

A Course in Fluid Mechanics with Vector Field Theory

by

*Dennis C. Prieve
Department of Chemical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213*

An electronic version of this book in Adobe PDF® format was made available
to students of 06-703, Department of Chemical Engineering,
Carnegie Mellon University, Fall, 2000.

Copyright © 2000 by Dennis C. Prieve

Table of Contents

ALGEBRA OF VECTORS AND TENSORS	1
VECTOR MULTIPLICATION.....	1
<i>Definition of Dyadic Product</i>	2
DECOMPOSITION INTO SCALAR COMPONENTS.....	3
SCALAR FIELDS.....	4
GRADIENT OF A SCALAR	4
<i>Geometric Meaning of the Gradient.....</i>	6
<i>Applications of Gradient</i>	7
CURVILINEAR COORDINATES.....	7
<i>Cylindrical Coordinates</i>	7
<i>Spherical Coordinates.....</i>	8
DIFFERENTIATION OF VECTORS W.R.T. SCALARS	10
VECTOR FIELDS	12
<i>Fluid Velocity as a Vector Field.....</i>	12
PARTIAL & MATERIAL DERIVATIVES	13
CALCULUS OF VECTOR FIELDS.....	15
GRADIENT OF A SCALAR (EXPLICIT).....	15
DIVERGENCE, CURL, AND GRADIENT.....	16
<i>Physical Interpretation of Divergence</i>	17
<i>Calculation of $\nabla \cdot \mathbf{v}$ in R.C.C.S.</i>	17
<i>Evaluation of $\nabla \times \mathbf{v}$ and $\nabla \mathbf{v}$ in R.C.C.S.....</i>	19
<i>Evaluation of $\nabla \cdot \mathbf{v}$, $\nabla \times \mathbf{v}$ and $\nabla \mathbf{v}$ in Curvilinear Coordinates</i>	20
<i>Physical Interpretation of Curl.....</i>	21
VECTOR FIELD THEORY	24
DIVERGENCE THEOREM	24
<i>Corollaries of the Divergence Theorem</i>	25
<i>The Continuity Equation</i>	25
<i>Reynolds Transport Theorem</i>	27
STOKES THEOREM	29
<i>Velocity Circulation: Physical Meaning</i>	29
DERIVABLE FROM A SCALAR POTENTIAL	31
THEOREM III.....	32
TRANSPOSE OF A TENSOR, IDENTITY TENSOR.....	33
DIVERGENCE OF A TENSOR	34
INTRODUCTION TO CONTINUUM MECHANICS*	36
CONTINUUM HYPOTHESIS	36
CLASSIFICATION OF FORCES.....	38
HYDROSTATIC EQUILIBRIUM.....	39
FLOW OF IDEAL FLUIDS.....	40
EULER'S EQUATION	40
KELVIN'S THEOREM	43
IRROTATIONAL FLOW OF AN INCOMPRESSIBLE FLUID	45
<i>Potential Flow Around a Sphere.....</i>	48
<i>d'Alembert's Paradox</i>	54

STREAM FUNCTION	56
Two-D Flows.....	57
Axisymmetric Flow (Cylindrical)	58
Axisymmetric Flow (Spherical).....	59
Orthogonality of $\psi=\text{const}$ and $\phi=\text{const}$	60
Streamlines, Pathlines and Streaklines	60
Physical Meaning of Streamfunction	61
Incompressible Fluids	63
VISCOUS FLUIDS	65
Tensorial Nature of Surface Forces.....	65
Generalization of Euler's Equation.....	69
Momentum Flux	71
Response of Elastic Solids to Normal (Uniaxial) Stress	73
Response of Elastic Solids to Shear Stress	75
Generalized Hooke's Law	76
Response of a Viscous Fluid to Pure Shear	78
Generalized Newton's Law of Viscosity	79
Navier-Stokes Equation	80
Boundary Conditions.....	81
EXACT SOLUTIONS OF N-S EQUATIONS.....	83
Problems with Zero Inertia	83
<i>Flow in Long Straight Conduit of Uniform Cross Section</i>	84
<i>Flow of Thin Film Down Inclined Plane</i>	87
Problems with Non-Zero Inertia.....	92
<i>Rotating Disk*</i>	92
CREEPING FLOW APPROXIMATION.....	94
Cone-and-Plate Viscometer	94
Creeping Flow Around a Sphere ($Re \rightarrow 0$).....	100
<i>Scaling</i>	100
<i>Velocity Profile</i>	102
<i>Displacement of Distant Streamlines</i>	105
<i>Pressure Profile</i>	106
Correcting for Inertial Terms	109
Perturbation Expansion	110
Flow Around Cylinder as $Re \rightarrow 0$	113
BOUNDARY-LAYER APPROXIMATION	115
Flow Around Cylinder as $Re \rightarrow \infty$	115
Mathematical Nature of Boundary Layers.....	116
Matched-Asymptotic Expansions	120
MAE's Applied to 2-D Flow Around Cylinder	125
<i>Outer Expansion</i>	126
<i>Inner Expansion</i>	127
<i>Boundary Layer Thickness</i>	130
Prandtl's B.L. Equations for 2-D Flows	132
Alternate Method: Prandtl's Scaling Theory	135
Solution for a Flat Plate.....	140
<i>Time Out: Flow Next to Suddenly Accelerated Plate</i>	142
<i>Time In: Boundary Layer on Flat Plate</i>	144
<i>Boundary-Layer Thickness</i>	145

<i>Drag on Plate</i>	148
SOLUTION FOR A SYMMETRIC CYLINDER	149
<i>Boundary-Layer Separation</i>	152
<i>Drag Coefficient and Behavior in the Wake of the Cylinder</i>	154
THE LUBRICATION APPROXIMATION	156
TRANSLATION OF A CYLINDER ALONG A PLATE	162
CAVITATION	165
SQUEEZING FLOW	166
REYNOLDS EQUATION	170
TURBULENCE	178
GENERAL NATURE OF TURBULENCE	178
TURBULENT FLOW IN PIPES	179
TIME-SMOOTHING	181
TIME-SMOOTHING OF CONTINUITY EQUATION	182
TIME-SMOOTHING OF THE NAVIER-STOKES EQUATION	182
ANALYSIS OF TURBULENT FLOW IN PIPES	184
PRANDTL'S MIXING LENGTH THEORY	186
PRANDTL'S "UNIVERSAL" VELOCITY PROFILE	189
PRANDTL'S UNIVERSAL LAW OF FRICTION	191
ELECTROHYDRODYNAMICS	194
ORIGIN OF CHARGE	194
GOUY-CHAPMAN MODEL OF DOUBLE LAYER	195
ELECTROSTATIC BODY FORCES	198
ELECTROKINETIC PHENOMENA	200
SMOLUCHOWSKI'S ANALYSIS (CA. 1918)	201
ELECTRO-OSMOSIS IN CYLINDRICAL PORES	204
ELECTROPHORESIS	205
STREAMING POTENTIAL	205
SURFACE TENSION	208
MOLECULAR ORIGIN	208
BOUNDARY CONDITIONS FOR FLUID FLOW	210
INDEX	213

Algebra of Vectors and Tensors

Whereas heat and mass are scalars, fluid mechanics concerns transport of momentum, which is a vector. Heat and mass fluxes are vectors, momentum flux is a tensor. Consequently, the mathematical description of fluid flow tends to be more abstract and subtle than for heat and mass transfer. In an effort to make the student more comfortable with the mathematics, we will start with a review of the algebra of vectors and an introduction to tensors and dyads. A brief review of vector addition and multiplication can be found in Greenberg,* pages 132-139.

Scalar - a quantity having magnitude but no direction (e.g. temperature, density)

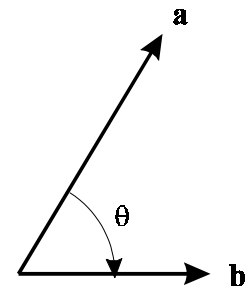
Vector - (a.k.a. 1st rank tensor) a quantity having magnitude and direction (e.g. velocity, force, momentum)

(2nd rank) **Tensor** - a quantity having magnitude and **two** directions (e.g. momentum flux, stress)

VECTOR MULTIPLICATION

Given two arbitrary vectors **a** and **b**, there are three types of vector products are defined:

	Notation	Result	Definition
Dot Product	$\mathbf{a} \cdot \mathbf{b}$	scalar	$ab \cos \theta$
Cross Product	$\mathbf{a} \times \mathbf{b}$	vector	$ab \sin \theta \mathbf{n}$



where θ is an interior angle ($0 \leq \theta \leq \pi$) and **n** is a unit vector which is normal to both **a** and **b**. The sense of **n** is determined from the "right-hand-rule"♦

Dyadic Product **ab** tensor

In the above definitions, we denote the magnitude (or length) of vector **a** by the scalar *a*. Boldface will be used to denote vectors and italics will be used to denote scalars. Second-rank tensors will be denoted with double-underlined boldface; e.g. tensor **T**.

* Greenberg, M.D., *Foundations Of Applied Mathematics*, Prentice-Hall, 1978.

♦ The "right-hand rule": with the fingers of the right hand initially pointing in the direction of the first vector, rotate the fingers to point in the direction of the second vector; the thumb then points in the direction with the correct sense. Of course, the thumb should have been normal to the plane containing both vectors during the rotation. In the figure above showing **a** and **b**, $\mathbf{a} \times \mathbf{b}$ is a vector pointing *into* the page, while $\mathbf{b} \times \mathbf{a}$ points *out of* the page.

Definition of Dyadic Product

Reference: Appendix B from Happel & Brenner.[♥] The word “dyad” comes from Greek: “dy” means two while “ad” means adjacent. Thus the name dyad refers to the way in which this product is denoted: the two vectors are written adjacent to one another with no space or other operator in between.

There is no geometrical picture that I can draw which will explain what a dyadic product is. It's best to think of the dyadic product as a purely mathematical abstraction having some very useful properties:

Dyadic Product \mathbf{ab} - that mathematical entity which satisfies the following properties
(where \mathbf{a} , \mathbf{b} , \mathbf{v} , and \mathbf{w} are any four vectors):

1. $\mathbf{ab} \cdot \mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v})$ [which has the direction of \mathbf{a} ; note that $\mathbf{ba} \cdot \mathbf{v} = \mathbf{b}(\mathbf{a} \cdot \mathbf{v})$ which has the direction of \mathbf{b} . Thus $\mathbf{ab} \neq \mathbf{ba}$ since they don't produce the same result on post-dotting with \mathbf{v} .]
2. $\mathbf{v} \cdot \mathbf{ab} = (\mathbf{v} \cdot \mathbf{a})\mathbf{b}$ [thus $\mathbf{v} \cdot \mathbf{ab} \neq \mathbf{ab} \cdot \mathbf{v}$]
3. $\mathbf{ab} \times \mathbf{v} = \mathbf{a}(\mathbf{b} \times \mathbf{v})$ which is another dyad
4. $\mathbf{v} \times \mathbf{ab} = (\mathbf{v} \times \mathbf{a})\mathbf{b}$
5. $\mathbf{ab} : \mathbf{vw} = (\mathbf{a} \cdot \mathbf{w})(\mathbf{b} \cdot \mathbf{v})$ which is sometimes known as the *inner-outer product* or the *double-dot product*.*
6. $\mathbf{a}(\mathbf{v} + \mathbf{w}) = \mathbf{av} + \mathbf{aw}$ (distributive for addition)
7. $(\mathbf{v} + \mathbf{w})\mathbf{a} = \mathbf{va} + \mathbf{wa}$
8. $(s + t)\mathbf{ab} = s\mathbf{ab} + t\mathbf{ab}$ (distributive for scalar multiplication--also distributive for dot and cross product)
9. $s\mathbf{ab} = (s\mathbf{a})\mathbf{b} = \mathbf{a}(s\mathbf{b})$

[♥] Happel, J., & H. Brenner, *Low Reynolds Number Hydrodynamics*, Noordhoff, 1973.

* Brenner defines this as $(\mathbf{a} \cdot \mathbf{v})(\mathbf{b} \cdot \mathbf{w})$. Although the two definitions are *not* equivalent, either can be used -- as long as you are consistent. In these notes, we will adopt the definition above and ignore Brenner's definition.

DECOMPOSITION INTO SCALAR COMPONENTS

Three vectors (say \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3) are said to be **linearly independent** if none can be expressed as a linear combination of the other two (e.g. \mathbf{i} , \mathbf{j} , and \mathbf{k}). Given such a set of three LI vectors, any vector (belonging to E^3) can be expressed as a linear combination of this **basis**:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

where the v_i are called the **scalar components** of \mathbf{v} . Usually, for convenience, we choose **orthonormal** vectors as the basis:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

although this is not necessary. δ_{ij} is called the **Kronecker delta**. Just as the familiar dot and cross products can be written in terms of the scalar components, so can the dyadic product:

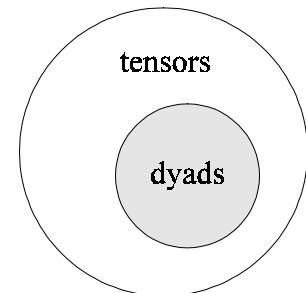
$$\begin{aligned} \mathbf{vw} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)(w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3) \\ &= (v_1\mathbf{e}_1)(w_1\mathbf{e}_1) + (v_1\mathbf{e}_1)(w_2\mathbf{e}_2) + \dots \\ &= v_1w_1\mathbf{e}_1\mathbf{e}_1 + v_1w_2\mathbf{e}_1\mathbf{e}_2 + \dots \text{ (nine terms)} \end{aligned}$$

where the $\mathbf{e}_i\mathbf{e}_j$ are nine distinct **unit dyads**. We have applied the definition of dyadic product to perform these two steps: in particular items 6, 7 and 9 in the list above.

