, describes a circle around another element is plotted on figure 1.15. The mutual Ampère force varies from one arbitrary unit of repulsion at $\alpha=0$ or 180° to two units of attraction at $\alpha=90°$ or 270°. In each quadrant there exists an angular position for which the force is zero, and changes from attraction to repulsion, or vice versa. Grassmann did not like this unexpected variation of the elemental force with angular position, but was unable to provide an argument which proved Ampère's force law to be wrong.

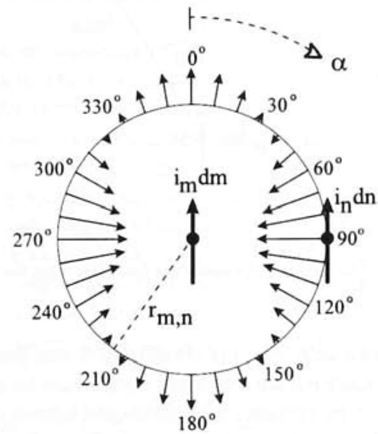

Figure 1.15 : Polar diagram of the Ampère force between two parallel current elements with a constant distance of separation

Lorentz established that Grassmann's formulae could be treated as empirical laws for electric charges drifting in vacuum. To do this he changed the definition of a current element. In the Ampère-Neumann electrodynamics this was a piece of matter of length dl, carrying an electric current i, such that the current element could be expressed as i.dl. Lorentz made the current element the product of an electric charge q and a relative velocity v, which could be written qv. The change in the definition of the current element had major physical consequences which will be discussed in later chapters.

Grassmann had no experimental results in 1845 with which he could support his force law. Instead he used mathematical and verbal arguments. His mathematical treatment is difficult to understand as he was still groping for an easy method of handling vectors.

The verbal logic which Grassmann put forward is interesting and worth summarizing. He had recognized that an infinitely long current behaves like a closed loop in its interaction with other currents. That is to say, the force which an infinitely long current will exert on an element of another current is always perpendicular to the latter, as proved by the wire-arc experiment of Ampère. Grassmann then extended this principle to an angle-current (Winkelstrom). This is an infinitely long current forming the two arms of an angle, the current coming from infinity in one straight conductor and returning to infinity in another. Subsequently he relied on the force on any current element lying in the plane of the angle-current to be perpendicular to the element, whatever its position or orientation in that plane. As Ampère had done, Grassmann assumed the wire to consist of a very large number of straight and short matter elements. Each element could be thought of as lying to one side of the apex of the angle-current. The idea is further explained by figure 1.16 which deals with a clever arrangement in which the closed circuit ABCDE consists merely of five elements. Element AB is part of the angle-current aAb and the element BC of angle-current bBc, and so on. It will be appreciated that each of the infinite rays a, b, c, d, and e carry outgoing and incoming currents of the same magnitude superimposed on each other and therefore, in fact, carry no current at all. With this mental picture Grassmann considered the force exerted on an external current element, $i_m dm$ in figure 1.16, and particularly the interaction of this external current element with any of the elements of a closed circuit such as ABCDE. He was convinced that each of the circuit elements would, independently of the others, generate a perpendicular force on the external element, because each of the elements of ABCDE was also part of a separate angle current. The algebraic sum of the perpendicular forces on $i_m dm$ was then the total force exerted by the circuit ABCDE on the external elements. This was the basis on which Grassmann justified the directions in his force law Eq.1.56.

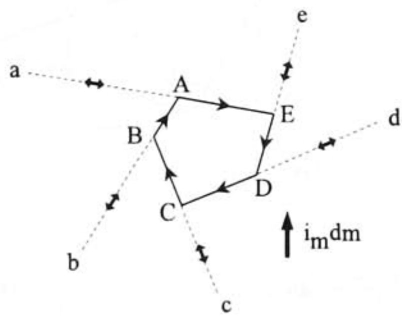

Figure 1.16 : Grassmann's angle-current representation of a closed circuit

Grassmann's argument requires substantiation in two respects. It is not clear whether forward and reverse currents can be superimposed to cancel their effects. Secondly, the five current elements of figure 1.16 cannot be fitted together as shown, for every element must have a volume, and the volumes would overlap.

