

# Superluminal phase and group velocities: A tutorial on Sommerfeld's phase, group, and front velocities for wave motion in a medium, with applications to the "instantaneous superluminality" of electrons.

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## 1 Introduction

In 1905, Einstein published his historic paper on special relativity. Shortly afterwards, Sommerfeld [1] answered criticisms of Einstein's work, viz., that the phase and group velocities of electromagnetic waves can become superluminal, since these two kinds of velocities can exceed the vacuum speed of light inside a dielectric medium. Note that Einstein considered wave propagation solely *in the vacuum*, whereas his critics considered wave propagation *in media*.

Sommerfeld pointed out that while it is true that both the phase and the group velocities in media can in fact exceed  $c$ , the front velocity, defined as the velocity of a *discontinuous jump* in the initial wave amplitude from zero to a finite value, cannot exceed  $c$ . It is Sommerfeld's principle of the *non-superluminality of the front velocity* that prevents a violation of the Einstein's basic principle of causality in special relativity, i.e., that no effect can ever precede its cause.

In subsequent work, Sommerfeld and Brillouin [1] showed that the "front" is accompanied by two kinds of "precursors", now known as the "Sommerfeld", or the "high-frequency", precursor, and the "Brillouin", or the "low-frequency", precursor. These precursors are weak ringing waveforms that *follow* the abrupt onset of the front, but they *precede* the gradual onset of the strong main signal.

## 2 Superluminal phase velocities

It is well known that the phase velocity of electromagnetic waves can become superluminal under certain circumstances. A simple example is the superluminality of the phase velocity of an electromagnetic wave traveling within a

rectangular waveguide in its fundamental  $TE_{01}$  mode. We shall give below yet another, more impressive, example, namely, the superluminality of the phase velocity of X-rays in all materials. As was first noticed by Einstein, the superluminality of the phase velocity of X-rays in all kinds of crystals leads to the phenomenon of total *external* reflection of X-rays impinging at grazing incidence from the vacuum upon the surfaces of these crystals.

The definition of the phase velocity is best given through an example. Consider a monochromatic electromagnetic plane wave traveling down the  $z$  axis of a homogeneous dielectric medium

$$E(z, t) = E_0 \cos(kz - \omega t) \quad (1)$$

$$B(z, t) = B_0 \cos(kz - \omega t) \quad (2)$$

where  $\omega$  is the angular frequency of the wave and  $k$  is its wavenumber. The phasefronts  $\phi(z, t)$  of this wave are defined through the relationship

$$\phi(z, t) = kz - \omega t = k\left(z - \frac{\omega}{k}t\right) = k(z - v_{\text{phase}}t) = \text{const} \quad (3)$$

where the phase velocity  $v_{\text{phase}}$  is defined as follows:

$$v_{\text{phase}} = \frac{\omega}{k} \quad (4)$$

Thus a given phasefront of the electromagnetic wave satisfies the relationship

$$z - v_{\text{phase}}t = \frac{\text{const}}{k} = \text{const}' = z_0 \quad (5)$$

It is customary to define the index of refraction  $n(\omega)$  of the medium through the relationship

$$k = k(\omega) = n(\omega) \frac{\omega}{c} \quad (6)$$

where  $c$  is the vacuum speed of light. Thus the phase velocity is related to the index of refraction by

$$v_{\text{phase}} = \frac{\omega}{k(\omega)} = \frac{c}{n(\omega)} \quad (7)$$

For a typical transparent medium, such as a piece of glass, the index of refraction is greater than unity, so that the phase velocity of light in glass is less than the vacuum speed of light.

However, the index of refraction function  $n(\omega)$  is in general determined by the dispersive properties of the medium, and can be less than unity. For example, consider a medium consisting of Lorentz oscillators which obey the simple harmonic equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2x = eE/m \quad (8)$$

where  $\gamma$  is the damping constant of the oscillator,  $\omega_0$  is its resonance frequency,  $e$  is the charge of an electron, and  $m$  is its mass. The solution of the equation

of motion (8) of the simple harmonic oscillator, when it is being driven by the monochromatic electric field  $E$  written in its complex exponential form,

$$E = E_0 \exp(ikz - i\omega t) \quad (9)$$

is given by

$$x = \frac{eE/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad (10)$$

The polarization  $P$  of the medium is therefore given by

$$P = n_{\text{atoms}}ex = \frac{n_{\text{atoms}}e^2E/m}{\omega_0^2 - \omega^2 - i\gamma\omega} = \varepsilon_0\chi E \quad (11)$$

when  $n_{\text{atoms}}$  is the number density of atoms, that is, the number density of Lorentz oscillators. Solving for the susceptibility of the medium  $\chi$ , one then finds that

$$\chi = \frac{n_{\text{atoms}}e^2/m\varepsilon_0}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad (12)$$

where the plasma frequency is defined as

$$\omega_p = (n_{\text{atoms}}e^2/m\varepsilon_0)^{1/2} \quad (13)$$

The dielectric constant of the medium is related to the susceptibility by the definition

$$\varepsilon = 1 + \chi \quad (14)$$

Therefore the index of refraction for a medium of Lorentz oscillators is given by

$$n = \sqrt{\varepsilon} = \sqrt{1 + \chi} = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}} \quad (15)$$

Einstein noticed that for sufficiently high frequency X-rays, i.e., those X-rays which have a frequency above the highest possible X-ray transition frequency of the atoms in a crystal, namely, those transitions in which the most tightly bound electron in the ground state of the atom (i.e, the electron in the 1S state, or K shell, closest to the nucleus) is knocked out by the X-ray into the continuum, one can approximate the Lorentz model for the index of refraction (15) by its high-frequency form

$$n_{\text{X-ray}} = n(\omega \rightarrow \infty) = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}} \rightarrow 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} < 1 \quad (16)$$

*In other words, the phase velocity of sufficiently high-frequency X-rays in any medium that can be modeled by Lorentz oscillators will always be superluminal. Thus it follows that the index of refraction of any kind of crystal for X-rays of sufficiently high energy will always be slightly less than unity. However, from (16), it can be shown that the group velocity for these same X-rays in the same*

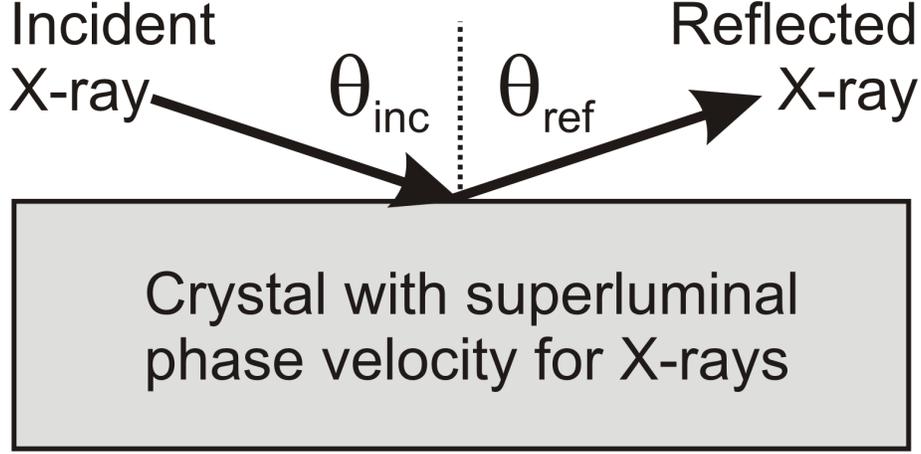


Figure 1: Einstein's total *external* reflection of grazing X-rays from a crystal of Lorentz oscillators. The phase velocity for sufficiently energetic X-rays inside the crystal will always be superluminal.

crystal will always be subluminal, since the group index (see (37)), which, at sufficiently high frequencies, is given by

$$n_{\text{X-ray}} + \omega \left. \frac{dn}{d\omega} \right|_{\text{X-ray}} \rightarrow 1 + \frac{1}{2} \frac{\omega_p^2}{\omega^2} > 1 \quad (17)$$

will always exceed unity.