More generally any n th rank tensor (in E^3) can be expressed as a linear combination of the 3^n **unit n -ads**. For example, if $n=2$, $3^n=9$ and an n -ad is a dyad. Thus a general **second-rank tensor** can be decomposed as a linear combination of the 9 unit dyads:

$$\underline{\underline{\mathbf{T}}} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + \dots = \sum_{i=1,3} \sum_{j=1,3} T_{ij}\mathbf{e}_i\mathbf{e}_j$$

Although a dyad (e.g. \mathbf{vw}) is an example of a second-rank tensor, not all 2nd rank tensors $\underline{\underline{\mathbf{T}}}$ can be expressed as a dyadic product of two vectors. To see why, note that a general second-rank tensor has nine scalar components which need not be related to one another in any way. By contrast, the 9 scalar components of dyadic product above involve only six distinct scalars (the 3 components of \mathbf{v} plus the 3 components of \mathbf{w}).



After a while you get tired of writing the summation signs and limits. So an abbreviation was adopted whereby repeated appearance of an index implies summation over the three allowable values of that index:

$$\underline{\underline{\mathbf{T}}} = T_{ij}\mathbf{e}_i\mathbf{e}_j$$

This is sometimes called the **Cartesian (implied) summation convention**.

SCALAR FIELDS

Suppose I have some scalar function of position (x,y,z) which is *continuously differentiable*, that is

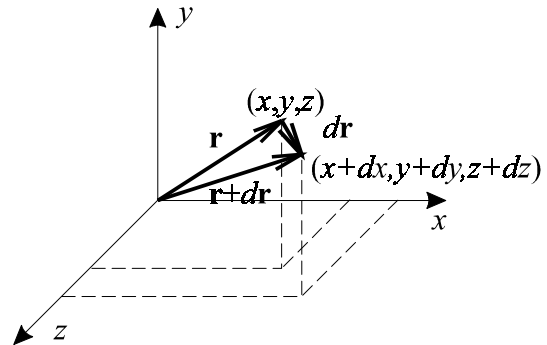
$$f = f(x,y,z)$$

and $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous throughout some 3-D region in space. This function is called a **scalar field**. Now consider f at a second point which is differentially close to the first. The difference in f between these two points is called the **total differential** of f :

$$f(x+dx, y+dy, z+dz) - f(x,y,z) \equiv df$$

For any continuous function $f(x,y,z)$, df is linearly related to the position displacements, dx , dy and dz . That linear relation is given by the Chain Rule of differentiation:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$



Instead of defining position using a particular coordinate system, we could also define position using a **position vector** \mathbf{r} :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The scalar field can be expressed as a function of a vector argument, representing position, instead of a set of three scalars:

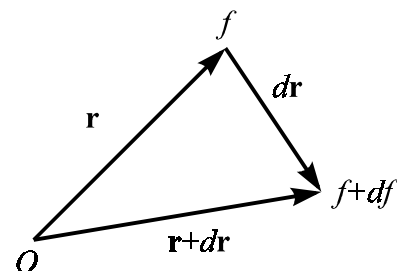
$$f = f(\mathbf{r})$$

Consider an arbitrary displacement away from the point \mathbf{r} , which we denote as $d\mathbf{r}$ to emphasize that the magnitude $|d\mathbf{r}|$ of this displacement is sufficiently small that $f(\mathbf{r})$ can be linearized as a function of position around \mathbf{r} . Then the total differential can be written as

$$df = f(\mathbf{r} + d\mathbf{r}) - f(\mathbf{r})$$

GRADIENT OF A SCALAR

We are now in a position to define an important vector associated with this scalar field. The **gradient** (denoted as ∇f) is defined such that the dot product of it and a differential displacement vector gives the total differential:



$$df \equiv d\mathbf{r} \cdot \nabla f$$

EXAMPLE: Obtain an explicit formula for calculating the gradient in Cartesian* coordinates.

Solution:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{r} + d\mathbf{r} = (x+dx)\mathbf{i} + (y+dy)\mathbf{j} + (z+dz)\mathbf{k}$$

subtracting:

$$d\mathbf{r} = (dx)\mathbf{i} + (dy)\mathbf{j} + (dz)\mathbf{k}$$

$$\nabla f = (\nabla f)_x \mathbf{i} + (\nabla f)_y \mathbf{j} + (\nabla f)_z \mathbf{k}$$

$$d\mathbf{r} \cdot \nabla f = [(dx)\mathbf{i} + \dots] \cdot [(\nabla f)_x \mathbf{i} + \dots]$$

$$df = (\nabla f)_x dx + (\nabla f)_y dy + (\nabla f)_z dz \quad (1)$$

Using the Chain rule: $df = (\partial f / \partial x) dx + (\partial f / \partial y) dy + (\partial f / \partial z) dz \quad (2)$

According to the definition of the gradient, (1) and (2) are identical. Equating them and collecting terms:

$$[(\nabla f)_x - (\partial f / \partial x)] dx + [(\nabla f)_y - (\partial f / \partial y)] dy + [(\nabla f)_z - (\partial f / \partial z)] dz = 0$$

Think of dx , dy , and dz as three independent variables which can assume an infinite number of values, even though they must remain small. The equality above must hold for all values of dx , dy , and dz . The only way this can be true is if each individual term separately vanishes:**

So $(\nabla f)_x = \partial f / \partial x$, $(\nabla f)_y = \partial f / \partial y$, and $(\nabla f)_z = \partial f / \partial z$,

leaving
$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

Other ways to denote the gradient include:

$$\nabla f = \text{grad} f = \partial f / \partial \mathbf{r}$$

*Named after French philosopher and mathematician René Descartes (1596-1650), pronounced "day-**cart**", who first suggested plotting $f(x)$ on rectangular coordinates

** For any particular choice of dx , dy , and dz , we might obtain zero by cancellation of positive and negative terms. However a small change in one of the three without changing the other two would cause the sum to be nonzero. To ensure a zero-sum for *all* choices, we must make each term vanish independently.

Geometric Meaning of the Gradient

- 1) direction: $\nabla f(\mathbf{r})$ is normal to the $f=\text{const}$ surface passing through the point \mathbf{r} in the direction of increasing f . ∇f also points in the direction of steepest ascent of f .
- 2) magnitude: $|\nabla f|$ is the rate of change of f with distance along this direction

What do we mean by an " $f=\text{const}$ surface"? Consider an example.

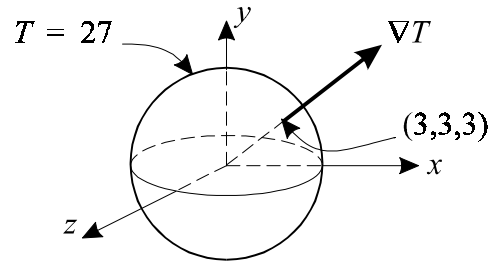
Example: Suppose the steady state temperature profile in some heat conduction problem is given by:

$$T(x,y,z) = x^2 + y^2 + z^2$$

Perhaps we are interested in ∇T at the point $(3,3,3)$ where $T=27$. ∇T is normal to the $T=\text{const}$ surface:

$$x^2 + y^2 + z^2 = 27$$

which is a sphere of radius $\sqrt{27}$. ♣



Proof of 1). Let's use the definition to show that these geometric meanings are correct.

$$df = d\mathbf{r} \cdot \nabla f$$

Consider an arbitrary f . A portion of the $f=\text{const}$ surface containing the point \mathbf{r} is shown in the figure at right. Choose a $d\mathbf{r}$ which lies entirely on $f=\text{const}$. In other words, the surface contains both \mathbf{r} and $\mathbf{r}+d\mathbf{r}$, so

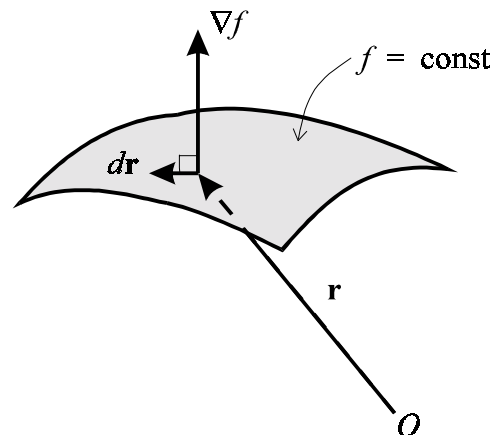
$$f(\mathbf{r}) = f(\mathbf{r}+d\mathbf{r})$$

and
$$df = f(\mathbf{r}+d\mathbf{r}) - f(\mathbf{r}) = 0$$

Substituting this into the definition of gradient:

$$df = 0 = d\mathbf{r} \cdot \nabla f = |d\mathbf{r}| |\nabla f| \cos\theta$$

Since $|d\mathbf{r}|$ and $|\nabla f|$ are in general not zero, we are



♣ A vertical bar in the left margin denotes material which (in the interest of time) will be omitted from the lecture.

forced to the conclusion that $\cos\theta=0$ or $\theta=90^\circ$. This means that ∇f is normal to $d\mathbf{r}$ which lies in the surface.

2) can be proved in a similar manner: choose $d\mathbf{r}$ to be parallel to ∇f . Does ∇f point toward higher or lower values of f ?

Applications of Gradient

- find a vector pointing in the direction of steepest ascent of some scalar field
- determine a normal to some surface (needed to apply b.c.'s like $\mathbf{n} \cdot \mathbf{v} = 0$ for a boundary which is impermeable)
- determine the rate of change along some arbitrary direction: if \mathbf{n} is a unit vector pointing along some path, then

$$\mathbf{n} \cdot \nabla f = \frac{\partial f}{\partial s}$$

is the rate of change of f with distance (s) along this path given by \mathbf{n} . $\partial f / \partial s$ is called the ***directed derivative*** of f .

CURVILINEAR COORDINATES

In principle, all problems in fluid mechanics and transport could be solved using Cartesian coordinates. Often, however, we can take advantage of symmetry in a problem by using another coordinate system. This advantage takes the form of a reduction in the number of independent variables (e.g. PDE becomes ODE). A familiar example of a non-Cartesian coordinate system is:

Cylindrical Coordinates

$$r = (x^2 + y^2)^{1/2} \quad x = r \cos \theta$$

$$\theta = \tan^{-1}(y/x) \quad y = r \sin \theta$$

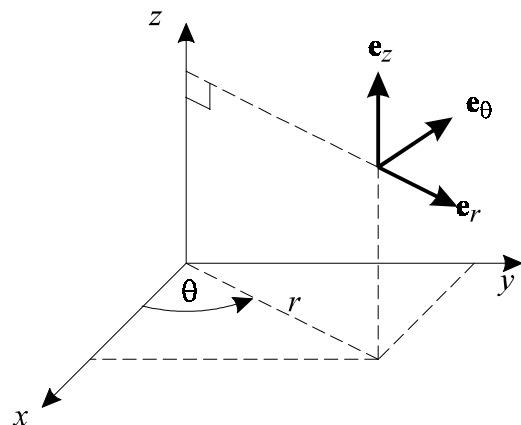
$$z = z \quad z = z$$

Vectors are decomposed differently. Instead of

$$\text{in R.C.C.S.:} \quad \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

in cylindrical coordinates, we write

$$\text{in cyl. coords.:} \quad \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z$$



where \mathbf{e}_r , \mathbf{e}_θ , and \mathbf{e}_z are new unit vectors pointing the r , θ and z directions. We also have a different set of nine unit dyads for decomposing tensors:

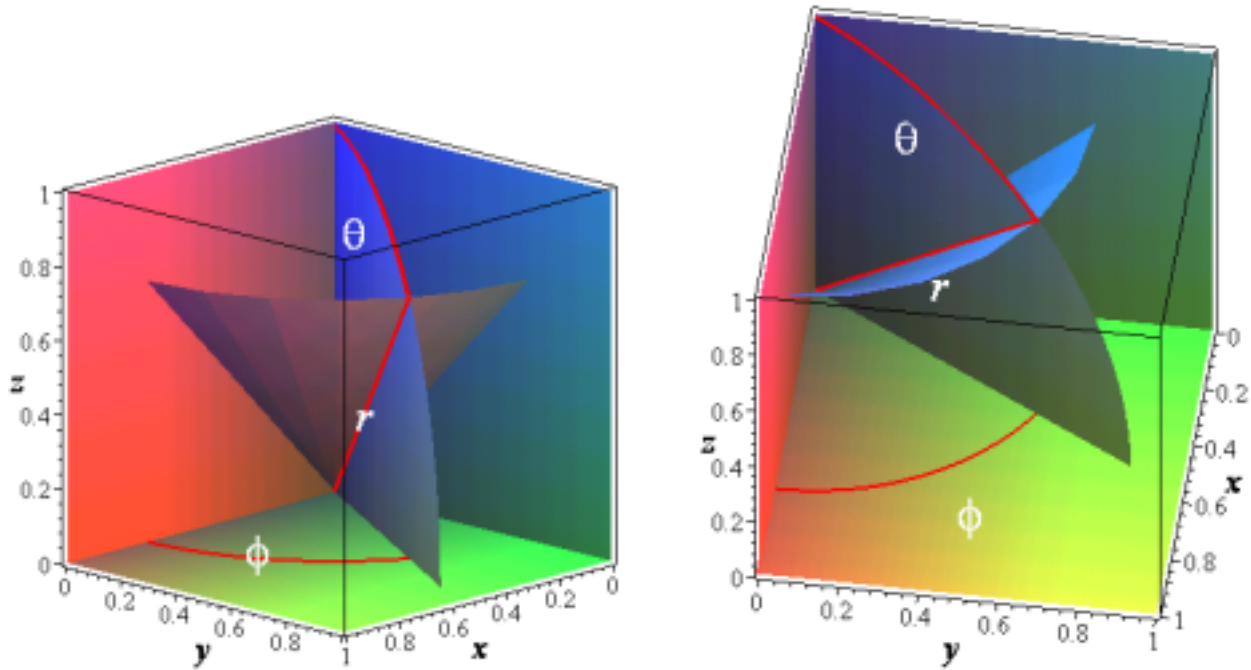
$$\mathbf{e}_r\mathbf{e}_r, \mathbf{e}_r\mathbf{e}_\theta, \mathbf{e}_r\mathbf{e}_z, \mathbf{e}_\theta\mathbf{e}_r, \text{ etc.}$$

Like the Cartesian unit vectors, the unit vectors in cylindrical coordinates form an orthonormal set of basis vectors for E^3 . Unlike Cartesian unit vectors, the orientation of \mathbf{e}_r and \mathbf{e}_θ depend on position. In other words:

$$\mathbf{e}_r = \mathbf{e}_r(\theta)$$

$$\mathbf{e}_\theta = \mathbf{e}_\theta(\theta)$$

Spherical Coordinates



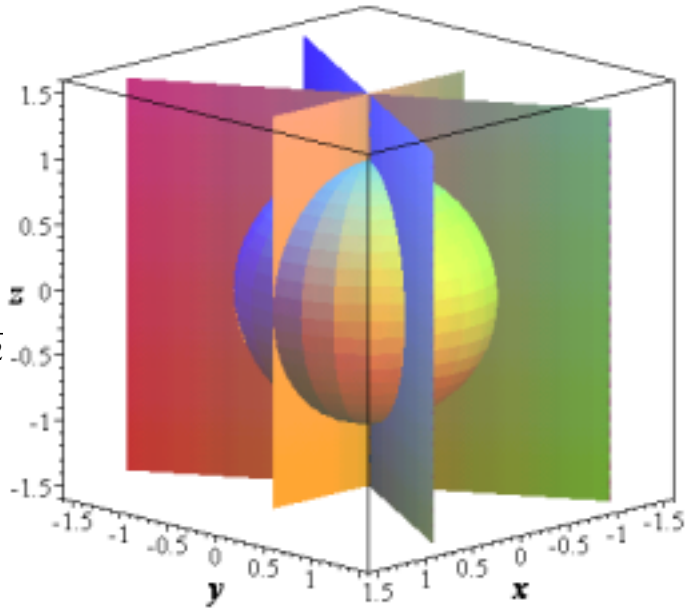
Spherical coordinates (r, θ, ϕ) are defined relative to Cartesian coordinates as suggested in the figures above (two views of the same thing). The green surface is the xy -plane, the red surface is the xz -plane, while the blue surface (at least in the left image) is the yz -plane. These three planes intersect at the origin $(0,0,0)$, which lies deeper into the page than $(1,1,0)$. The straight red line, drawn from the origin to the point $(r, \theta, \phi)^*$ has length r . The angle θ is the angle the red line makes with the z -axis (the red circular arc labelled θ has radius r and is subtended by the angle θ). The angle ϕ (measured in the xy -plane) is the angle the second blue plane (actually it's one quadrant of a disk) makes with the xy -plane (red). This plane which is a quadrant of a disk is a

* This particular figure was drawn using $r = 1$, $\theta = \pi/4$ and $\phi = \pi/3$.

$\phi = \text{const}$ surface: all points on this plane have the same ϕ coordinate. The second red (circular) arc labelled ϕ is also subtended by the angle ϕ .

A number of other $\phi = \text{const}$ planes are shown in the figure at right, along with a sphere of radius $r=1$. All these planes intersect along the z -axis, which also passes through the center of the sphere.

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= +\sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right) \\ z &= r \cos \theta & \phi &= \tan^{-1} (y/x) \end{aligned}$$



The position vector in spherical coordinates is given by

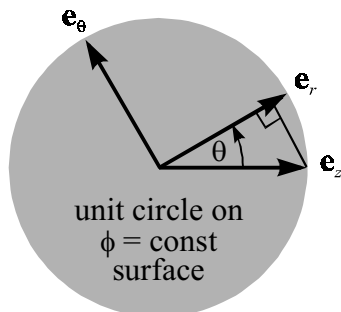
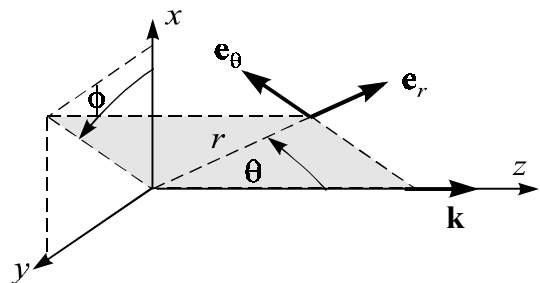
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = r \mathbf{e}_r(\theta, \phi)$$

In this case all three unit vectors depend on position:

$$\mathbf{e}_r = \mathbf{e}_r(\theta, \phi), \mathbf{e}_\theta = \mathbf{e}_\theta(\theta, \phi), \text{ and } \mathbf{e}_\phi = \mathbf{e}_\phi(\phi)$$

where \mathbf{e}_r is the unit vector pointing the direction of increasing r , holding θ and ϕ fixed; \mathbf{e}_θ is the unit vector pointing the direction of increasing θ , holding r and ϕ fixed; and \mathbf{e}_ϕ is the unit vector pointing the direction of increasing ϕ , holding r and θ fixed.

These unit vectors are shown in the figure at right. Notice that the surface $\phi = \text{const}$ is a plane containing the point \mathbf{r} itself, the projection of the point onto the xy -plane and the origin. The unit vectors \mathbf{e}_r and \mathbf{e}_θ lie in this plane as well as the Cartesian unit vector \mathbf{k} (sometimes denoted \mathbf{e}_z).



If we tilt this $\phi = \text{const}$ plane into the plane of the page (as in the sketch at left), we can more easily see the relationship between these three unit vectors:

$$\mathbf{e}_z = (\cos \theta) \mathbf{e}_r - (\sin \theta) \mathbf{e}_\theta$$

This is obtained by determined from the geometry of the right triangle in the figure at left. When any of the unit vectors is position

dependent, we say the coordinates are:

curvilinear - at least one of the basis vectors is position dependent

This will have some profound consequences which we will get to shortly. But first, we need to take “time-out” to define:

DIFFERENTIATION OF VECTORS W.R.T. SCALARS

Suppose we have a vector \mathbf{v} which depends on the scalar parameter t :

$$\mathbf{v} = \mathbf{v}(t)$$

For example, the velocity of a satellite depends on time. What do we mean by the “derivative” of a vector with respect to a scalar. As in the Fundamental Theorem of Calculus, we define the derivative as:

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \rightarrow 0} \left\{ \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \right\}$$

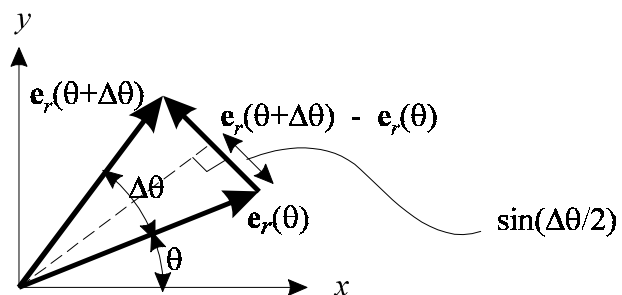
Note that $d\mathbf{v}/dt$ is also a vector.

EXAMPLE: Compute $d\mathbf{e}_r/d\theta$ in cylindrical coordinates.

Solution: From the definition of the derivative:

$$\frac{d\mathbf{e}_r}{d\theta} = \lim_{\Delta\theta \rightarrow 0} \left\{ \frac{\mathbf{e}_r(\theta + \Delta\theta) - \mathbf{e}_r(\theta)}{\Delta\theta} \right\} = \lim_{\Delta\theta \rightarrow 0} \left\{ \frac{\Delta\mathbf{e}_r}{\Delta\theta} \right\}$$

Since the location of the tail of a vector is not part of the definition of a vector, let's move both vectors to the origin (keeping the orientation fixed). Using the parallelogram law, we obtain the difference vector. Its magnitude is:



$$|\mathbf{e}_r(\theta + \Delta\theta) - \mathbf{e}_r(\theta)| = 2 \sin \frac{\Delta\theta}{2}$$

Its direction is parallel to $\mathbf{e}_\theta(\theta + \Delta\theta/2)$, so:

$$\mathbf{e}_r(\theta + \Delta\theta) - \mathbf{e}_r(\theta) = 2 \sin \frac{\Delta\theta}{2} \mathbf{e}_\theta\left(\theta + \frac{\Delta\theta}{2}\right)$$

Recalling that $\sin x$ tends to x as $x \rightarrow 0$, we have

$$\lim_{\Delta\theta \rightarrow 0} \{\mathbf{e}_r(\theta + \Delta\theta) - \mathbf{e}_r(\theta)\} = \Delta\theta \mathbf{e}_\theta(\theta)$$

Dividing this by $\Delta\theta$, we obtain the derivative:

$$d\mathbf{e}_r/d\theta = \mathbf{e}_\theta$$

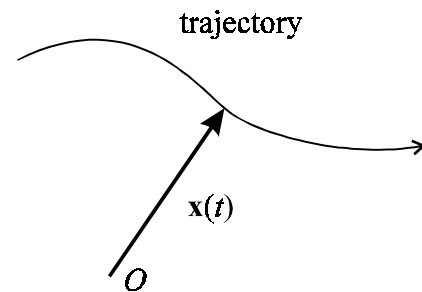
Similarly,

$$d\mathbf{e}_\theta/d\theta = -\mathbf{e}_r$$

One important application of “differentiation with respect to a scalar” is the calculation of velocity, given position as a function of time. In general, if the position vector is known, then the velocity can be calculated as the rate of change in position:

$$\mathbf{r} = \mathbf{r}(t)$$

$$\mathbf{v} = d\mathbf{r}/dt$$



Similarly, the acceleration vector \mathbf{a} can be calculated as the derivative of the velocity vector \mathbf{v} :

$$\mathbf{a} = d\mathbf{v}/dt$$

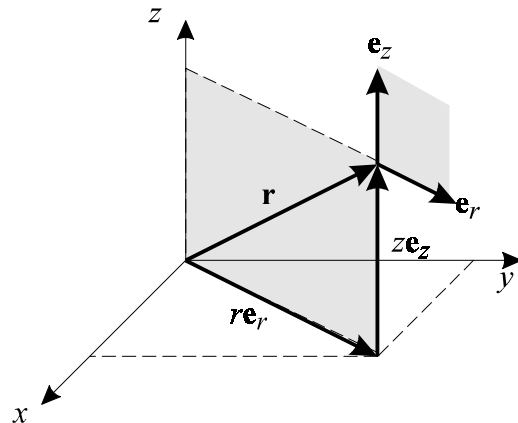
EXAMPLE: Given the trajectory of an object in cylindrical coordinates

$$r = r(t), \theta = \theta(t), \text{ and } z = z(t)$$

Find the velocity of the object.

Solution: First, we need to express \mathbf{r} in terms of the unit vectors in cylindrical coordinates. Using the figure at right, we note by inspection that*

$$\mathbf{r}(r, \theta, z) = r\mathbf{e}_r(\theta) + z\mathbf{e}_z$$



Now we can apply the Chain Rule:

*Recalling that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Cartesian coordinates, you might be tempted to write $\mathbf{r} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$ in cylindrical coordinates. Of course, this temptation gives the wrong result (in particular, the units of length in the second term are missing).

$$d\mathbf{r} = \underbrace{\left(\frac{\partial \mathbf{r}}{\partial r}\right)_{\theta,z}}_{\mathbf{e}_r} dr + \underbrace{\left(\frac{\partial \mathbf{r}}{\partial \theta}\right)_{r,z}}_{r \underbrace{\frac{d\mathbf{e}_r(\theta)}{d\theta}}_{\mathbf{e}_\theta}} d\theta + \underbrace{\left(\frac{\partial \mathbf{r}}{\partial z}\right)_{r,\theta}}_{\mathbf{e}_z} dz = \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta + \mathbf{e}_z dz$$

Dividing by dt , we obtain the velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \underbrace{\frac{dr(t)}{dt}}_{v_r} \mathbf{e}_r + r \underbrace{\frac{d\theta(t)}{dt}}_{v_\theta} \mathbf{e}_\theta + \underbrace{\frac{dz(t)}{dt}}_{v_z} \mathbf{e}_z$$

VECTOR FIELDS

A vector field is defined just like a scalar field, except that it's a vector. Namely, a vector field is a position-dependent vector:

$$\mathbf{v} = \mathbf{v}(\mathbf{r})$$

Common examples of vector fields include force fields, like the gravitational force or an electrostatic force field. Of course, in this course, the vector field of greatest interest is:

Fluid Velocity as a Vector Field

Consider steady flow around a submerged object. What do we mean by “fluid velocity?” There are two ways to measure fluid velocity. First, we could add tracer particles to the flow and measure the position of the tracer particles as a function of time; differentiating position with respect to time, we would obtain the velocity.♦ A mathematical “tracer particle” is called a “material point.”