Grassmann accepted Ampère's proof of the inverse square of distance relationship and the proportionality of the force to the product of the current intensities and element lengths. The sin θ of Eq.1.59 stems from the Biot-Savart law, but Grassmann arrived at it independently. The agreement between Grassmann's law, Eq.1.59, and the expansion of the triple vector product Eq.1.58 may be shown with figure 1.17. In this diagram $i_n dn$ is the resolved component of the general element at N in the plane of $i_m dm$ and $r_{m,n}$. Furthermore if

$$k = i_m \, i_n \, \frac{dm \;\; dn}{r_{m,n}^2}$$

then k cos ε is the magnitude of the Newtonian vector of Eq.1.58 and k $\cos\alpha_n$ is the magnitude of the relativistic vector of the same equation. Applying the sine rule to the force triangle of figure 1.17 gives

$$\frac{\Delta F_n}{\sin\theta} = \frac{k\ \cos\alpha_n}{\sin((\pi/2) - \alpha_n)}$$

Therefore

$$\Delta F_n = k\ \sin\theta$$

which proves the magnitude equality of Eqs.1.58 and 1.59.

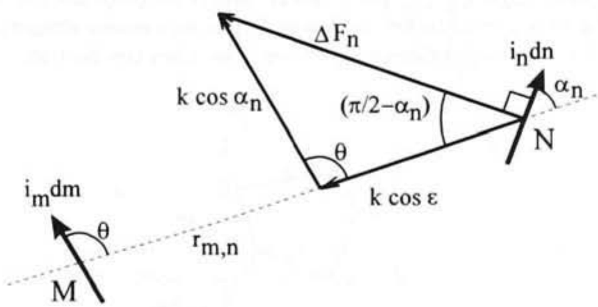

Figure 1.17 : Force triangle used to prove equality of Eqs. 1.58 and 1.59

Grassmann's new electrodynamics had little impact on his contemporaries. It would now be completely forgotten but for the fact that it fitted in well with the 20th century field, relativity, and electron theories. Although it was based on many of Ampère's ideas and experiments, his rejection of the concept of balanced action and reaction between each pair of current elements meant that Grassmann's law was not a part of the Newtonian electrodynamics.

Weber's Force Law and Electrodynamic Potential

Finding mathematical laws capable of quantifying Faraday's 1831 discovery of electromagnetic induction took much longer than Ampère's deduction of a force law from the discovery of electromagnetism. No less than fourteen years elapsed before Neumann in Köenigsberg published his laws of induction. Other scientists had been studying this problem at the same time. Among them were Fechner [1.16] and Weber [1.17] in Leipzig. Neumann had derived his theory of electromagnetic induction without hypothesis as to the nature of the electric fluid. He did, however, have to invent a new force, which he called the electromotive force, to distinguish it from the ponderomotive force which moved the metallic conductor.

The Leipzig school apparently appealed to only one kind of force on charged particles that possessed mass and constituted the electric fluid. Electrolysis had revealed the existence of discrete charged particles in an electric current, yet Fechner and Weber did not know of the rigid connection of positive charges (ions) to the metal lattice. That not all electrodynamic forces in a metal are mechanical forces on the lattice must have something to do with the two types of bonds that (a) exist between positive and negative charges and (b) between charges and the solid body. This issue of bonding between charges and ponderable matter has still not been satisfactorily resolved even at the end of the twentieth century.