Next, Einstein pointed out that Snell's law, when applied to the vacuum-medium interface in Figure 1 using (16), will lead to a critical angle given by

$$\sin \theta_{\text{crit}} = n_{\text{X-ray}} \cdot \sin 90^\circ = n_{\text{X-ray}} < 1 \quad (18)$$

Since the index of refraction is less than unity, there always will exist a solution for the critical angle for the total *external* reflection of grazing-incidence X-rays

$$\theta_{\text{crit}} = \sin^{-1} n_{\text{X-ray}} \quad (19)$$

The complement of this critical angle,  $\phi_{\text{crit}} = 90^\circ - \theta_{\text{crit}}$ , which, for grazing incidence X-rays, is given by

$$\sin \theta_{\text{crit}} = \cos \phi_{\text{crit}} \approx 1 - \frac{1}{2} \phi_{\text{crit}}^2 \approx 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} \quad (20)$$

so that the complement of the critical angle is given approximately by

$$\phi_{\text{crit}} \approx \frac{\omega_p}{\omega} \ll 1 \quad (21)$$

Now the plasma frequency lies in the ultraviolet part of the electromagnetic spectrum, whereas X-rays lie at much higher frequencies than ultraviolet frequencies. For example, for a 1 keV X-ray reflecting from gold, whose plasma

frequency corresponds to an energy of approximately 30 eV [2],  $\phi_{\text{crit}}$  is known to be 3.72 degrees [3], which agrees with (21) within a factor of two.

Satellite X-ray telescopes use grazing-incidence optics, which is based on Einstein's total-external-reflection effect, in order to form X-ray images of distant astrophysical objects [3]. This demonstrates that superluminal phase velocities have important applications.

It is a common misconception that superluminal phase velocities are unobservable [4]. However, as the above example of Einstein's total external reflection of X-rays shows, there exists at least one striking counter-example that can serve to dispel this misconception.

### 3 Superluminal signaling is not possible using superluminal phase velocities

Can one send a true signal faster than light by means of a superluminal phase velocity? The answer is no, since the phase velocity is the velocity of the crests (i.e., the phasefronts) of a continuous-wave, monochromatic, electromagnetic wave. Since the amplitude and phase of a continuous wave is not changing with time, there can be no information contained within such a waveform. As in radio, one must introduce a truly *time-dependent modulation* of a continuous "carrier" waveform (using either AM or FM modulation), before any true signal can be sent via the carrier wave.

In the case of quantum mechanics, the wavefunction of an electron can be written in terms of an amplitude and a phase factor as follows:

$$\psi(\mathbf{r}, t) = A(\mathbf{r}, t) e^{i\phi(\mathbf{r}, t)} \quad (22)$$

For the special case of a monochromatic plane wave traveling in the  $z$  direction

$$A(z, t) = A_0 \quad (23)$$

is a constant, and

$$\phi(z, t) = kz - \omega t \quad (24)$$

so that the electron wavefunction in a momentum eigenstate has the form

$$\psi(z, t) = A_0 e^{i(kz - \omega t)} \quad (25)$$

Therefore the phase velocity of the electron is determined through the relationship

$$\psi(z, t) = A_0 e^{i(kz - \omega t)} = A_0 e^{ik(z - \frac{\omega}{k}t)} = A_0 e^{ik(z - v_{\text{phase}}t)} \quad (26)$$

so that once again

$$v_{\text{phase}} = \frac{\omega}{k} \quad (27)$$

Now the Born interpretation tells us that the probability density for finding the electron at  $(z, t)$  is given by

$$|\psi(z, t)|^2 = \left| A_0 e^{i(kz - \omega t)} \right|^2 = |A_0|^2 = \text{constant} \quad (28)$$

and is therefore constant for a plane wave. Since the overall phase factor of the wavefunction is not an observable quantity, it can be argued that the phase velocity of the electron is not an observable quantity. However, the overall phase factor picked up by the wavefunction can in fact be observed in interference experiments.

However, in the case of a classical electromagnetic plane wave, the phase velocity *is* an observable quantity, just as the speed of the crests of the ripples of water waves on the surface of a pond is obviously an observable quantity. Furthermore, the Poynting vector for a classical electromagnetic plane wave is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{k}} (E_0 H_0) \cos^2(kz - \omega t) = \hat{\mathbf{k}} (E_0 H_0) \left( \frac{1}{2} - \frac{1}{2} \cos(2kz - 2\omega t) \right) \quad (29)$$

which clearly has a second-harmonic component that moves at the phase velocity  $\omega/k$ , which can in principle be observed. No such second-harmonic component exists in the case of the electron, as is evident by inspection of (28). However, although the phase velocity of an electromagnetic plane wave is in principle an observable quantity, no true signal can be transmitted by means of it.

## 4 Superluminal group velocities

While it is well known that phase velocities can become superluminal, it is less well known that group velocities can also become superluminal. There is a common misconception that the group velocity is the “signal” velocity of physics, which relates a cause to its effect. However, as we shall presently see, the group velocity is *not* the velocity that relates a cause to its effect. Only the front velocity can fulfill this role.

Consider a wavepacket propagating along the  $z$  axis. In quantum theory, such a wavepacket, for example a Gaussian wavepacket containing a single electron within it, can be represented by the Fourier integral

$$\psi(z, t) = \int_{-\infty}^{\infty} d\omega \tilde{\psi}(\omega) e^{ik(\omega)z - i\omega t} \quad (30)$$

Suppose that the wavepacket is strongly peaked in its amplitude  $\tilde{\psi}(\omega)$  at some frequency  $\omega_0$ . It is natural then to perform a Taylor series expansion of the wavenumber  $k(\omega)$  around  $\omega_0$ , which yields

$$k(\omega) = k(\omega_0) + (\omega - \omega_0) \left. \frac{dk}{d\omega} \right|_{\omega_0} + \dots \quad (31)$$

One can therefore approximate the Fourier integral (30) as follows:

$$\begin{aligned}
\psi(z, t) &= \int_{-\infty}^{\infty} d\omega \tilde{\psi}(\omega) \exp\left(ik(\omega_0)z + i(\omega - \omega_0) \left.\frac{dk}{d\omega}\right|_{\omega_0} z + \dots - i\omega t\right) \\
&= \int_{-\infty}^{\infty} d\omega \tilde{\psi}(\omega) e^{ik(\omega_0)z + i(\omega - \omega_0) \left.\frac{dk}{d\omega}\right|_{\omega_0} z + \dots - i(\omega - \omega_0)t} e^{-i\omega_0 t} \\
&= e^{ik(\omega_0)z - i\omega_0 t} \int_{-\infty}^{\infty} d\omega \tilde{\psi}(\omega) e^{i(\omega - \omega_0) \left.\frac{dk}{d\omega}\right|_{\omega_0} z - i(\omega - \omega_0)t + \dots} \\
&= e^{ik(\omega_0)z - i\omega_0 t} \int_{-\infty}^{\infty} d\omega \tilde{\psi}(\omega) e^{i(\omega - \omega_0) \left.\frac{dk}{d\omega}\right|_{\omega_0} \left(z - \left.\frac{dk}{d\omega}\right|_{\omega_0}^{-1} t\right) + \dots} \\
&\propto \psi(z - v_{\text{group}}t)
\end{aligned} \tag{32}$$

where the group velocity is identified as

$$v_{\text{group}} = \left.\frac{dk}{d\omega}\right|_{\omega_0}^{-1} = \left.\frac{d\omega}{dk}\right|_{\omega_0} \tag{33}$$

The meaning of the group velocity is that it is the velocity with which the peak of the wavepacket moves.