Material point - fluid element - a given set of fluid molecules whose location may change with time.♣

♦ Actually, this only works for steady flows. In unsteady flows, pathlines, streaklines and streamlines differ (see “Streamlines, Pathlines and Streaklines” on page 65).

♣ In a molecular-level description of gases or liquids, even nearby molecules have widely different velocities which fluctuate with time as the molecules undergo collisions. We will reconcile the molecular-level description with the more common continuum description in Chapter 4. For now, we just state that by “location of a material point” we mean the location of the center of mass of the molecules. The “point” needs to contain a statistically large number of molecules so that $\mathbf{r}(t)$ converges to a smooth continuous function.

Suppose the trajectory of a material point is given by:

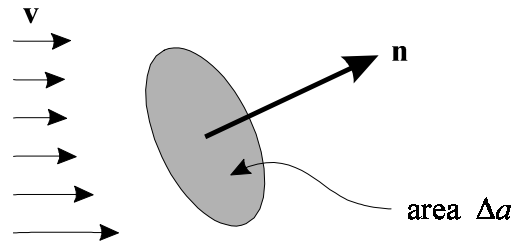
$$\mathbf{r} = \mathbf{r}(t)$$

Then the fluid velocity at any time is $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ (3)

A second way to measure fluid velocity is similar to the “bucket-and-stopwatch method.” We measure the volume of fluid crossing a surface per unit time:

$$\mathbf{n} \cdot \mathbf{v} = \lim_{\Delta a \rightarrow 0} \left\{ \frac{\Delta q}{\Delta a} \right\}$$

where Δa is the area of a surface element having a unit normal \mathbf{n} and Δq is the volumetric flowrate of fluid crossing Δa in the direction of \mathbf{n} .

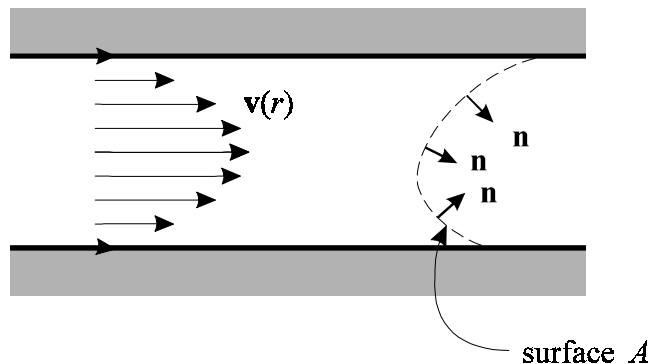


When Δa is small enough so that this quotient has converged in a mathematical sense and Δa is small enough so that the surface is locally planar so we can denote its orientation by a unit normal \mathbf{n} , we can replace Δa by da and Δq by dq and rewrite this definition as:

$$dq = \mathbf{n} \cdot \mathbf{v} da \quad (4)$$

This is particularly convenient to compute the volumetric flowrate across an arbitrary curved surface, given the velocity profile. We just have to sum up the contribution from each surface element:

$$Q = \int_A \mathbf{n} \cdot \mathbf{v} da$$



PARTIAL & MATERIAL DERIVATIVES

Let $f = f(\mathbf{r}, t)$

represent some unsteady scalar field (e.g. the unsteady temperature profile inside a moving fluid). There are two types of time derivatives of unsteady scalar fields which we will find convenient to define. In the example in which f represents temperature, these two time derivatives correspond to the rate of change (denoted generically as df/dt) measured with a thermometer which either is held stationary in the moving fluid or drifts along with the local fluid.

partial derivative - rate of change at a fixed *spatial* point:

$$\frac{\partial f}{\partial t} = \left(\frac{df}{dt} \right)_{d\mathbf{r}=\mathbf{0}}$$

where the subscript $d\mathbf{r}=\mathbf{0}$ denotes that we are evaluating the derivative along a path* on which the spatial point \mathbf{r} is held fixed. In other words, there is no displacement in position during the time interval dt . As time proceeds, different material points occupy the spatial point \mathbf{r} .

material derivative (a.k.a. substantial derivative) - rate of change within a particular *material* point (whose spatial coordinates vary with time):

$$\frac{Df}{Dt} = \left(\frac{df}{dt} \right)_{d\mathbf{r}=\mathbf{v}dt}$$

where the subscript $d\mathbf{r} = \mathbf{v} dt$ denotes that a displacement in position (corresponding to the motion of the velocity) occurs: here \mathbf{v} denotes the local fluid velocity. As time proceeds, the moving material occupies different spatial points, so \mathbf{r} is not fixed. In other words, we are following along with the fluid as we measure the rate of change of f .

A relation between these two derivatives can be derived using a generalized vectorial form of the Chain Rule. First recall that for steady (independent of t) scalar fields, the Chain Rule gives the total differential (in invariant form) as

$$df \equiv f(\mathbf{r} + d\mathbf{r}) - f(\mathbf{r}) = d\mathbf{r} \cdot \nabla f$$

When t is a variable, we just add another contribution to the total differential which arises from changes in t , namely dt . The Chain Rule becomes

$$df = f(\mathbf{r} + d\mathbf{r}, t + dt) - f(\mathbf{r}, t) = \frac{\partial f}{\partial t} dt + d\mathbf{r} \cdot \nabla f$$

The first term has the usual Chain-Rule form for changes due to a scalar variable; the second term gives changes due to a displacement in vectorial position \mathbf{r} . Dividing by dt holding \mathbf{R} fixed yields the material derivative:

* By “path” I mean a constraint among the independent variables, which in this case are time and position (e.g. x, y, z and t). For example, I might vary one of the independent variables (e.g. x) while holding the others fixed. Alternatively, I might vary one of the independent variables (e.g. t) while prescribing some related changes in the others (e.g. $x(t)$, $y(t)$ and $z(t)$). In the latter case, I am prescribing (in parametric form) a trajectory through space, hence the name “path.”

$$\frac{Df}{Dt} \equiv \left(\frac{df}{dt} \right)_{d\mathbf{r}=\mathbf{v}dt} = \underbrace{\frac{\partial f}{\partial t} \left(\frac{dt}{dt} \right)_{d\mathbf{r}=\mathbf{v}dt}}_1 + \underbrace{\left(\frac{d\mathbf{r}}{dt} \right)_{d\mathbf{r}=\mathbf{v}dt}}_{\mathbf{v}} \cdot \nabla f$$

But $(d\mathbf{r}/dt)$ is just \mathbf{v} , leaving:
$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$$

This relationship holds for a tensor of any rank. For example, the material derivative of the velocity vector is the acceleration \mathbf{a} of the fluid, and it can be calculated from the velocity profile according to

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

We will define $\nabla \mathbf{v}$ in the next section.

Calculus of Vector Fields

Just like there were three kinds of vector multiplication which can be defined, there are three kinds of differentiation with respect to position.

Shortly, we will provide explicit definitions of these quantities in terms of surface integrals. Let me introduce this type of definition using a more familiar quantity:

	Notation	Result
Divergence	$\nabla \cdot \mathbf{v}$	scalar
Curl	$\nabla \times \mathbf{v}$	vector
Gradient	$\nabla \mathbf{v}$	tensor

GRADIENT OF A SCALAR (EXPLICIT)

Recall the previous definition for gradient:

$$f = f(\mathbf{r}): \quad df = d\mathbf{r} \cdot \nabla f \quad (\text{implicit def'n of } \nabla f)$$

Such an implicit definition is like defining $f'(x)$ as that function associated with $f(x)$ which yields:

$$f = f(x): \quad df = (dx) f' \quad (\text{implicit def'n of } f')$$

An equivalent, but explicit, definition of derivative is provided by the Fundamental Theorem of the Calculus:

$$f'(x) \equiv \lim_{\Delta x \rightarrow 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\} = \frac{df}{dx} \quad (\text{explicit def'n of } f')$$

We can provide an analogous definition of ∇f

$$\nabla f \equiv \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_A \mathbf{n} f da \right\} \quad (\text{explicit def'n of } \nabla f)$$

where f = any scalar field

A = a set of points which constitutes any closed surface enclosing the point \mathbf{r} at which ∇f is to be evaluated

V = volume of region enclosed by A

da = area of a differential element (subset) of A

\mathbf{n} = unit normal to da , pointing out of region enclosed by A

$\lim (V \rightarrow 0)$ = limit as all dimensions of A shrink to zero (in other words, A collapses about the point at which ∇f is to be defined.)

What is meant by this surface integral? Imagine A to be the skin of a potato. To compute the integral:

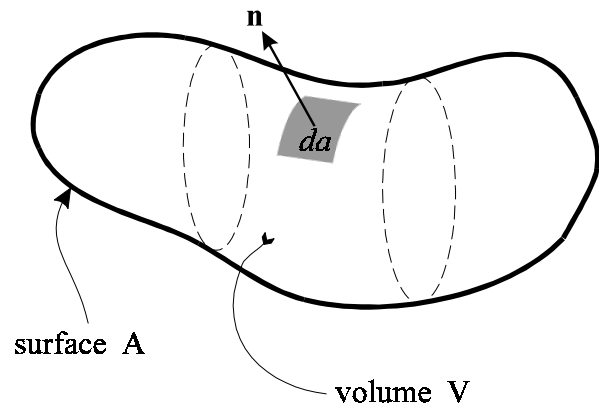
- 1) Carve the skin into a number of elements. Each element must be sufficiently small so that
 - element can be considered planar (i.e. \mathbf{n} is practically constant over the element)
 - f is practically constant over the element
- 2) For each element of skin, compute $\mathbf{n} f da$
- 3) Sum yields integral

This same type of definition can be used for each of the three spatial derivatives of a vector field:

DIVERGENCE, CURL, AND GRADIENT

Divergence

$$\nabla \cdot \mathbf{v} \equiv \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_A \mathbf{n} \cdot \mathbf{v} da \right\}$$



Curl

$$\nabla \times \mathbf{v} \equiv \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_A \mathbf{n} \times \mathbf{v} da \right\}$$

Gradient

$$\nabla \mathbf{v} \equiv \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_A \mathbf{n} \mathbf{v} da \right\}$$

Physical Interpretation of Divergence

Let the vector field $\mathbf{v} = \mathbf{v}(\mathbf{r})$ represent the steady-state velocity profile in some 3-D region of space. What is the physical meaning of $\nabla \cdot \mathbf{v}$?

- $\mathbf{n} \cdot \mathbf{v} da = dq =$ volumetric flowrate out through da (cm^3/s). This quantity is positive for outflow and negative for inflow.
- $\int_A \mathbf{n} \cdot \mathbf{v} da =$ net volumetric flowrate out of enclosed volume (cm^3/s). This is also positive for a net outflow and negative for a net inflow.
- $(1/V) \int_A \mathbf{n} \cdot \mathbf{v} da =$ flowrate out per unit volume (s^{-1})
- $\nabla \cdot \mathbf{v} = \begin{cases} > 0 \text{ for an expanding gas (perhaps } T \uparrow \text{ or } p \downarrow) \\ = 0 \text{ for an incompressible fluid (no room for accumulation)} \\ < 0 \text{ for a gas being compressed} \end{cases}$
- $\nabla \cdot \mathbf{v} =$ volumetric rate of expansion of a differential element of fluid per unit volume of that element (s^{-1})

Calculation of $\nabla \cdot \mathbf{v}$ in R.C.C.S.

Given:

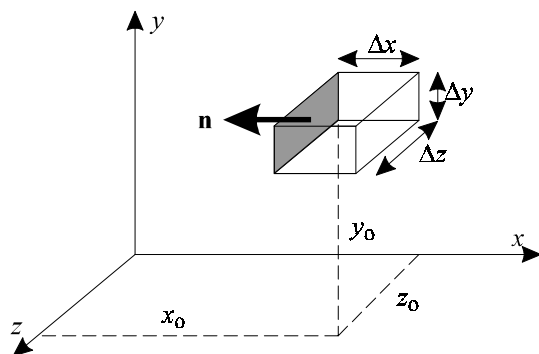
$$\mathbf{v} = v_x(x,y,z)\mathbf{i} + v_y(x,y,z)\mathbf{j} + v_z(x,y,z)\mathbf{k}$$

Evaluate $\nabla \cdot \mathbf{v}$ at (x_o, y_o, z_o) .

Solution: Choose A to be surface of rectangular parallelopiped of dimensions $\Delta x, \Delta y, \Delta z$ with one corner at x_o, y_o, z_o .