In its experimental consequences, Weber's work, in the end, added little to that of Coulomb, Ampère, and Neumann. Weber based his force law on the same mutual simultaneous far-action principle underlying all of the Newtonian electrodynamics. What stands out in Weber's writings is that he was the first to take notice of the atomicity of electricity. He formulated a new model of the metallic current element in terms of mobile charged particles. As the following chapters will show, the search for a satisfactory model of the Ampèrian current element, compatible with all of solid state physics, is still in progress today. Weber's theory will be reviewed to illustrate some problems which have to be faced in order to find the correct microscopic current element model.

At the instigation of Fechner, Weber searched for a force expression which was mathematically equivalent to Ampère's well-proven law. Accepting the two-fluid model of electric current, Weber used positive and negative charges streaming in opposite directions, past each other, through and along current elements. Then he hypothesized that the interaction forces between the charged mass particles would not only depend on their relative positions, but also on their relative motions. From the start he intended to unify electrostatics with electrodynamics, and therefore Weber's force law had to include Coulomb's law, Eq.1.3, for two charges at relative rest with respect to each other.

Weber proposed the following empirical laws for the force $\Delta F_{e,e'}$ between two electric charges e and e', and the mutual potential $P_{e,e'}$, associated with this force

$$\Delta F_{e,e'} = \frac{e\ e'}{r^2}\left[1 - \frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2 + \frac{r}{c^2}\frac{d^2r}{dt^2}\right] \qquad (1.61)$$

$$\Delta P_{e,e'} = \frac{e,e'}{r}\left[1 - \frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2\right] \qquad (1.62)$$

where r is the distance between the charges, t is time, and c is a dimensional constant. The Ampère-Neumann electrodynamics was formulated in fundamental electromagnetic units (e.m.u.). The Weber equations, in contrast, are given in fundamental electrostatic units (e.s.u.) because they contain Coulomb's law which defines the e.s.u.-system of units. According to this system, the product of two electrostatic units of charge divided by their distance (in centimeters) squared gives the interaction force in dynes. Therefore the factor outside the main bracket in Eq.1.61 has the dimension of a force. The terms inside the bracket have to be dimensionless numbers. This means c must have the dimension of velocity. Weber went on to show that if Eq.1.61 was to be in agreement with Ampère's force law, Eq.1.24, when the current in each element is carried by a single positive electric charge travelling with constant velocity with respect to the metallic conductor element containing it, and furthermore if this law was also to agree with the results of Ampère's experiments, then the constant had to have the value $c = 3\times10^{10}$ cm/s. This constant became known as the velocity of light and it always emerges when the laws of electrostatics are combined with those of electrodynamics.

It was Weber's research into the fundamental measures of electromagnetism which revealed the importance of the c-factor. Using the same unit of force in the laws of Coulomb and Ampère, he calculated that 3×10^{10} electrostatic units of charge had to pass through the current element every second to represent the flow of one electromagnetic unit of current. The same rate of charge transfer is obtained when one charge passing through the current element travels at the velocity of light. This is how the velocity of light made its first appearance in the literature and Newtonian electrodynamics. In 1857 Kirchhoff [1.18] proved with Weber's law that electrical disturbances travel with the velocity of light along transmission lines. Readers of modern textbooks are often misled to believe that Maxwell was the first to discover the role which the velocity of light plays in electromagnetism.

Weber attributed no particular importance to c, however today it appears truly astonishing that the velocity of light should have sprung up in a simultaneous far-action theory such as his. Although the charges to which Eq.1.61 relates, move relative to each other and their distance r is a function of time, the forces of repulsion or attraction between the charges are assumed to change simultaneously with r. The formula does not allow for an energy propagation delay which could be linked to the velocity of light.

Weber [1.17] proved in detail how his force law, Eq.1.61, can be transformed to Ampère's force law, Eq.1.24. His transformation is a long mathematical process and teaches little. Of course Weber could not have guessed the form of Eq.1.61. In fact he derived it from the work of Coulomb and Ampère. His method of derivation is very instructive and worth repeating in brief outline. A more complete derivation can be found in reference [1.12].