Now for an optical medium, we saw earlier that

$$k(\omega) = \frac{n(\omega)\omega}{c} \tag{34}$$

where  $n(\omega)$  is medium's refractive index. It follows that

$$\frac{dk(\omega)}{d\omega} = \frac{1}{c} \left( n(\omega) + \omega \frac{dn(\omega)}{d\omega} \right) \tag{35}$$

and therefore that the group velocity is given by

$$v_{\text{group}} = \left.\frac{dk(\omega)}{d\omega}\right|_{\omega_0}^{-1} = \frac{c}{n(\omega) + \omega \left.\frac{dn(\omega)}{d\omega}\right|_{\omega_0}} \tag{36}$$

The denominator of this expression for the group velocity

$$n_{\text{group}} = n(\omega) + \omega \left.\frac{dn(\omega)}{d\omega}\right|_{\omega_0} \tag{37}$$

is called the “group index”. By inspection of the group index, it is apparent that it can vanish whenever

$$n(\omega) + \omega \frac{dn(\omega)}{d\omega} = 0 \tag{38}$$

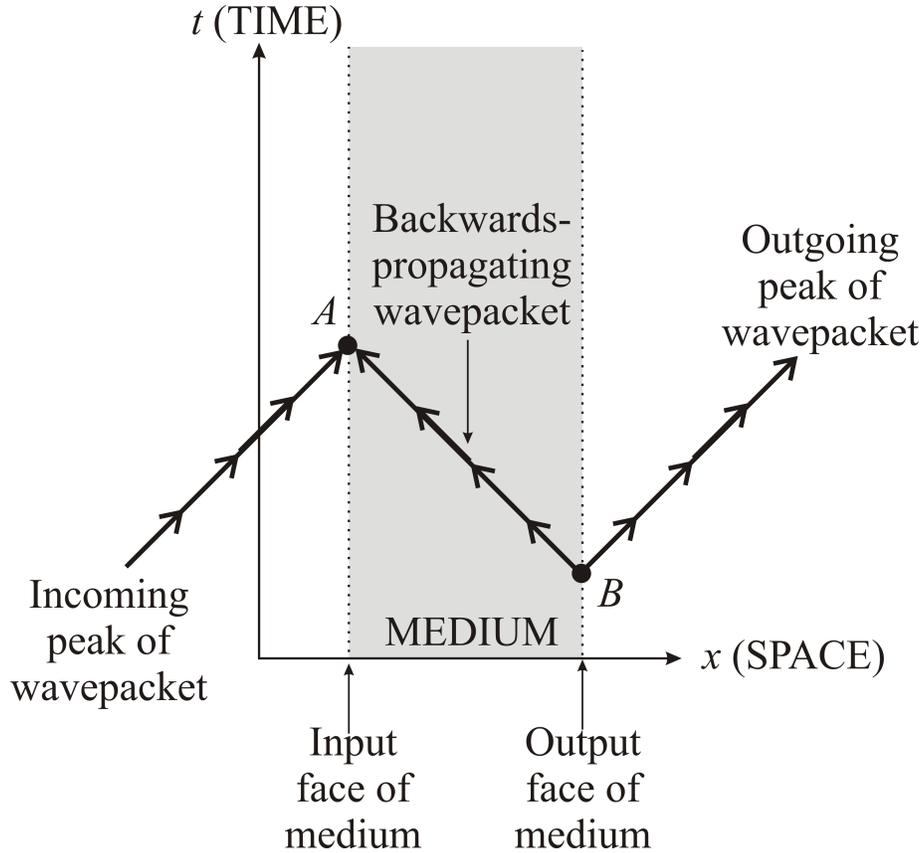


Figure 2: Feynman-like space-time diagram for the motion of the peaks of a wavepacket propagating with a negative group velocity inside a superluminal medium (in *gray*).

This can happen whenever there is anomalous dispersion

$$\frac{dn(\omega)}{d\omega} < 0 \quad (39)$$

i.e., whenever the index of refraction decreases with increasing frequency. Whenever this can happen, the group velocity can become infinite, which is obviously a kind of superluminal behavior.

By inspection of the denominator of (36), it is also clear that the group velocity can become *negative* whenever the group index is negative, i.e., whenever

$$n(\omega) + \omega \frac{dn(\omega)}{d\omega} < 0 \quad (40)$$

The meaning of a negative group velocity is this: *Before* the peak of an incoming wavepacket has entered the entrance face of the medium, the peak of

an outgoing wavepacket has *already* left the exit face of medium. This highly counter-intuitive, superluminal behavior in fact does not violate causality, and has in fact been observed in many experiments [5]. It can be understood with the help of the Feynman-like space-time diagram in Figure 2.

By taking an arbitrary time-slice between the two events  $A$  and  $B$  in this space-time diagram, one sees that there exist *three* wavepackets at this moment of time. The first wavepacket is the one coming in from the left towards the input face of the medium, the second wavepacket is the one propagating backwards within the medium from the output face of the medium towards the input face of the medium, and the third wavepacket is the one leaving the output face of the medium, and going out towards the right. Event  $A$  corresponds to an event in which the first and second wavepackets annihilate with each other in a “pair annihilation” event at the input face, and event  $B$  corresponds to an event in which the second and third wavepackets are created together in a “pair creation” event at the output face. Note that the pair-creation event precedes the pair-annihilation event.

In computer simulations of this phenomenon, it is observed that the early, analytic tail of a Gaussian wavepacket penetrates deeply into the medium. This early tail of the incoming wavepacket then triggers the emission of the pair of wavepackets at the pair-creation event  $B$  at the output face of the medium. The backwards-propagating wavepacket is then observed to subsequently annihilate with the incoming wavepacket at the pair-annihilation event  $A$  at the input face of the medium. In the special case of an inverted two-level medium excited far off of resonance by the incident wavepacket, the medium loans energy from the inverted two-level system in order to produce the two new wavepackets at  $B$ . This the loan is repaid later to the medium at  $A$ .

Not only can the group velocity become negative, but under certain circumstances it also can become infinite, such as in the special case (38). We shall call this important special case “instantaneous superluminality”, and we shall see that it can naturally arise in some quantum many-body problems, for example, in the quantum many-electron problem.

## 5 Two examples of “instantaneous superluminality” in the case of electrons

Here we illustrate the phenomenon of “instantaneous superluminality” using two examples. The first example is the case of single electrons escaping from the interior to the exterior of a normal metallic conductor, and the second example is the case of Cooper pairs of electrons propagating from one end of a superconducting island to the other.