So we partition A into the six faces of the parallelopiped. The integral will be computed separately over each face:

$$\int_A \mathbf{n} \cdot \mathbf{v} da = \int_{A_1} \mathbf{n} \cdot \mathbf{v} da + \int_{A_2} \mathbf{n} \cdot \mathbf{v} da + \cdots + \int_{A_6} \mathbf{n} \cdot \mathbf{v} da$$



Surface A_1 is the $x=x_o$ face:

$$\mathbf{n} = -\mathbf{i}$$

$$\mathbf{n} \cdot \mathbf{v} = -\mathbf{i} \cdot \mathbf{v} = -v_x(x_o, y, z)$$

$$\int_{A_1} \mathbf{n} \cdot \mathbf{v} da = \int_{z_o}^{z_o + \Delta z} \int_{y_o}^{y_o + \Delta y} -v_x(x_o, y, z) dy dz$$

Using the Mean Value Theorem: $= -v_x(x_o, y', z') \Delta y \Delta z$

where

$$y_o \leq y' \leq y_o + \Delta y$$

and

$$z_o \leq z' \leq z_o + \Delta z$$

Surface A_2 is the $x=x_o + \Delta x$ face:

$$\mathbf{n} = +\mathbf{i}$$

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{i} \cdot \mathbf{v} = v_x(x_o + \Delta x, y, z)$$

$$\int_{A_2} \mathbf{n} \cdot \mathbf{v} da = \int_{z_o}^{z_o + \Delta z} \int_{y_o}^{y_o + \Delta y} v_x(x_o + \Delta x, y, z) dy dz$$

Using the Mean Value Theorem: $= v_x(x_o + \Delta x, y'', z'') \Delta y \Delta z$

where

$$y_o \leq y'' \leq y_o + \Delta y$$

and

$$z_o \leq z'' \leq z_o + \Delta z$$

The sum of these two integrals is:

$$\int_{A_1} + \int_{A_2} = [v_x(x_o + \Delta x, y'', z'') - v_x(x_o, y', z')] \Delta y \Delta z$$

Dividing by $V = \Delta x \Delta y \Delta z$:

$$\frac{1}{V} \int_{A_1 + A_2} \mathbf{n} \cdot \mathbf{v} da = \frac{v_x(x_o + \Delta x, y'', z'') - v_x(x_o, y', z')}{\Delta x}$$

Letting Δy and Δz tend to zero:

$$\lim_{\Delta y, \Delta z \rightarrow 0} \left\{ \frac{1}{V} \int_{A_1 + A_2} \mathbf{n} \cdot \mathbf{v} da \right\} = \frac{v_x(x_o + \Delta x, y_o, z_o) - v_x(x_o, y_o, z_o)}{\Delta x}$$

Finally, we take the limit as Δx tends to zero:

$$\lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_{A_1 + A_2} \mathbf{n} \cdot \mathbf{v} da \right\} = \frac{\partial v_x}{\partial x} \Big|_{x_o, y_o, z_o}$$

Similarly, from the two $y=\text{const}$ surfaces, we obtain:

$$\lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_{A_3 + A_4} \mathbf{n} \cdot \mathbf{v} da \right\} = \frac{\partial v_y}{\partial y} \Big|_{x_o, y_o, z_o}$$

and from the two $z=\text{const}$ surfaces:

$$\lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_{A_5 + A_6} \mathbf{n} \cdot \mathbf{v} da \right\} = \frac{\partial v_z}{\partial z} \Big|_{x_o, y_o, z_o}$$

Summing these three contributions yields the divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Evaluation of $\nabla \times \mathbf{v}$ and $\nabla \mathbf{v}$ in R.C.C.S.

In the same way, we could use the definition to determine expressions for the curl and the gradient.

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}$$

The formula for curl in R.C.C.S. turns out to be expressible as a determinant of a matrix:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}$$

But remember that the determinant is just a mnemonic device, **not the definition** of curl. The gradient of the vector \mathbf{v} is

$$\nabla \mathbf{v} = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j$$

where $x_1 = x$, $x_2 = y$, and $x_3 = z$, $v_1 = v_x$, etc.

Evaluation of $\nabla \cdot \mathbf{v}$, $\nabla \times \mathbf{v}$ and $\nabla \mathbf{v}$ in Curvilinear Coordinates

Ref: Greenberg, p175

These surface-integral definitions can be applied to any coordinate system. On HWK #2, we obtain $\nabla \cdot \mathbf{v}$ in cylindrical coordinates.

More generally, we can express divergence, curl and gradient in terms of the **metric coefficients** for the coordinate systems. If u, v, w are the three scalar coordinate variables for the curvilinear coordinate system, and

$$x = x(u, v, w)$$

$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

can be determined, then the three metric coefficients — h_1 , h_2 and h_3 — are given by

$$h_1(u, v, w) = \sqrt{x_u^2 + y_u^2 + z_u^2}$$

$$h_2(u, v, w) = \sqrt{x_v^2 + y_v^2 + z_v^2}$$

$$h_3(u, v, w) = \sqrt{x_w^2 + y_w^2 + z_w^2}$$

where letter subscripts denote partial differentials while numerical subscripts denote component, and the general expressions for evaluating divergence, curl and gradient are given by

gradient of scalar:
$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial v} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial w} \mathbf{e}_3$$

divergence of vector:

$$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (h_2 h_3 v_1) + \frac{\partial}{\partial v} (h_1 h_3 v_2) + \frac{\partial}{\partial w} (h_1 h_2 v_3) \right]$$

curl of vector:

$$\nabla \times \mathbf{v} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$$

These formulas have been evaluated for a number of common coordinate systems, including R.C.C.S., cylindrical and spherical coordinates. The results are tabulated in Appendix A of BSL (see pages 738-741). These pages are also available online:

[rectangular coords.](#)

[cylindrical coords:](#)

[spherical coords:](#)

Physical Interpretation of Curl

To obtain a physical interpretation of $\nabla \times \mathbf{v}$, let's consider a particularly simple flow field which is called ***solid-body rotation***. Solid-body rotation is simply the velocity field a solid would experience if it was rotating about some axis. This is also the velocity field eventually found in viscous fluids undergoing steady rotation.

Imagine that we take a container of fluid (like a can of soda pop) and we rotate the can about its axis. After a transient period whose duration depends on the dimensions of the container, the steady-state velocity profile becomes solid-body rotation.

A material point imbedded in a solid would move in a circular orbit at a constant angular speed equal to Ω radians per second. The corresponding velocity is most easily described using cylindrical coordinates with the z -axis oriented perpendicular to the plane of the orbit and passing through the center of the orbit. Then the orbit lies in a $z = \text{const}$ plane. The radius of the orbit is the radial coordinate r which is also constant. Only the θ -coordinate changes with time and it increases linearly so that $d\theta/dt = \text{const} = \Omega$.

In parametric form in cylindrical coordinates, the trajectory of a material point is given by

$$r(t) = \text{const}, z(t) = \text{const}, \theta(t) = \Omega t$$

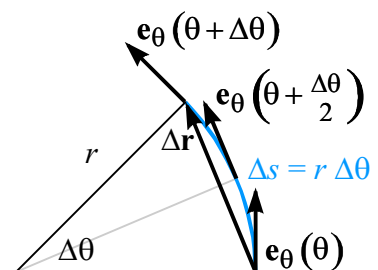
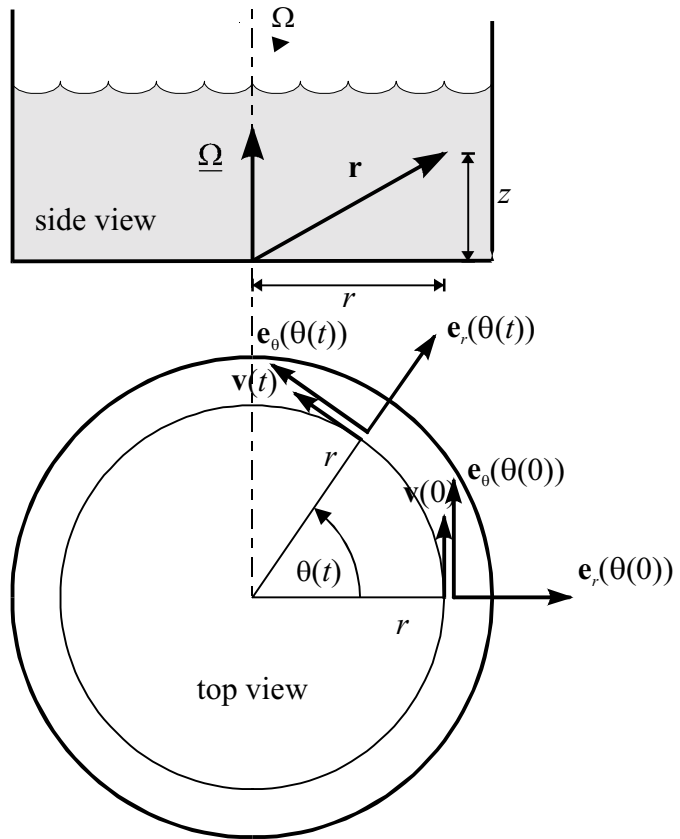
The velocity can be computed using the formulas developed in the example on page 12:

$$\mathbf{v} = \underbrace{\frac{dr(t)}{dt}}_0 \mathbf{e}_r + r \underbrace{\frac{d\theta(t)}{dt}}_{r\Omega} \mathbf{e}_\theta + \underbrace{\frac{dz(t)}{dt}}_0 \mathbf{e}_z = r\Omega \mathbf{e}_\theta$$

Alternatively, we could deduce \mathbf{v} from the definition of derivative of a vector with respect to a scalar:

$$\mathbf{v} = \frac{D\mathbf{r}}{Dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{r d\theta \mathbf{e}_\theta}{dt} = r \frac{d\theta}{dt} \mathbf{e}_\theta = r\Omega \mathbf{e}_\theta$$

More generally, in invariant form (i.e. in any coordinate system) the velocity profile corresponding to solid-body rotation is given by



$$\mathbf{v}(\mathbf{r}_p) = \underline{\underline{\Omega}} \times \mathbf{r}_p \quad (5)$$

where $\underline{\underline{\Omega}}$ is called the **angular velocity vector** and \mathbf{r}_p is the position vector* whose origin lies somewhere along the axis of rotation. The magnitude of $\underline{\underline{\Omega}}$ is the rotation speed in radians per unit time. It's direction is the axis of rotation and the sense is given by the "right-hand rule." In cylindrical coordinates, the angular velocity is

$$\underline{\underline{\Omega}} = \Omega \mathbf{e}_z$$

and the position vector is

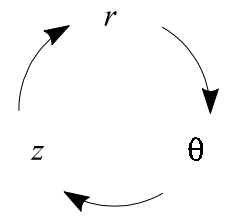
$$\mathbf{r}_p = r\mathbf{e}_r + z\mathbf{e}_z$$

Taking the cross product of these two vectors (keeping the order the same as in (5)):

$$\mathbf{v}(\mathbf{r}) = r\underbrace{\Omega \mathbf{e}_z \times \mathbf{e}_r}_{\mathbf{e}_\theta} + z\underbrace{\Omega \mathbf{e}_z \times \mathbf{e}_z}_{\mathbf{0}} = r\Omega \mathbf{e}_\theta$$

To obtain this result we have used the fact that the cross product of any two parallel vectors vanishes (because the sine of the angle between them is zero — recall definition of cross product on p1).

The cross product of two distinct unit vectors in any right-handed coordinate system yields a vector parallel to the third unit vector with a sense that can be remembered using the figure at right. If the cross product of the two unit vectors corresponds to a "clockwise" direction around this circle, the sense is positive; in a "counter-clockwise" direction, the sense is negative. In this case, we are crossing \mathbf{e}_z with \mathbf{e}_r which is clockwise; hence the cross product is $+\mathbf{e}_\theta$.



Now that we have the velocity field, let's compute the curl. In cylindrical coordinates, the formula for the curl is obtained from p739 of BSL:

$$\nabla \times \mathbf{v} = \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \mathbf{e}_\theta + \left(\frac{1}{r} \frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \mathbf{e}_z$$

Substituting

$$v_r = 0$$

$$v_\theta = r\Omega$$

$$v_z = 0$$

we obtain

$$\nabla \times \mathbf{v} = 2\Omega \mathbf{e}_z = 2\underline{\underline{\Omega}}$$

* The subscript "p" was added here to avoid confusing the cylindrical coordinate r with the magnitude of the position vector. Note that in cylindrical coordinates, $|\mathbf{r}_p| = \sqrt{r^2 + z^2} \neq r$.