Weber published his force law in 1846, the year between the two Neumann memoirs [1.9, 1.10], and he was obviously not aware of Neumann's researches on induction, to which he referred extensively in later years. Weber began his force law deduction as follows:

> "To lay down a guideline for this study, which is based on experience, we consider three specific facts resting partly on direct observation and partly on the indirect measurements underlying Ampère's fundamental law.
>
> (1) The first fact is that two current elements lying on the same straight line either repel or attract each other, depending on whether their currents flow in the same or opposite directions.

(2) The second fact is that two parallel current elements perpendicular to the line joining them either attract or repel each other, depending on whether their currents flow in the same or opposite directions.

(3) The third fact is that a current element, which lies on a straight line with a wire element, induces similarly directed or opposed current, depending on whether its own current intensity decreases or increases.

These three facts are not direct results of experiments, because the action of an element on another cannot be observed, but they accurately correspond to observed phenomena to the extent that they almost have the same validity. The first two facts are already incorporated in Ampère's basic formula of electrodynamics and the third has been added by Faraday's discovery (of induction)."

Figure 1.18 depicts Weber's model of two interacting current elements. Each element need only contain one positive and one negative charge. The two charges in each element move toward each other, along the line of the element, with velocity v relative to the metal. They are allowed to pass each other without appreciable deviation because, as Weber explained, we are not dealing with the actual happenings in the conductor but only with an action at a distance theory in which the charges are treated as if they could pass each other on a line. The current intensity of the element is taken to be ev, where e is the positive charge.

Four Coulomb-type interactions have to be considered, two of which are repulsions of like charges and the remaining two are attractions of unlike charges. All four sets of forces act along r, the line connecting the center points M and N of the two elements. All that was known about the forces at Weber's time was their strengths as given by Coulomb's law, Eq.1.3, for the case where they are at rest with respect to each other. Weber deemed it probable that relative motion between the charges would modify the actions, and Coulomb's law would give the limiting value of the forces when the relative velocities tended to zero. He considered it to be his task to determine the departure from Coulomb's law as a function of the relative motion between the charges.

Of the three facts on which Weber claimed he had built his theory, (1) and (2) referred to Ampère ponderomotive forces on the conductor metal, but (3) involved a Neumann-type electromotive force on charge. Weber convinced himself, however, that the total force experienced by the metal of the current element was the vector sum of all the electric forces on charges within it by charges located elsewhere. Since charges cannot penetrate the surface of the conductor, forces on charges which are transverse to the wire axis will be directly transmitted to the metal. This mechanism of force transmission to the wire material is not available to electromotive forces along current elements to generate Faraday's induced currents. With regard to these latter forces Weber seemed to argue that, since they are only of a transient nature and their magnitude is related to the ratio of the mass of the moving charges to the much greater mass of the stationary metal, they may be ignored in the calculation of mechanical forces on the metal. Weber apparently did not appreciate that the electromotive forces in a homopolar generator (Faraday disk) are not of a transient nature.

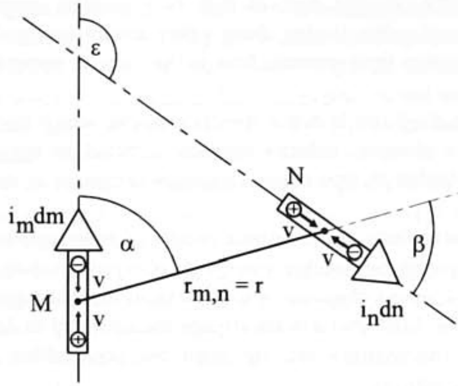

Figure 1.18 : Weber's model of two current elements

The lack of a credible mechanism of force transmission from freely moving charges to the body of the metal is the weakest point of Weber's electrodynamics. This is particularly true for longitudinal electrodynamic forces which generate tension in wire conductors.