# Faraday's "ice pail" experiment

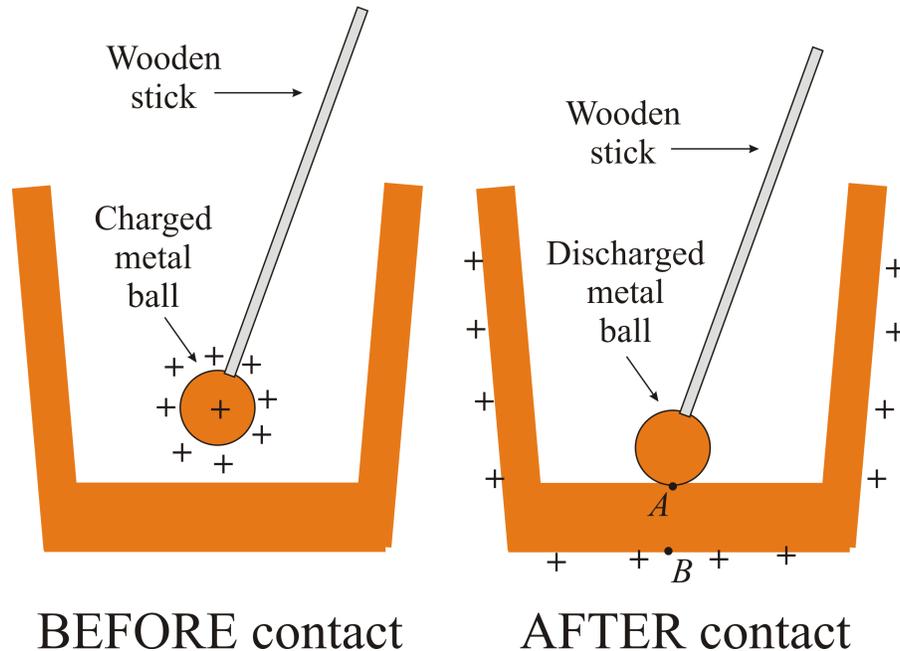


Figure 3: Faraday's "ice-pail" experiment. A charged metal ball at the end of a wooden stick is slowly and carefully lowered into the interior of a metallic ice pail, until it contacts the bottom of the pail. Upon contact, the charge on the ball disappears from the ball, and also from the interior of the ice pail, and suddenly reappears on the exterior surface of the pail. Note that the metallic bottom of the pail can be made arbitrarily thick. How suddenly does the charge disappear from the ball and reappear on the exterior surface of the ice pail?

## 5.1 The deposition of charge into the interior of a normal metallic body

Recall Faraday's "ice-pail" experiment, which is sketched in Figure 3. Charge is being delivered into the interior of a normal metallic body (i.e., the "ice pail") by means of a charged metal ball attached to the end of wooden stick (i.e., an insulated rod). The charged ball is being slowly lowered through the top opening of the ice pail until it contacts the bottom of the pail. Upon contact, the initial charge on the ball is observed to disappear from the ball, and also from the interior surface of the ice pail. Furthermore, the charge from the ball is observed to *suddenly* reappear on the exterior surface of the pail. How suddenly does this charge transfer process occur?

There are two possible answers to this question. The first possible answer is that the charge of the ball will initially escape from the ball along the in-

side surface of the pail, and then will finally reappear on the outside surface of the pail, after the escaping charge has propagated as *surface* electrical currents climbing up and over the rim of the pail. Such surface currents will propagate at the speed of light, since magnetic fields will be generated by these currents, and therefore will cause an electromagnetic wave to propagate along the surface of the metal. Therefore this *surface* kind of charge transfer will not be instantaneous, but will be retarded by the speed of light.

The second possible answer is that the charge of the ball will try to escape as a *volume* electrical current directly from the point of contact of the ball with the bottom of the pail to the nearest possible point on the exterior of the pail (i.e., from point *A* to point *B* and its surroundings in Figure 3). Note that shortest distance is that of the straight line joining points *A* and *B*, which lies entirely within the volume of the metallic bottom of the pail. These *volume* currents, which will flow inside the metal of the bottom of the pail, will be driven by the electric field lines emanating from the charge on the ball. Since no magnetic field can be generated within the interior of any metal beyond a skin depth of the surface of the metal, no propagating electromagnetic wave can be generated within the volume of the metal inside the bottom of the pail. Rather, these volume currents will be driven by the *instantaneous* Coulomb electric field lines emanating from the initial charge on the ball. Therefore this *volume* kind of charge transfer will occur directly from *A* to *B*. It represents a kind of *instantaneous action-at-distance*, and will not be retarded by the speed of light.

This latter kind of *instantaneous* charge transfer process is highly counter-intuitive, and has never been observed before. Does it really exist? In order to understand it better, consider the following “thought experiment” depicted in Figure 4, in which a pulsed electron beam suddenly deposits charge at the center of a copper sphere through a radial hole. A grounded cylindrical sleeve surrounding the incoming electron beam prevents the copper sphere from seeing the approaching electrons, until they actually strike the center of the sphere.

The continuity equation applied to the copper sphere states that

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \quad (41)$$

where  $\mathbf{j}$  is the electrical current density flowing at any point within the sphere and  $\rho$  is the charge density at that point.

Now let us assume that Ohm’s law holds at every point inside the sphere, so that

$$\mathbf{j} = \sigma \mathbf{E} \quad (42)$$

where  $\sigma$  is the conductivity of the copper sphere, which is composed of a homogeneous and isotropic copper material, and where  $\mathbf{E}$  is the local electric field inside the body, which is driving the electrical currents flowing from the center to the surface of the sphere. Actually, all that we need to assume here is that the local current density is *linearly related* to the local electric field that is driving the currents.

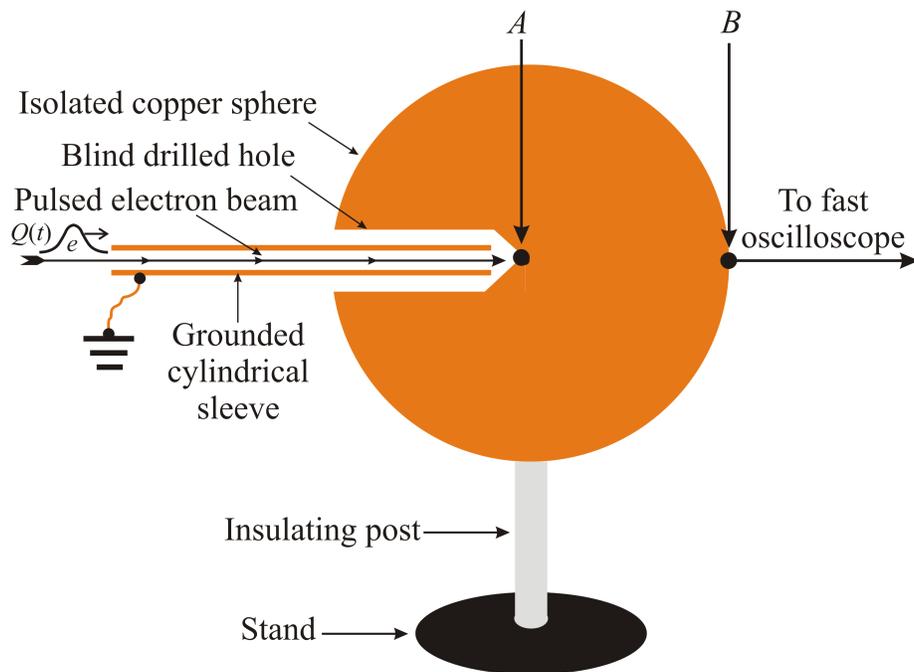


Figure 4: A pulsed electron beam enters a copper sphere (in *orange*) through an insulated, grounded copper sleeve inserted within a radial hole, and stops at the center (point  $A$ ) of the sphere. The grounded copper sleeve prevents the copper sphere from seeing the incident electrons before they strike point  $A$ . Note that the radius of the sphere can be made arbitrarily large. How suddenly does the charge disappear from point  $A$  and reappear at the surface, for example, at point  $B$ ?