Thus the curl turns out to be twice the angular velocity of the fluid elements. While we have only shown this for a particular flow field, the results turns out to be quite general:

$$\nabla \times \mathbf{v} = 2\mathbf{\underline{\underline{\Omega}}}$$

Vector Field Theory

There are three very powerful theorems which constitute “vector field theory:”

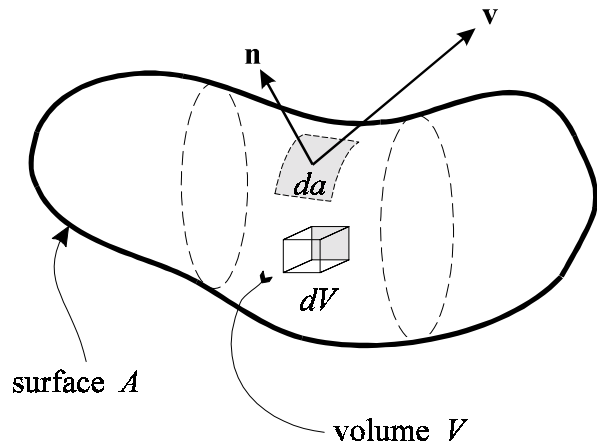
- Divergence Theorem
- Stokes Theorem
- Irrotational \Leftrightarrow Conservative \Leftrightarrow Derivable from potential

DIVERGENCE THEOREM

This is also known as “*Gauss* Divergence Theorem*” or “*Green’s Formula*” (by Landau & Lifshitz). Let \mathbf{v} be any (continuously differentiable) vector field and choose A to be any (piecewise smooth, orientable) closed surface; then

$$\oint_A \mathbf{n} \cdot \mathbf{v} \, da = \int_V \nabla \cdot \mathbf{v} \, dV$$

where V is the region enclosed by A and \mathbf{n} is the outward pointing unit normal to the differential surface element having area da . Although we will not attempt to prove this theorem, we can offer the following rationalization. Consider the limit in which all dimensions of the region are very small, i.e. $V \rightarrow 0$. When the region is sufficiently small, the integrand (which is



* Carl Friedrich Gauss (1777-1855), German mathematician, physicist, and astronomer. Considered the greatest mathematician of his time and the equal of Archimedes and Isaac Newton, Gauss made many discoveries before age twenty. Geodetic survey work done for the governments of Hanover and Denmark from 1821 led him to an interest in space curves and surfaces.

assumed to vary continuously with position)* is just a constant over the region:

$$\nabla \cdot \mathbf{v} = \text{const. inside } V$$

$$\int_A \mathbf{n} \cdot \mathbf{v} da = \int_V \nabla \cdot \mathbf{v} dV = (\nabla \cdot \mathbf{v}) \left(\int_V dV \right) = (\nabla \cdot \mathbf{v}) V$$

Solving for the divergence, we get the definition back (recalling that this was derived for $V \rightarrow 0$):

$$\nabla \cdot \mathbf{v} = \frac{1}{V} \int_A \mathbf{n} \cdot \mathbf{v} da$$

Thus the divergence theorem is at least consistent with the definition of divergence.

Corollaries of the Divergence Theorem

Although we have written the Divergence Theorem for vectors (tensors of rank 1), it can also be applied to tensors of other rank:

$$\int_A \mathbf{n} f da = \int_V \nabla f dV$$

$$\int_A \mathbf{n} \cdot \underline{\underline{\tau}} da = \int_V \nabla \cdot \underline{\underline{\tau}} dV$$

One application of the divergence theorem is to simplify the evaluation of surface or volume integrals. However, we will use GDT mainly to derive invariant forms of the equations of motion:

Invariant: independent of coordinate system.

To illustrate this application, let's use GDT to derive the continuity equation in invariant form.

The Continuity Equation

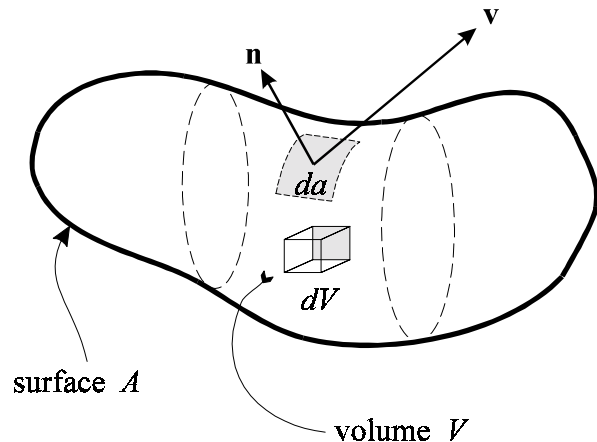
Let $\rho(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ be the density and fluid velocity. What relationship between them is imposed by conservation of mass?

* This is a consequence of \mathbf{v} being “continuously differentiable”, which means that all the partial derivatives of all the scalar components of \mathbf{v} exist and are continuous.

For any system, conservation of mass means:

$$\left\{ \begin{array}{l} \text{rate of acc.} \\ \text{of total mass} \end{array} \right\} = \left\{ \begin{array}{l} \text{net rate of} \\ \text{mass entering} \end{array} \right\}$$

Let's now apply this principle to an arbitrary system whose boundaries are fixed spatial points. Note that this system, denoted by V can be macroscopic (it doesn't have to be differential). The boundaries of the system are the set of fixed spatial points denoted as A . Of course, fluid may readily cross these mathematical boundaries.



Subdividing V into many small volume elements:

$$dm = \rho dV$$

$$M = \int dm = \int_V \rho dV$$

$$\frac{dM}{dt} = \frac{d}{dt} \left(\int_V \rho dV \right) = \int_V \frac{\partial \rho}{\partial t} dV$$

where we have switched the order of differentiation and integration. This last equality is only valid if the boundaries are independent of t . Now mass enters through the surface A . Subdividing A into small area elements:

\mathbf{n} = outward unit normal

$$\mathbf{n} \cdot \mathbf{v} da = \text{vol. flowrate out through } da \text{ (cm}^3/\text{s)}$$

$$\rho(\mathbf{n} \cdot \mathbf{v}) da = \text{mass flowrate out through } da \text{ (g/s)}$$

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{mass leaving} \end{array} \right\} = \int_A \rho(\mathbf{n} \cdot \mathbf{v}) da = \int_A \mathbf{n} \cdot (\rho \mathbf{v}) da = \int_V \nabla \cdot (\rho \mathbf{v}) dV$$

The third equality was obtained by applying GDT. Substituting into the general mass balance:

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot (\rho \mathbf{v}) dV$$

Since the two volume integrals have the same limits of integration (same domain), we can combine them:

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0$$

Since V is arbitrary, and since this integral must vanish for all V , the integrand must vanish at every point:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

which is called the **equation of continuity**. Note that we were able to derive this result in its most general vectorial form, without recourse to any coordinate system and using a **finite** (not differential) control volume. In the special case in which ρ is a constant (i.e. depends on neither time nor position), the continuity equation reduces to:

$$\nabla \cdot \mathbf{v} = 0 \qquad \rho = \text{const.}$$

Recall that $\nabla \cdot \mathbf{v}$ represents the rate of expansion of fluid elements. “ $\nabla \cdot \mathbf{v} = 0$ ” means that any flow into a fluid element is matched by an equal flow out of the fluid element: accumulation of fluid inside any volume is negligible small.

Reynolds Transport Theorem

In the derivation above, the boundaries of the system were fixed spatial points. Sometimes it is convenient to choose a system whose boundaries move. Then the accumulation term in the balance will involve time derivatives of volume integrals whose limits change with time. Similar to Liebnitz rule for differentiating an integral whose limits depend on the differentiation variable, it turns out that:*

* If the domain V were not arbitrary, we would not be able to say the integrand vanishes for every point in the domain. For example:

$$\begin{aligned} \int_0^{2\pi} \cos \theta \, d\theta &= \int_0^{2\pi} \sin \theta \, d\theta = 0 \\ \int_0^{2\pi} (\cos \theta - \sin \theta) \, d\theta &= 0 \end{aligned}$$

does not imply that $\cos \theta = \sin \theta$ since the integral vanishes over certain domains, but not all domains.

* For a proof, see G:163-4.

$$\frac{d}{dt} \left(\int_{V(t)} S(\mathbf{r}, t) dV \right) = \int_{V(t)} \frac{\partial S}{\partial t} dV + \int_{A(t)} S(\mathbf{r}, t) (\mathbf{n} \cdot \mathbf{w}) da \quad (6)$$

where \mathbf{w} is the local velocity of the boundary and $S(t)$ is a tensor of any rank. If $\mathbf{w} = \mathbf{0}$ at all points on the boundary, the boundary is stationary and this equation reduces to that employed in our derivation of the continuity equation. In the special case in which \mathbf{w} equals the local fluid velocity \mathbf{v} , this relation is called the **Reynolds Transport Theorem**.♦

EXAMPLE: rederive the continuity equation using a control volume whose boundaries move with the velocity of the fluid.

Solution: If the boundaries of the system move with the same velocity as local fluid elements, then fluid elements near the boundary can never cross it since the boundary moves with them. Since fluid is not crossing the boundary, the system is *closed*.* For a closed system, conservation of mass requires:

$$\frac{d}{dt} \left\{ \begin{array}{l} \text{mass of} \\ \text{system} \end{array} \right\} = 0$$

or

$$\frac{dM}{dt} = \frac{d}{dt} \int_{V(t)} \rho dV = 0 \quad (7)$$

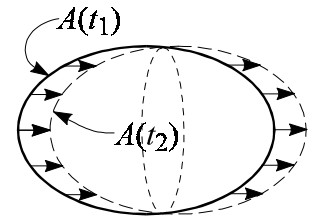
Notice that we now have to differentiate a volume integral whose limits of integration depend on the variable with respect to which we are differentiating. Applying (6) with $\mathbf{w} = \mathbf{v}$ (i.e. applying the Reynolds Transport Theorem):

$$\frac{d}{dt} \left(\int_{V(t)} \rho(\mathbf{r}, t) dV \right) = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} \rho(\mathbf{r}, t) (\mathbf{n} \cdot \mathbf{v}) da$$

which must vanish by (7). Applying the divergence theorem, we can convert the surface integral into a volume integral. Combining the two volume integrals, we have

♦ Osborne Reynolds (1842-1912), Engineer, born in Belfast, Northern Ireland, UK. Best known for his work in hydrodynamics and hydraulics, he greatly improved centrifugal pumps. The Reynolds number takes its name from him.

* When we say “closed,” we mean no *net* mass enters or leaves the system; individual molecules might cross the boundary as a result of Brownian motion. However, in the absence of concentration gradients, as many molecules enter the system by Brownian motion as leave it by Brownian motion. \mathbf{v} is the mass-averaged velocity.



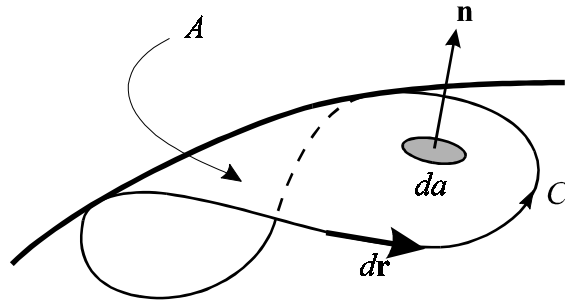
$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0$$

which is the same as we had in the previous derivation, except that V is a function of time. However, making this hold for all time and all initial V is really the same as holding for all V . The rest of the derivation is the same as before.

STOKES THEOREM

Let \mathbf{v} be any (continuously differentiable) vector field and choose A to be any (piecewise smooth, orientable) open surface. Then

$$\int_A \mathbf{n} \cdot (\nabla \times \mathbf{v}) da = \oint_C \mathbf{v} \cdot d\mathbf{r}$$



where C is the closed curve forming the edge of A (has direction) and \mathbf{n} is the unit normal to A whose sense is related to the direction of C by the “right-hand rule”. The above equation is called **Stokes Theorem**.*

Velocity Circulation: Physical Meaning

The contour integral appearing in Stokes’ Theorem is an important quantity called **velocity circulation**. We will encounter this quantity in a few lectures when we discuss Kelvin’s Theorem. For now, I’d like to use Stokes Theorem to provide some physical meaning to velocity circulation. Using Stokes Theorem and the Mean Value Theorem, we can write the following:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} \stackrel{\text{Stokes' Theorem}}{=} \int_A \mathbf{n} \cdot (\nabla \times \mathbf{v}) da \stackrel{\text{Mean Value Theorem}}{=} \langle \mathbf{n} \cdot (\nabla \times \mathbf{v}) \rangle A = 2 \langle \Omega_n \rangle A$$

Finally, we note from the meaning of curl that $\nabla \times \mathbf{v}$ is twice the angular velocity of fluid elements, so that $\mathbf{n} \cdot \nabla \times \mathbf{v}$ is the normal component of the angular velocity (i.e. normal to the surface A). Thus velocity circulation is twice the average angular speed of fluid elements times the area of the surface whose edge is the closed contour C .

* Sir George Gabriel Stokes (1819-1903): British (Irish born) mathematician and physicist, known for his study of hydrodynamics. Lucasian professor of mathematics at Cambridge University 1849-1903 (longest-serving Lucasian professor); president of Royal Society (1885-1890).

Example: Compare “velocity circulation” and “angular momentum” for a thin circular disk of fluid undergoing solid-body rotation about its axis.