The relative velocity of two charges separated by the distance r is (dr/dt). This differential coefficient is positive when the charges move apart and negative when they come together. In order to make the magnitude of that component of the inter-particle force, caused by relative motion, the same for approaching and receding motions, Weber assumed the second term of his force law, Eq.1.61 to be proportional to the square of the relative velocity.

According to Weber's rule of the stronger velocity-dependent interactions of charges moving in the same direction, the second term, furthermore, had to reduce the Coulomb forces. This determined its negative sign. The second term now had to be proportional to $-(dr/dt)^2$. The full mathematical analysis leading to the $(1/2c^2)$ factor is outlined in reference [1.12].

The third term of Weber's force law followed from the knowledge that side-by-side elements exert larger forces on each other than in-line elements. This refers to fact (2) of the three facts on which Weber's theory was based. To account for the stronger interaction of parallel elements, which are arranged perpendicular to the line joining them, it was necessary to call upon the relative acceleration d^2r/dt^2 contained in the third term of Weber's force law, Eq.1.61. The detailed mathematical steps of this long deduction can also be found in reference [1.12].

For two current elements $i_m dm$ and $i_n dn$, the electrostatic interaction will be zero, and Weber's force law reduces to the $(dr/dt)^2$ and the d^2r/dt^2 terms. It is only when considering the interaction force between any two charges e and e' that the full formula Eq.1.61 is required. In this formula the force is given in dynes when e and e' are inserted in e.s.u. of charge, r is in centimeters, and t in seconds. As explained before, the velocity c in Eq.1.61 is the velocity of light in vacuum.

Writing Weber's electrodynamic potential Eq.1.62 of two charged particles as

$$\Delta P_{e,e'} = \frac{e\,e'}{r} - \frac{e\,e'}{2\,r\,c^2}\left(\frac{dr}{dt}\right)^2 \tag{1.63}$$

it is seen to be the difference between the electrostatic potential and a term containing the relative velocity between the charges. The term involving the square of the relative velocity is akin to a quantity of kinetic energy given by

$$\frac{1}{2}\,m_e\,v_r^2$$

where m_e is some non-material electromagnetic mass and v_r the relative velocity between the charges. Weber's potential thus defines the electromagnetic mass as

$$m_e = \frac{e\,e'}{r\,c^2} \tag{1.64}$$

which leads to the following mass-energy relationship

$$m_e\,c^2 = \frac{e\,e'}{r} \tag{1.65}$$

This represents the first time that a term of the form mc^2 appeared in the scientific literature, and should be compared with Einstein's mass-energy equation ($E=mc^2$). The left-hand side of Eq.1.65 will be recognized as being similar to the magnetic field energy (with a different electromagnetic mass) of relativistic electromagnetism which is being transported by the Poynting vector, and yet the right side is a Newtonian potential energy.

Because of the inclusion of some kinetic energy in Weber's potential, the inter-particle force is not simply the negative gradient of the potential, as it was in Neumann's virtual work theory, but the Lagrange-force defined by the differential operator

$$\left[-\frac{\partial}{\partial r} + \frac{d}{dt}\,\frac{\partial}{\partial v_r}\right]$$

that is

$$\Delta F_{e,e'} = \left[-\frac{\partial}{\partial r} + \frac{d}{dt}\frac{\partial}{\partial v_r} \right] \left\{ \frac{e\ e'}{r} \left(1 - \frac{1}{2c^2} v_r^2 \right) \right\}$$

$$= \frac{e\ e'}{r^2} \left\{ 1 - \frac{1}{2c^2} \left(\frac{dr}{dt} \right)^2 + \frac{r}{c^2}\frac{d^2 r}{dt^2} \right\} \tag{1.66}$$

which is the Weber force law. It should be noted that, just as with photons, Weber's electromagnetic rest mass is zero.

The gathering of experimental facts in confirmation of Ampère's longitudinal forces led to a revival of the Weber electrodynamics in the decade of the 1980s. The hope was that a far-action electrodynamics would emerge which was based on forces between moving charges. Ampère's electrodynamics had offered no clues as to how the longitudinal forces could eventually be justified at the atomic level.