Substituting (42) into (41), and using the fact that  $\sigma$  is a constant, one gets

$$\nabla \cdot (\sigma \mathbf{E}) + \frac{\partial \rho}{\partial t} = \sigma (\nabla \cdot \mathbf{E}) + \frac{\partial \rho}{\partial t} = \sigma \left( \frac{\rho}{\varepsilon_0} \right) + \frac{\partial \rho}{\partial t} = 0 \quad (43)$$

where we have used Maxwell's first equation  $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ . Therefore the charge density  $\rho$  obeys the linear, first-order partial differential equation

$$\frac{\partial \rho}{\partial t} = - \left( \frac{1}{\tau} \right) \rho = - \left( \frac{\sigma}{\varepsilon_0} \right) \rho \quad (44)$$

which implies an exponential decay with a time constant (i.e., the ‘‘Jeans’’ time scale [6])

$$\tau = \frac{\varepsilon_0}{\sigma} \quad (45)$$

The solution to (44) is the exponential decay law

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) \exp(-t/\tau) \quad (46)$$

According to this solution, if, initially at  $t = 0$ , the material is neutral at any point  $\mathbf{r}$  in the interior of the conducting body, i.e., if

$$\rho(\mathbf{r}, t = 0) = 0 \quad (47)$$

then it follows that at the same point  $\mathbf{r}$ , the body must remain neutral at all later times  $t$ , i.e.,

$$\rho(\mathbf{r}, t) = 0 \quad (48)$$

for all  $t > 0$ . In other words, the body at all interior points  $\mathbf{r}$  within its volume, during the entire charge transfer process from point  $A$  to point  $B$ , must remain electrically neutral.

However, suppose that at a single point  $A$ , such as at the center in the interior of the copper sphere in Figure 4 at time  $t = 0$ , the conducting body at this point were suddenly to be made non-neutral, for example, by charge being deposited at point  $A$  by a sudden charge deposition by the pulsed electron beam (or, similarly, by a sudden contact of the charged ball with the bottom of the ice pail at point  $A$  in Figure 3), so that

$$\rho(\mathbf{r}_A, t = 0) \neq 0 \quad (49)$$

Then at all later times  $t > 0$  following this sudden charge deposition, the charge density at this point will decay exponentially as follows:

$$\rho(\mathbf{r}_A, t) = \rho(\mathbf{r}_A, 0) \exp(-t/\tau) \quad (50)$$

However, all other interior points other than  $A$  that were initially electrically neutral, must remain electrically neutral at all later times  $t > 0$ . This implies that currents originating from the decay of the charge at  $A$  cannot accumulate any charge at intermediate points interior to the *volume* of the body, since the

divergence of the current density must vanish at all interior points due to the solution (48). Therefore the only points where the charge can accumulate due to the nonvanishing current density originating from the point charge deposition at  $A$  would be at the *surface* on the exterior of the body, such as at point  $B$  in Figure 4.

According to the solution (50), the charge at point  $A$  will disappear, and will extremely quickly reappear at the surface, for example, at point  $B$ . Let us put in some numbers to see how quickly this happens. For the case of copper, the measured value of the conductivity is

$$\sigma_{\text{Cu}} = 59.6 \times 10^6 \text{ S} \cdot \text{m}^{-1} \quad (51)$$

Therefore the decay time for the charge density at  $A$  is predicted to occur over the extremely short time scale [7]

$$\tau = \frac{\varepsilon_0}{\sigma_{\text{Cu}}} = 1.48 \times 10^{-19} \text{ s} \approx 0.15 \text{ attoseconds} \quad (52)$$

This is the time that it takes light to cross a distance of

$$c\tau = 4.45 \times 10^{-11} \text{ m} \approx 45 \text{ picometers} \quad (53)$$

which is about the size of the Bohr radius (approximately 50 picometers). Therefore for any macroscopically-sized copper sphere, the disappearance of the electron charge at point  $A$  and its sudden reappearance at an arbitrarily far-away point  $B$  in the case of an arbitrarily large sphere, would be a clear example of superluminality.

Note that no radiation can be produced in the configuration depicted in Figure 4 due to the spherical symmetry of the time-varying currents and charges, so that no retardation effect can occur associated with the production of an electromagnetic wave propagating in the interior at the vacuum speed of  $c$  *within* the volume of the metal.

However, the spherical symmetry of Figure 4 is not a necessary condition for the phenomenon of “instantaneous superluminality” to occur. In particular, the breaking of the spherical symmetry of the conducting body by the hole drilled into the copper sphere shown in Figure 4 for the purpose of the admission of the electron beam, is unimportant, as evidenced by the known observations in Faraday’s “ice pail” experiment depicted in Figure 3. The configuration of Faraday’s ice pail experiment clearly does not possess the near-spherical symmetry of the configuration shown in Figure 4. In particular, note that the small, radial hole in Figure 4 can be replaced by the very large opening at the top of the ice pail in Figure 3. Nevertheless, the charge deposited by the metallic ball is observed to disappear from the *inside* the ice pail, and to reappear suddenly on the *outside* of the ice pail.

If the above analysis, which is based on the continuity equation and on the linearity of Ohm’s law, turns out to be correct, this charge transfer process should have an unusual, instantaneously superluminal, component arising from the internal, volume currents produced by the Coulomb field within the metal, in