Solution: Choosing cylindrical coordinates with the z -axis aligned with axis of rotation. Solid-body rotation corresponds to the following velocity profile (see page 23):

$$\mathbf{v} = r\Omega\mathbf{e}_\theta$$

and

$$\nabla \times \mathbf{v} = 2\Omega\mathbf{e}_z$$

Finally the unit normal to the disk surface is $\mathbf{n} = \mathbf{e}_z$. Then the velocity-circulation integral becomes

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{n} \cdot (\nabla \times \mathbf{v}) da = \int_A \mathbf{e}_z \cdot (2\Omega\mathbf{e}_z) da = 2\Omega\pi R^2$$

According to L&L Vol I* page 25, the angular momentum \mathbf{L} of a mass m undergoing motion at velocity \mathbf{v} is the lever arm \mathbf{r} times the linear momentum ($\mathbf{p} = m\mathbf{v}$): i.e. $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Summing this over differential fluid mass in our disk with $dm = \rho dV$, the net angular momentum of the disk is:

$$\mathbf{L} = \int_V (\mathbf{r} \times \mathbf{v}) \rho dV = \rho\Delta z \int_A (\mathbf{r} \times \mathbf{v}) da$$

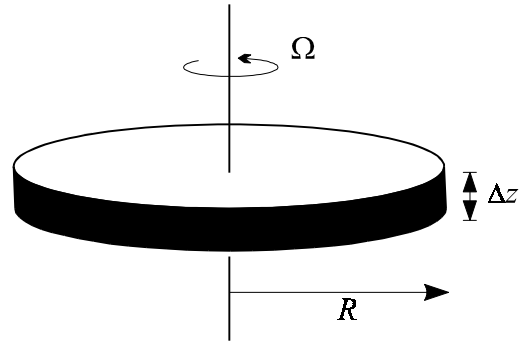
Since the disk is of uniform thickness Δz and density ρ , we can write the second equation above. If the disk is sufficiently thin that we can neglect the z contribution to the position vector, then we can approximate $\mathbf{r} = r\mathbf{e}_r$ in cylindrical coordinates.♦ Substituting into the second integral above

$$\mathbf{L} = \rho\Delta z \int_A (r^2\Omega\mathbf{e}_z) da = \rho\Delta z\Omega\mathbf{e}_z \int_0^R r^2 2\pi r dr = \frac{\pi}{2} R^4 \rho\Delta z\Omega\mathbf{e}_z$$

Dividing this by the velocity circulation integral:

* Landau & Lifshitz, *Mechanics and Electrodynamics* (Course of Theoretical Physics: Vol. 1), Pergamon, 1959.

♦ Actually this assumption isn't necessary since any z -component of \mathbf{r} will produce an r -component in the cross-product and this r -component will integrate to zero as long as V is a body of rotation about the same axis.



$$\frac{L_z}{\oint_C \mathbf{v} \cdot d\mathbf{r}} = \frac{\frac{\pi}{2} R^4 \rho \Delta z \Omega}{2\Omega \pi R^2} = \frac{1}{4} R^2 \rho \Delta z = \frac{1}{4\pi} \underbrace{\pi R^2 \Delta z}_V \rho = \frac{M}{4\pi}$$

where M is the mass of fluid in the disk. This could be rewritten as

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 4\pi \frac{L_z}{M}$$

So the velocity-circulation integral is just proportional to the angular momentum per unit mass.

DERIVABLE FROM A SCALAR POTENTIAL

A very special class of vector fields consists of those vectors for which a scalar field exists such that the vector can be represented as the gradient of the scalar:

Suppose: $\mathbf{v} = \mathbf{v}(\mathbf{r})$ and $f = f(\mathbf{r})$

If f exists such that: $\mathbf{v} = \nabla f$

for all \mathbf{r} in some domain, then $f(\mathbf{r})$ is called the *scalar potential* of \mathbf{v} and \mathbf{v} is said to be *derivable from a potential* in that domain.

An example of a vector field which is “derivable from a potential” is the gravitational force near sea level:

$$\mathbf{F}_{\text{grav}} = -Mg\mathbf{k}$$

and the associated potential energy is:

$$\phi(z) = Mgz$$

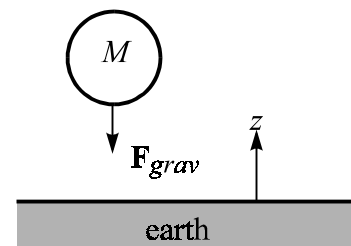
Note that

$$\nabla \phi = Mg\mathbf{k}$$

is identical to the force, except for the sign (introduced by convention). This example also suggests why ϕ is called the “potential” of \mathbf{v} . Not every vector field has a potential. Which do? To answer this, let's look at some special properties of such vector fields.

Property I: if $\mathbf{v} = \nabla \phi$ then $\nabla \times \mathbf{v} = \mathbf{0}$ (irrotational)

Proof: Recall that $\nabla \times (\nabla \phi) = \mathbf{0}$ (see HWK #2, Prob. 4e). A vector which has this property is said to be *irrotational*. This name is an allusion to $\nabla \times \mathbf{v}$ representing the rotation rate if \mathbf{v} is the fluid velocity. $\nabla \times \mathbf{v} = \mathbf{0}$ means the fluid elements are not rotating.



Property II:

$$\text{if } \mathbf{v} = \nabla\phi \text{ then } \oint_C \mathbf{v} \cdot d\mathbf{r} = 0 \quad (\text{conservative})$$

for any closed contour in the region.

Proof: Using Property I, we know that $\nabla \times \mathbf{v} = \mathbf{0}$. Then we can deduce the value of this closed-contour integral from Stokes' Theorem:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{n} \cdot \underbrace{(\nabla \times \mathbf{v})}_{\mathbf{0}} da = 0$$

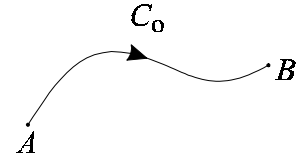
A vector field which has this property is said to be **conservative**. This name is an allusion to the special case in which \mathbf{v} represents a force, like gravity. Then $\mathbf{v} \cdot d\mathbf{r}$ (force times displacement) represents the work required to move the object through the force field. Saying that the contour integral vanishes means that the work required to lift a weight can be recovered when the weight falls. In other words, energy is conserved.

If C is open, $\mathbf{v} = \nabla\phi$ is still quite useful:

Property III: let C_o be an **open** contour connecting points A and B .

$$\text{If } \mathbf{v} = \nabla\phi \text{ then } \int_{C_o} \mathbf{v} \cdot d\mathbf{r} = \phi(\mathbf{r}_B) - \phi(\mathbf{r}_A)$$

for any contour connecting A and B .



Proof: Note that $\nabla\phi \cdot d\mathbf{r} = d\phi$ (from our definition of gradient). Then

$$\int_{C_o} \mathbf{v} \cdot d\mathbf{r} = \int_{C_o} d\phi = \phi(\mathbf{r}_B) - \phi(\mathbf{r}_A)$$

We call this property **path independence**. Of course, Property II is just a special case of this for which $A=B$ so that $\phi(\mathbf{r}_B) - \phi(\mathbf{r}_A) = 0$.

THEOREM III

We have just shown that properties I and II are implied by $\mathbf{v} = \nabla\phi$; it turns out that the converse is also true, although I'm not going to prove it here. We can distill these properties and their converse into a single statement:

$$\left\{ \begin{array}{l} \nabla \times \mathbf{v} = \mathbf{0} \\ \text{for all } \mathbf{r} \\ \text{in Region} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \phi = \phi(\mathbf{r}) \text{ exists} \\ \text{such that } \mathbf{v} = \nabla\phi \\ \text{in Region} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \oint_C \mathbf{v} \cdot d\mathbf{r} = 0 \text{ for every} \\ C \\ \text{closed } C \text{ in Region} \end{array} \right\}$$

TRANPOSE OF A TENSOR, IDENTITY TENSOR

The *transpose* of a tensor $\underline{\underline{\tau}}$ is denoted $\underline{\underline{\tau}}^t$ and is defined so that:

$$\mathbf{v} \cdot \underline{\underline{\tau}} = \underline{\underline{\tau}}^t \cdot \mathbf{v}$$

and

$$\underline{\underline{\tau}} \cdot \mathbf{v} = \mathbf{v} \cdot \underline{\underline{\tau}}^t$$

for all vectors \mathbf{v} . For example:

if $\underline{\underline{\tau}} = \mathbf{ab}$

then $\underline{\underline{\tau}}^t = \mathbf{ba}$

More generally, in terms of scalar components of $\underline{\underline{\tau}}$, we can write the relationship between a tensor and its transpose as:

$$\tau_{ij}^t = \tau_{ji}$$

Symmetric Tensor:

$$\underline{\underline{\tau}}^t = \underline{\underline{\tau}}$$

An example of a symmetric tensor is the dyad \mathbf{aa} .

Identity Tensor: Also known as the *Idem Factor*. Denoted as $\underline{\underline{\mathbf{I}}}$ and defined so that:

$$\mathbf{v} \cdot \underline{\underline{\mathbf{I}}} = \mathbf{v} = \underline{\underline{\mathbf{I}}} \cdot \mathbf{v}$$

for any vector \mathbf{v} . Clearly $\underline{\underline{\mathbf{I}}}$ is symmetric, but in addition, dotting it with another vector gives that vector back (like multiplying by one). In any coordinate system, $\underline{\underline{\mathbf{I}}}$ can be calculated from:

$$\underline{\underline{\mathbf{I}}} = \nabla \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}}$$

where \mathbf{r} is the position vector expressed in terms of the unit vectors in that coordinate system. Recalling from p5 that gradient can be thought of as the partial derivative with respect to position, $\underline{\underline{\mathbf{I}}}$ can be thought of as the derivative of the position vector with respect to itself. In R.C.C.S., recall that:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and

$$\nabla \mathbf{r} = \sum_i \sum_j \frac{\partial r_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j$$

where r_j is the j^{th} component of the position vector \mathbf{r} and x_i is the i^{th} coordinate. In Cartesian coordinates, the position vector components are related to the coordinates according to:

$$r_1 = x_1 = x, r_2 = x_2 = y, \text{ and } r_3 = x_3 = z:$$

then

$$\frac{\partial r_j}{\partial x_i} = \delta_{ij}$$

which is 0 if $i \neq j$ or 1 if $i = j$. This leaves:

$$\nabla \mathbf{r} = \sum_i \sum_j \mathbf{e}_i \mathbf{e}_j^*$$

so

$$\underline{\mathbf{I}} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

As a partial proof that $\underline{\mathbf{I}}$ has the desired properties which make it the identity tensor, consider dotting it with an arbitrary vector \mathbf{v} :

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{ii} + \mathbf{jj} + \mathbf{kk}) &= \mathbf{v} \cdot \mathbf{ii} + \mathbf{v} \cdot \mathbf{jj} + \mathbf{v} \cdot \mathbf{kk} \\ &= (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k} \\ &= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} = \mathbf{v} \end{aligned}$$

Thus we have shown that $\mathbf{v} \cdot \underline{\mathbf{I}} = \mathbf{v}$, as advertised.

DIVERGENCE OF A TENSOR

In presenting the corollaries to the Divergence Theorem, we have already introduced the divergence of a tensor. This quantity is defined just like divergence of a vector.

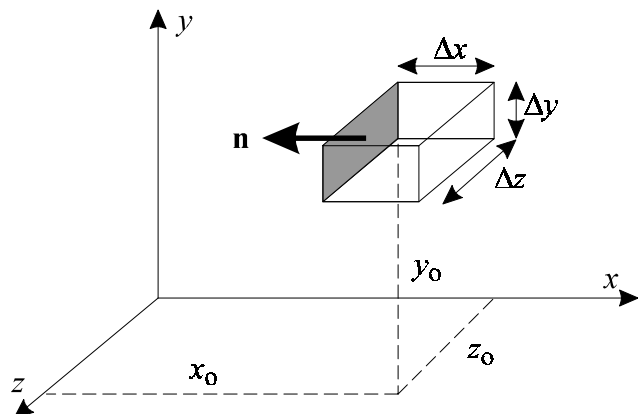
$$\nabla \cdot \underline{\boldsymbol{\tau}} \equiv \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_A \mathbf{n} \cdot \underline{\boldsymbol{\tau}} da \right\}$$

Note that this definition uses a pre-dot not a post-dot. In R.C.C.S.

$$\underline{\boldsymbol{\tau}} = \sum \sum \tau_{ij} \mathbf{e}_i \mathbf{e}_j$$

On the $x=x_0$ face:

$$\mathbf{n} = -\mathbf{e}_1$$



* This expression for the identity tensor is valid for any set of orthonormal unit vectors (not just the Cartesian ones for which we have derived it here).

$$\mathbf{n} \cdot \underline{\underline{\tau}} = \sum_i \sum_j \tau_{ij} \mathbf{e}_i \cdot \mathbf{e}_i \mathbf{e}_j = - \sum_j \tau_{1j} \bigg|_{x_o} \mathbf{e}_j$$

Similarly, on the $x=x_o+\Delta x$ face, we obtain:

$$\mathbf{n} = +\mathbf{e}_1$$

$$\mathbf{n} \cdot \underline{\underline{\tau}} = \sum_j \tau_{1j} \bigg|_{x_o + \Delta x} \mathbf{e}_j$$

After integrating over the area, we obtain:

$$\int_{A_1 + A_2} \mathbf{n} \cdot \underline{\underline{\tau}} da = \sum_j \left\{ \tau_{1j} \big|_{x_o + \Delta x} - \tau_{1j} \big|_{x_o} \right\} \mathbf{e}_j \Delta y \Delta z$$

Dividing by V :

$$\frac{1}{V} \int_{A_1 + A_2} \mathbf{n} \cdot \underline{\underline{\tau}} da = \sum_j \left\{ \frac{\tau_{1j} \big|_{x_o + \Delta x} - \tau_{1j} \big|_{x_o}}{\Delta x} \right\} \mathbf{e}_j$$

Taking the limit as $V \rightarrow 0$:

$$\lim_{V \rightarrow 0} \left\{ \frac{1}{V} \int_{A_1 + A_2} \mathbf{n} \cdot \underline{\underline{\tau}} da \right\} = \sum_j \frac{\partial \tau_{1j}}{\partial x} \mathbf{e}_j$$

Adding on similar contributions from the $y=\text{const}$ and $z=\text{const}$ faces:

$$\nabla \cdot \underline{\underline{\tau}} = \sum_i \sum_j \frac{\partial \tau_{ij}}{\partial x_i} \mathbf{e}_j$$

$$\nabla \cdot \underline{\underline{\tau}} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \mathbf{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \mathbf{j} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \mathbf{k}$$

Introduction to Continuum Mechanics*

In this course, we will treat fluids like continua; in other words, we are going to ignore the molecular granularity of matter. This is an assumption which we, as engineers, often make in describing transport of heat, mass, or momentum although we don't always state this assumption explicitly. To make the nature of this assumption clearer, it might help to discuss the alternative.