Modern Weber theorists are taking note of the fact that we now know that positive charges in the metal are frozen to the lattice, preventing them from moving relative to the conductor metal. The unusual Fechner hypothesis of the counter-directional flow of positive and negative charges, while the current consists only of the motion of the positive charges, has had to be dropped. Lorentz's electron theory of metals allows for the flow of conduction electrons. This is now defined as a negative current.

Assis [1.19] has shown that the fixed positive charges and mobile negative charges do not in any way change the appearance of Weber's force law, Eq.1.61 and electrodynamic potential, Eq.1.62. The atomic bonds which hold the lattice ions in place form an ideal means of transferring forces on positive charges to the body of the metal. No such bonds are available to transfer the forces on the conduction electrons to the metal lattice. These same electrons must also be free to respond to induced electromotive forces predicted by the Weber electrodynamics. Therefore if it seems impossible for the Weber theory to account physically for longitudinal electrodynamic tension in wires, then this difficulty arises from Weber's current element model.

Weber's theory, as well as Lorentz's modern electron theory [1.15], treat the current element as the product of a charge multiplied by a velocity. Ampère's law contains no velocity. A number of the experiments to be discussed in Chapter 2 suggest that all of Ampère's current element is anchored on the lattice site. It indicates that the Ampèrian current element may be some form of electric or magnetic dipole.

Kirchhoff's Circuit Theory

Gustav Kirchhoff (1824-1887) was Franz Neumann's most illustrious pupil. As a young man, and before Maxwell published any of his field theory papers, Kirchhoff developed what has become known as 'circuit theory'. This has proved most useful for electrical engineers and promises to continue to do so for many more years.

Today it is forgotten that circuit theory has its roots in the Newtonian electrodynamics of Coulomb, Ampère, Neumann, and Weber. It is, therefore, an action at a distance theory which, on close examination, disagrees with various aspects of modern field theory. For example, Kirchhoff [1.20-1.22] proved with circuit theory that voltage and current waves travel along wires with the velocity of light. This remarkable fact arose from multiple inductive and capacitive far-actions of huge numbers of current and conductor elements. Kirchhoff was in fact the first to derive delays in the transmission of electrical disturbances along conductors with a many-body interaction model. The strange aspect of the complex far-action calculations is that they predict the same time delays as the simple energy transport model of field theory.

In field theory it is asserted that the transmission of electrical signals along a two-wire line requires the flight of free electromagnetic energy between the wires. Moreover, this energy must travel with the velocity of light. One may speculate that the energy transport model represents the true physical state of affairs, and far-action theory is simply an abstract mathematical framework which furnishes the same signal propagation velocity. Should it be shown, however, by experiment that the free energy transport between the wires is fiction, then we would have little choice but to endow the far-action mechanism with a degree of physical reality. The experimental resolution of this question is an important aspect of our book.

First and foremost, circuit theory clarified the concepts of voltage and current in conjunction with Ohm's law and the definition of electrical resistance. In addition, the capacitance parameter took over the science of electrostatics, and the inductance parameters (self and mutual) did the same for electrodynamics. These are the reasons why with the three lumped circuit parameters of resistance, capacitance, and inductance, in addition to Kirchhoff's laws of the distribution of voltages and currents in electric networks, we can solve almost any problem in electrical engineering which does not involve the radiation of electromagnetic energy.

This completes the review of the Newtonian electrodynamics as it evolved in the nineteenth century. Before field theory became fully accepted there was a period in which retarded, and even advanced, potentials found favour. Retarded and advanced potentials are really the science of flying forces. The logistics of this very complex force transport however, was never satisfactorily resolved. Hence at the end of the twentieth century we are left with the Newtonian model of action at a distance, and field contact physics, based on the flight of energy, and championed by Maxwell, Lorentz, and Einstein.

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