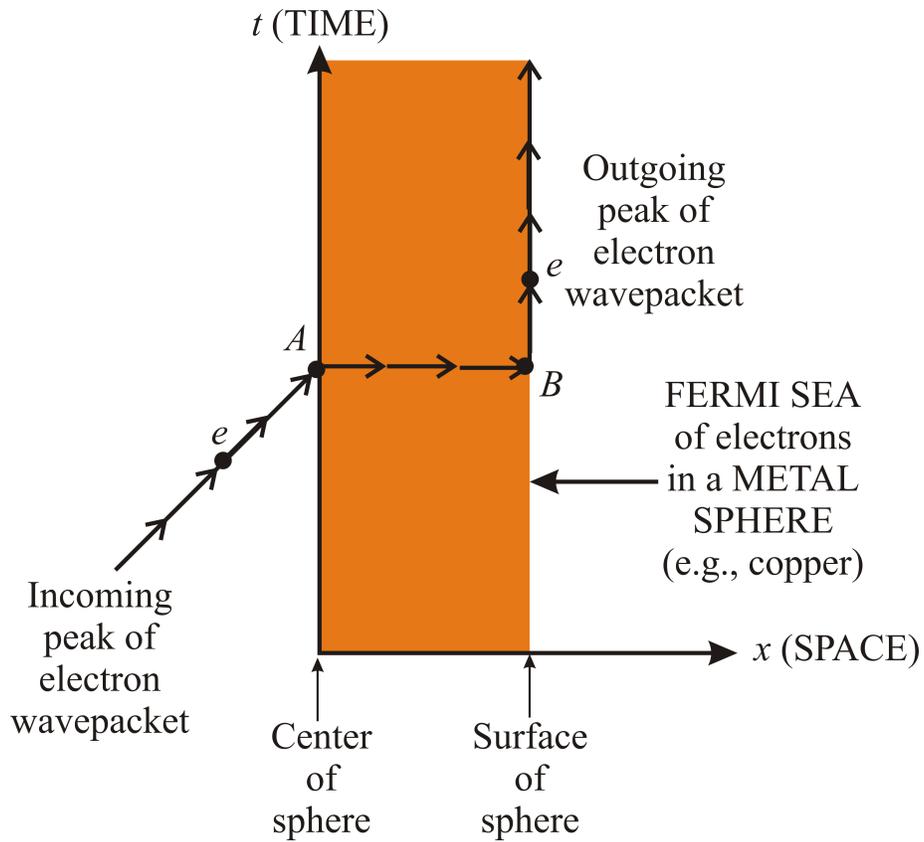


Figure 5: Feynman-like space-time diagram depicting the trajectory of the peak of a single electron wavepacket entering the center of the metallic sphere at point  $A$  in Figure 4. Here the group velocity of the electron must be infinite between  $A$  and  $B$  because the electron cannot join the Fermi sea due to the Pauli exclusion principle. It must therefore disappear at  $A$ , and must instantaneously reappear at  $B$ . Otherwise, charge conservation would be violated. The horizontal trajectory between  $A$  and  $B$  represents a virtual state of the electron.

addition to the usual, luminal component that one would expect to arise from the external, surface currents associated with the propagation of an electromagnetic wave.

At the quantum level, the above “instantaneous superluminality” effect is represented by the Feynman-like space-time diagram shown in Figure 5, which describes the charge transfer process for an individual electron which is initially approaching the center of the copper sphere at point  $A$  in Figure 4, with an energy less than the Fermi energy of copper. The electron will propagate superluminally via a virtual quantum state from point  $A$  to point  $B$  through the copper metal, inside which *there exists a Fermi sea of identical electrons*. This sea is in an entangled state, namely, the Slater determinant state. Entangled states lead to nonlocal, Einstein-Podolsky-Rosen (EPR) effects, in which instantaneous quantum correlations-at-a-distance can occur.

The electron which enters the metal at point  $A$  at the center of the sphere will be prevented by the Pauli exclusion principle from joining the Fermi sea of identical electrons in the interior of the metal [8]. Hence it has no choice but to reappear suddenly on the exterior surface of the metal, for example, at point  $B$  in Figure 4. Due to charge conservation, the disappearance of the electron at event  $A$  in Figure 5, and therefore the disappearance of its charge  $e$  at event  $A$ , must be *instantaneously* accompanied the *simultaneous* reappearance — in the reference frame of the center of mass of the Fermi sea — of an indistinguishable electron at event  $B$ , along with the reappearance of exactly same charge  $e$  at event  $B$ . Otherwise, charge conservation would be violated. Note that this will be true no matter how far apart  $A$  and  $B$  are from each other. Hence instantaneous actions-at-a-distance, in the form of Einstein-Podolsky-Rosen quantum correlations-at-a-distance, necessarily follow from the conservation of charge.

However, note here that the single electron approaching the center of the metal sphere will most probably go from the center to the surface of the sphere, and not from the surface to the center. This is because the density of allowed final states for the electron is much larger on the surface of the sphere than the density of allowed initial states at the center. Note also that here the charge transfer process is a dissipative one, and hence that it is an irreversible one.

Thus individual photons, which have earlier been observed to tunnel superluminally through a tunnel barrier [9], are not the only particles that can propagate through matter superluminally. Electrons can also be transferred superluminally through matter, for example, through a Fermi sea.

## 5.2 Experiment to observe “instantaneous superluminality” in a long aluminum bar

We are presently performing an experiment to test the highly counter-intuitive prediction of “instantaneous superluminality” for electron charge transfer in normal metals. A long, thick aluminum bar (six feet long, and five inches in diameter) has two blind holes (both about an inch deep, and half an inch in diameter [10]) drilled into either end of the bar, so that two miniature “Faraday ice pails” can be formed at the left and right ends of the bar, respectively

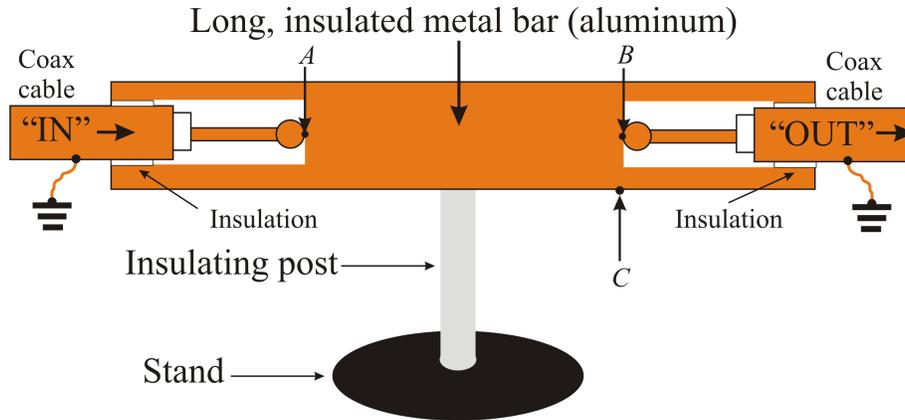


Figure 6: Schematic of an experiment to test the prediction of “instantaneous superluminality” in a long aluminum bar. The left cavity is for the insertion of the “IN” coaxial cable in the first miniature “Faraday ice pail” on the left, and the right cavity is for the insertion of the “OUT” coaxial cable in the second miniature “Faraday ice pail” on the right. A nanosecond pulse generator is connected to the “IN” coaxial cable, and a fast oscilloscope is connected to the “OUT” coaxial cable. *A* and *B* are points of electrical contact with the center conductors of the “IN” and “OUT” coaxial cables, respectively. A third coaxial cable (not shown) is connected to point *C* in order to detect the luminal signal for calibration purposes.

(see Figure 6). Two grounded coaxial cables are then inserted through thin, insulating sleeves into the two small cavities thus formed at the left and right ends of the long bar. Two small metal balls are soldered to the two ends of the center conductors of these cables, so that electrical contact can be made at the two points *A* and *B* deep inside these two miniature “Faraday ice pails”.

Charge is delivered to point *A* by a voltage pulse which is generated by means of a nanosecond pulse generator connected via a coaxial cable to the “IN” port on the left side of the aluminum bar. Most of the electric field lines emanating from point *A* that are generated by this pulse will intersect with the outer surface of bar, and will thus generate surface currents flowing along the exterior surface of the bar. Since time-varying magnetic fields will be generated by these surface currents, one expects an electromagnetic wave to propagate along the surface from the left to the right end of the bar, in the usual TEM mode of propagation. Thus a luminal pulse traveling at the vacuum speed of light should result from such surface currents.

However, some of the electric field lines emanating from point *A* will take the shortest possible path to reach point *B*, including the straight-line path that directly joins *A* to *B*. Such electric field lines will stay within the interior volume of the metal rod, and drive an internal current density through the ohmic relationship  $\mathbf{j} = \sigma\mathbf{E}$ , where  $\sigma$  is the conductivity of aluminum. The

analysis starting from the continuity equation given from (41) to (53) leads to the conclusion that the extremely short Jeans time scale (45) should hold for these volume currents.

One expects that a small fraction of the total number of electric field lines emanating from the charge deposited at point  $A$  will remain deep inside the volume of the metal rod at interior points during their journey from  $A$  to  $B$ . This includes the straight electric-field line that directly connects  $A$  to  $B$ . The fraction of internal electric field lines will be approximately given by the ratio of the solid angle subtended by the midsection of rod with respect to source point  $A$ , to  $4\pi$  steradians.

Therefore for the dimensions of our aluminum bar, we expect an attenuation of the voltage amplitude of about a factor of a thousand in the transmission of the instantaneously superluminal signal from  $A$  to  $B$ , relative to the voltage amplitude of the luminal signal, which can be picked up at point  $C$ . However, the resulting signal-to-noise ratio for detecting the pulse at  $B$  by means of a fast oscilloscope connected to the “OUT” coaxial cable, should still be large enough to allow for a significant detection of the instantaneously superluminal signal. For the purposes of calibration, a third coaxial cable will be connected to point  $C$  of Figure 6 on the surface of the rod near point  $B$ , so that the luminal surface-current signal can also be detected and displayed on separate channel of the fast oscilloscope for a direct comparison with the superluminal signal.

The data and the data analysis of this experiment will be presented elsewhere.

### 5.3 Superluminal charge transfer of Cooper pairs through a superconducting island

Figure 7 illustrates a second example of “instantaneous superluminality” for electrons. Consider a superconducting circuit consisting of a long superconducting island, which is connected to a charge source by means of a Josephson tunnel junction at point  $A$  on its left end, and to charge measuring device by means of an identical Josephson tunnel junction at point  $B$  on its right end.

The superconducting island is sandwiched tightly between upper and lower normal metallic films (made out of copper, for example), which are therefore in intimate electrical contact with it in a bimetallic structure. These upper and lower normal films serve to greatly decrease the island’s capacitance  $C_{\text{island}}$  [11].

The charging energy for depositing even just a single charge of  $Q_{\text{Cooper pair}} = 2e$  (i.e., the charge a single Cooper pair) onto the superconducting island is given by

$$U_{\text{charge}} = \frac{1}{2} \frac{Q_{\text{Cooper pair}}^2}{C_{\text{island}}} \quad (54)$$

Since the capacitance of the island can be made very low, the charging energy  $U_{\text{charge}}$  for adding even just a single Cooper pair to the island can be made very large when compared to the Josephson junction coupling energy

$$U_{\text{Josephson}} = I_{\text{J}} \Phi_0 \quad (55)$$

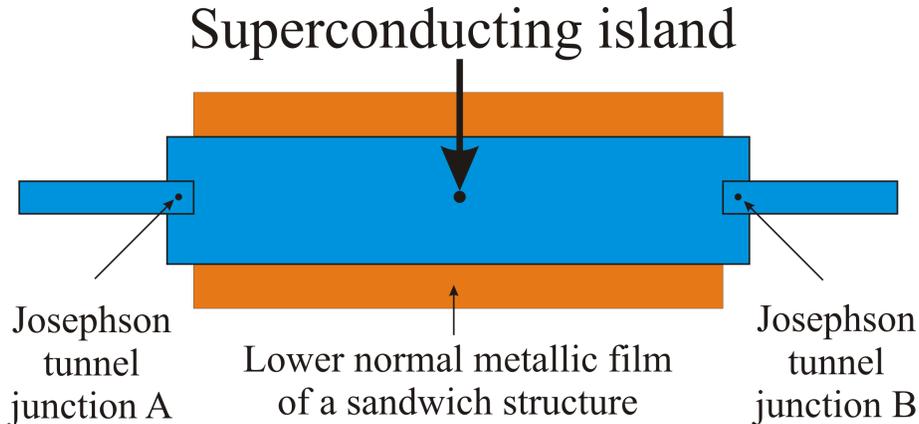


Figure 7: Charge is being transferred from point  $A$  to point  $B$  of a long superconducting island (in *blue*). A lower normal metallic film (in *orange*) lies in intimate electrical contact with it just beneath the island. Not shown is an upper normal metallic film lying just above the island in intimate contact with it, in a bimetallic “sandwich” structure. These normal films greatly diminishes the capacitance of the island, so that “Coulomb blockade” occurs.

where  $I_J$  is the critical current of the Josephson junctions and  $\Phi_0 = h/2e$  is the quantum of flux. Therefore it becomes highly energetically unfavorable for the superconducting island to become charged even by a single Cooper pair. Hence, due to the *discreteness* of electrical charge that arises from the quantization of charge, the island will remain electrically neutral at all times. This leads to a phenomenon called “Coulomb blockade”, in which all Cooper pairs are effectively “blockaded” from entering the island. Note that this effect is independent of the distance separating  $B$  from  $A$ .

As a consequence of the “Coulomb blockade” effect, a Cooper pair entering from the left at tunnel junction  $A$  will have to instantaneously exit to the right through tunnel junction  $B$  so as to maintain the charge neutrality of the superconducting island, no matter how far  $B$  is from  $A$ . This kind of instantaneous charge transfer from  $A$  to  $B$  leads to yet another example of a Feynman-like space-time diagram with the property of “instantaneous superluminality”, which is sketched in Figure 8. This diagram represents the instantaneous charge transfer of *Cooper pairs* of electrons through a Bose condensate of these pairs, a process that leaves the net charge of the condensate electrically neutral at all times (remember that the Cooper pairs initially present on the island are neutralized by the background ionic lattice).

An important difference between “instantaneous superluminality” in the case of superconductors, as compared to the case of normal conductors, is the fact that in former, the phenomenon is a non-dissipative one, whereas in the latter, it is a dissipative one (compare Figures 8 and 5). However, since an experiment can be performed at room temperature in normal metals, “instantaneous super-

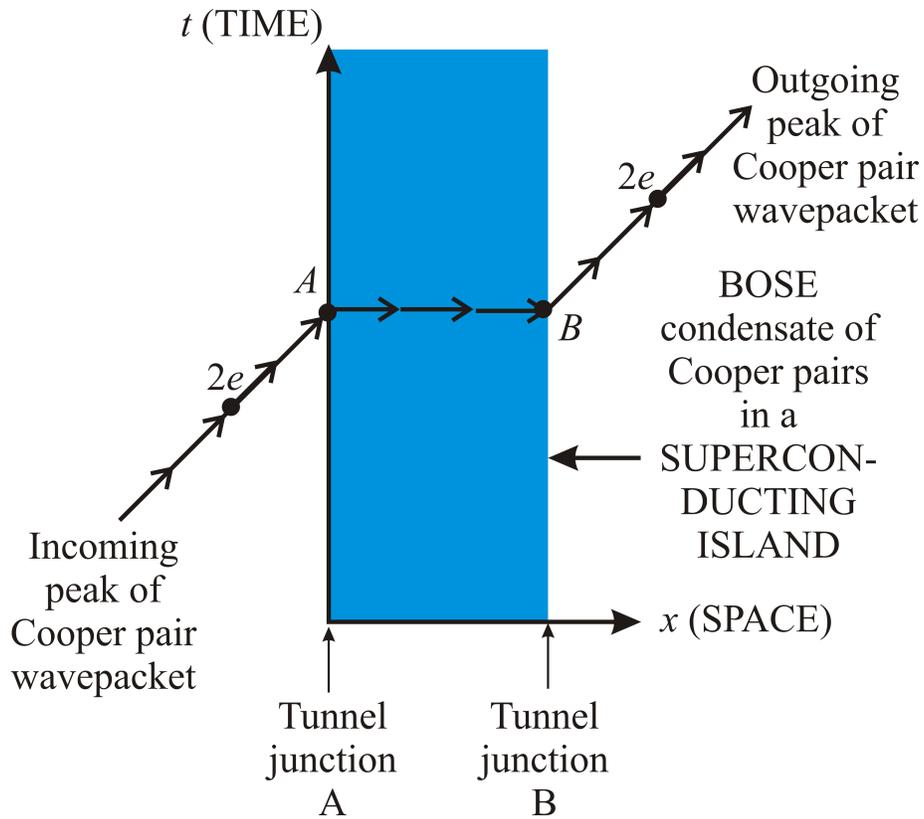


Figure 8: Feynman-like space-time diagram of the “instantaneously superluminal” charge transfer process of a Cooper pair through the superconducting island of Figure 7 from tunnel junction  $A$  to tunnel junction  $B$ . The horizontal trajectory between  $A$  and  $B$  represents a virtual state of the Cooper pair. This is another example of a quantum-mechanical “instantaneous action-at-a-distance”.

luminality” will be much easier to demonstrate in the former case than in the latter case. Nevertheless, experiments in the superconducting case could yield larger superluminal signals, since the dilution factor arising from the solid-angle considerations of the normal metal case would not apply.

## 6 Sommerfeld’s front velocity and causality in special relativity

In light of the predictions of the above superluminal phenomena in quantum many-electron systems, the question naturally arises: Do they violate relativity? In order to answer this question, we need to introduce the important concept of the “front velocity”, which is due to Sommerfeld [1].

Consider an electromagnetic carrier wave of the form

$$E(z, t) = E_0 \cos(kz - \omega t)\Theta(t - z/c) \quad (56)$$

$$B(z, t) = B_0 \cos(kz - \omega t)\Theta(t - z/c) \quad (57)$$

where the theta function is defined as follows:  $\Theta(t') = 0$  for all times  $t' < 0$  and  $\Theta(t') = 1$  for all times  $t' \geq 0$ . The instant  $t = 0$  corresponds to the sudden turn-on of a carrier wave, initiated, for example, by the pushing of the “ON” button of a continuous-wave signal generator located at  $z = 0$ . Thus one could characterize the electromagnetic wave of the form given by (56) and (57) as the “push-the-button” signal waveform, and one could think of the “front velocity” as the “push-the-button” velocity. This discontinuous kind of waveform can subsequently enter into any kind of medium, but the discontinuity of the theta function, that is, the the wave front associated with the original sudden turn-on of the carrier wave, will always travel within the medium at the vacuum speed of light.

The discontinuity represented by the theta function in (56) and (57) contains Fourier components at infinite frequency, or equivalently, at infinitely high energy. However the index of refraction of electromagnetic waves at infinite frequencies in all types of media will universally approach unity, i.e.,

$$n(\omega \rightarrow \infty) \rightarrow 1 \quad (58)$$

independent of the medium, since any medium will behave exactly like the vacuum when it is excited by electromagnetic waves at infinite frequencies. In other words, since we know that the speed of electromagnetic waves in the vacuum is exactly  $c$ , it follows that the phase velocity at infinite frequencies, which is equivalent to that of the front velocity, must universally also be exactly the vacuum speed of light  $c$ , independent of the nature of the medium.

In this way, Sommerfeld showed that it is the front velocity, and only the front velocity, that relates a cause to its effect in special relativity. The theta function in (56) and (57) is what guarantees that no effect can precede its cause. The light-cone structure of spacetime in relativity follows from the propagation

of “signals” at the front velocity, and not from the propagation of “signals” at the group velocity. Hence the “signal” velocity of physics, in the fundamental sense of a “signal” that connects a cause to its effect, is given by the front velocity, and not by the group velocity.

In the two specific examples of “instantaneous superluminality” in the case of electrons, one for normal metals and the other for superconductors, one must again ask: Do they violate relativistic causality? The answer is again no, because the front velocity in these two examples will be given by the velocity for electrons with infinitely high energies, i.e., with energies much larger than the Fermi energy of the normal metal, or of the BCS gap energy of superconductors. Such high energy electrons will pass through these metals with a front velocity equal to the vacuum speed of light. Again, relativistic causality is related only to “signals” which can be transported by these extremely high-energy electrons, and not by the infinite group velocities of the low-energy electrons depicted in the Feynman-like diagrams of Figures 5 and 8.

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- [3] [http://en.wikipedia.org/wiki/X-ray\\_telescope](http://en.wikipedia.org/wiki/X-ray_telescope)
- [4] It is commonly believed that superluminal phase velocities (or for that matter, subluminal phase velocities) for neutrinos (or for anti-neutrinos) will contribute only an unobservable overall phase factor to the wavefunction that can be ignored. See, for example, E.D. Commins and P.H. Bucksbaum, *Weak interactions of leptons and quarks* (Cambridge University Press, 1983), p. 381, and P. Langacker, J.P. Leveille, and J. Sheiman, *Phys. Rev. D* 27, 1228 (1983). However, note that observable superluminal (or subluminal) group velocities for anti-neutrinos (or for neutrinos), respectively, follow from these subluminal (or superluminal) phase velocities.
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- [6] J. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge, 1908), p. 358.

- [7] It has been objected by W.M. Saslow and G. Wilkinson, “Expulsion of free electronic charge from the interior of a metal”, *Am. J. Phys.* 39, 1244 (1971) that Ohm’s law breaks down at the extremely short Jeans time scale due to plasma oscillations in the metal. However, the *linearity* of Ohm’s law is not changed by Saslow and Wilkinson’s argument, and the “instantaneous superluminality” phenomenon will still follow, even when there exist plasma oscillations in the metal. In addition, H.C. Ohanian, “On the approach to electro- and magneto-static equilibrium”, *Am. J. Phys.* 51, 1020 (1983) has suggested that the diffusion equation replace Jeans’s equation (44). However, the time scale for Ohanian’s diffusion equation for electrons diffusing through large conducting bodies, will scale as the square of the sizes of these bodies, and would lead to absurdly long time scales for the approach to electrostatic equilibrium for these macroscopic conductors.
- [8] The Pauli exclusion principle applies in configuration space as well as in momentum space. The Fourier transform of the Fermi sea in momentum space leads to a Fermi sea in configuration space with a uniform density of electrons with a uniform negative charge density in the interior of the metal that compensates for the uniform positive charge density of the ionic lattice, except near the surface of the metal on the length scale  $1/k_F$  where  $k_F$  is the Fermi wavenumber. The incoming electron is excluded by the Pauli exclusion principle from the joining the Fermi sea in momentum space. Hence it will also be excluded from joining the Fermi sea in configuration space, except near the surface of the metal on a length scale of  $1/k_F$ .
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- [10] Note that the aspect ratio of the blind holes drilled into the aluminum bar (i.e., the ratio of the depth of the hole to its diameter) is around 2:1, which is comparable to the aspect ratio of the “ice pail” used by Faraday in his original experiments.
- [11] A single Cooper pair will be attracted to its own image charge, which will be formed inside the nearby normal metal film in the bimetallic sandwich structure of Figure 7. However, the BCS energy gap will prevent the breakup of the Cooper pair upon its attempt to cross over into the normal metal film in order to reach the vacuum side of the film, and thereby to charge up the capacitor formed by the superconducting island structure. Hence the effective capacitance of the bimetallic sandwich structure shown in Figure 7 will be much lower than that for a configuration without the bimetallic sandwich structure. The suppressed capacitance of the island structure should be on the order of  $C_{\text{island}} \simeq \varepsilon_0 \xi \simeq 10^{-19}$  f, where  $\xi \simeq 10$  nm is the coherence length of the superconductor.