Fluids are composed of molecules. In principle, if you tell me the initial location of every molecule in the fluid and its initial velocity, I can compute the position and velocity at some later time using Newton's laws of motion (i.e. $\mathbf{F} = m\mathbf{a}$). The difficulty with this approach is that the number of molecules in any volume of fluid of interest to us make such a detailed calculation impractical. For example:

$$1 \text{ cm}^3 \text{ of water} \rightarrow 3.3 \times 10^{22} \text{ molecules} \rightarrow 10 \text{ million years}$$

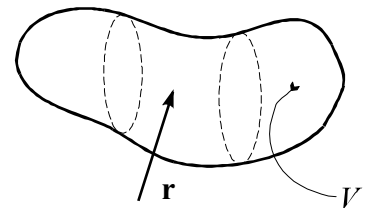
Even with a computer operating at 100 mfops, it would take 10 million years to do just one multiplication for each molecule. Molecules of a liquid collide on the average of once every 10^{-12} seconds. To describe one second of real behavior, I would need $10^{12} \times 10$ million years. Clearly, this is an absurd length of time. Although computers get faster every year, this will remain an absurdly long time for the foreseeable future. The alternative is:

CONTINUUM HYPOTHESIS

A detailed description at the molecular level is not required in order to predict macroscopic behavior of any material. For example, it is not necessary to know the precise location of every molecule of fluid; it turns out that all that is needed for most applications is the distribution of mass described by the density profile $\rho(\mathbf{r})$ of molecules in some region:

$$\rho(\mathbf{r}) = \lim_{V \rightarrow 0} \left\{ \frac{1}{V} \sum_i m_i \right\}$$

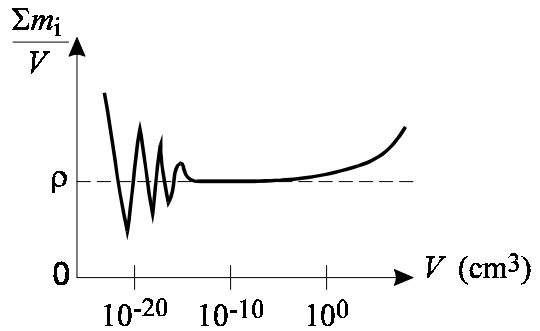
where m_i is the mass of molecule i , the sum is over all the molecules inside surface A , and V is the volume. In fluid mechanics, as in heat and mass transfer, we make an assumption known as the "continuum hypothesis."



* Reference: G.E. Mase, "Continuum Mechanics," Schaum's Outline Series, McGraw-Hill, 1970.

Basically this assumption is that the limit above will converge long before the dimensions of V shrink to molecular size.

Similarly, we don't need to know the translational, rotational, vibrational and electronic energy of each molecule. We usually need only to know the internal energy per unit volume as a function of position, which in turn, manifests itself macroscopically as temperature.



A more precise statement of the continuum hypothesis is:

Continuum Hypothesis - the region to be described can be subdivided into a set of (infinitesimal) volume elements, each of which simultaneously:

- 1) is small enough to be considered uniform (i.e. any spatial variations in properties -- such as ρ , \mathbf{v} , T , p -- inside the volume element are negligible); and
- 2) is large enough to contain a statistically large number of molecules.

In other words, we are assuming that dV exists such that the two conditions above are both satisfied. Materials which obey this “hypothesis” are said to behave as a **continuum**. Generally, the continuum hypothesis works well provided all the dimensions of the system are large compared to molecular size. An example of a situation in which the continuum hypothesis does **not** work is the flow of dilute gases in small pores, where the mean free path (for the collision of molecules) is comparable to the dimensions of the pore. This situation is called “Knudsen diffusion.”

The basic problem in continuum mechanics is to describe the response of material to stress. A quantitative statement of that response is known as:

Constitutive Equation - model which describes how a material will respond to stress.

Familiar examples of constitutive equations include:

- 1) Hooke's law of elasticity (solids)
- 2) Newton's law of viscosity (fluids)

Many materials, like toothpaste and polymer melts, have characteristics of both solids and fluids and do not obey either of these simple “laws.” Such fluids are called **viscoelastic**.

CLASSIFICATION OF FORCES

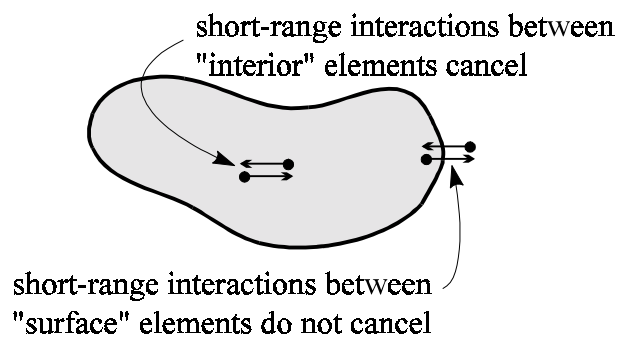
Having derived an equation (the Continuity equation) to describe the relationship among the variables which is imposed by conservation of mass, the remaining fundamental principle of physics is Newton's second law ($\sum_i \mathbf{F}_i = m\mathbf{a}$) which, as it turns out, is equivalent to conservation of momentum. To apply this principle, we will need to list the forces which can act on fluid systems. Forces tend to fall into one of two different categories, depending on the range over which they act: long-range (compared to molecular size) forces are computed as volume integrals (called "body forces") and short-range forces are computed as surface integrals (called "surface forces").

Of course, gravitational forces have the longest range of any known force. For example, gravitational forces between planets and the sun determine their orbits. In particular, all fluid elements (not just those at the system boundary) feel a gravitational force of interaction with the rest of the universe outside the system boundaries. Thus gravity is a "body force."

body forces: those which act on **every** fluid element in body (e.g. gravity):

$$d\mathbf{F}_g = (dm)\mathbf{g} = \rho\mathbf{g} dV$$

At the other end of the spectrum are forces which have very short range. If the range is of molecular dimensions, then only fluid elements experience a nonzero interaction with the universe outside the system. Although interior fluid elements might interact with one another through this short-range force, this interaction is not considered in a force balance, because the "action" and "reaction" forces cancel, leaving no net contribution to the force on the system. When only surface elements feel a particular force from outside, that force is called a "surface force."



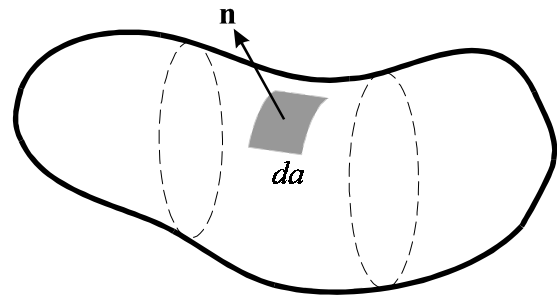
At the molecular scale, pressure arises from the momentum transferred during collisions between molecules outside and molecules inside the system. Since only surface molecules will be struck from outside, pressure is a surface force.

surface forces: those which act **only** on surfaces (including mathematical boundaries)

One example is **hydrostatic pressure**:

$$d\mathbf{F}_p = -p\mathbf{n} da$$

where $d\mathbf{F}_p$ is the force exerted *on* the system (through the surface element da) by the fluid outside, and \mathbf{n} is a unit **outward** (to system) normal. For a proof that this is the correct form for hydrostatic pressure, see Batchelor, Section 1.3.



Hydrostatic Equilibrium

If our material is a fluid and if it is at rest (no velocity and no acceleration), then gravity and hydrostatic pressure forces are usually the only forces acting on the system. At equilibrium, the forces must be balanced. Thus Newton's \clubsuit 2nd law, which generally requires $\sum_i \mathbf{F}_i = M\mathbf{a}$, reduces to $\sum_i \mathbf{F}_i = \mathbf{0}$ at mechanical equilibrium.

In our case, this means

$$\mathbf{F}_g + \mathbf{F}_p = \mathbf{0}$$

$$\mathbf{F}_g = \int_V \rho \mathbf{g} dV$$

$$\mathbf{F}_p = -\int_A \mathbf{n} p da = -\int_V \nabla p dV$$

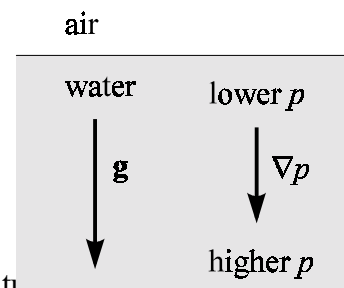
To obtain this last result, we applied one of the corollaries of the Divergence Theorem. Substituting back into the force balance and combining the two volume integrals leads to:

$$\mathbf{F}_g + \mathbf{F}_p = \int_V [\rho \mathbf{g} - \nabla p] dV = \mathbf{0}$$

Since V is arbitrary, we conclude that the integrand vanishes:

$$\nabla p = \rho \mathbf{g}$$

This says that the pressure increases in the direction of the acceleration of gravity (downward), which correctly describes



\clubsuit Sir Isaac Newton (1642–1727), English mathematician and natural philosopher (physicist); considered by many the greatest scientist of all time; invented differential calculus and formulated the theories of universal gravitation, terrestrial mechanics, and color.

(for example) how the pressure increases with depth in an ocean.

Flow of Ideal Fluids

Now let's consider fluids in motion. The simplest analysis is for:

ideal fluid - deformation of fluid elements is an isentropic process (i.e. adiabatic and reversible):

$$\mu = 0 \text{ and } k = 0$$

where μ is the viscosity and k is the thermal conductivity. Generally this means that any viscous forces are negligible (since viscous forces represent friction arising between fluid elements and friction gives rise to irreversibility). Furthermore, to keep the process adiabatic, the thermal conductivity must also be negligible.

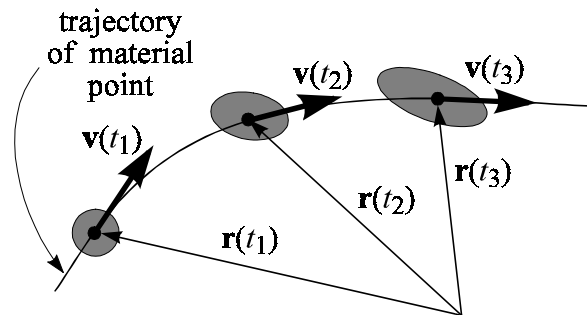
EULER'S EQUATION

Suppose these conditions on the fluid are met. Thus consider the isentropic deformation of an ideal fluid for an arbitrary macroscopic system. In addition to pressure and gravity, we must also consider inertia when the system accelerates. Newton's law requires:

$$M\mathbf{a} = \sum_i \mathbf{F}_i$$

Let $\mathbf{r}(t)$ denote the trajectory of one particular fluid element inside the system. Then the velocity of the fluid element is:

$$\mathbf{v} = \frac{D\mathbf{r}}{Dt}$$



while the acceleration is:

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt}$$

We use the material derivative here, since we are following a particular material point. Multiplying the acceleration by the mass of the fluid element gives the inertia:

$$(dm)\mathbf{a} = \rho(dV) \frac{D\mathbf{v}}{Dt}$$

To get the net inertia of the entire system, we must repeat this calculation for each of the fluid elements composing the system and add them up:

$$\langle M\mathbf{a} \rangle = \int_V \rho \frac{D\mathbf{v}}{Dt} dV$$

I've put carets (<...>) around the $M\mathbf{a}$ to indicate that this is the position-average inertia of the system (since the local $(dm)\mathbf{a}$ varies from point to point within a general system). Newton's second law requires us to equate this with the net force acting on the system:

$$\int_V \rho \frac{D\mathbf{v}}{Dt} dV = \mathbf{F}_p + \mathbf{F}_g = \int_V \rho \mathbf{g} dV - \underbrace{\int \mathbf{n} p da}_\int_V \nabla p dV$$

Using the divergence theorem to convert the surface integral into a volume integral, we have three volume integrals over the same domain. Combining these three volume integrals leaves:

$$\int_V \left[\rho \frac{D\mathbf{v}}{Dt} - \rho \mathbf{g} + \nabla p \right] dV = \mathbf{0}$$

Since this must hold for any choice of V , the integrand must vanish at each point in the domain. After dividing by ρ :

$$\frac{D\mathbf{v}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p \quad (8)$$

which is called ***Euler's Equation*** (1755).*

Significance: When combined with a statement of continuity, Euler's equation of motion provides as many equations as unknowns.

Another relationship among the unknowns is the continuity equation (see page 27), which comes from the mass balance.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

* Euler, Leonhard, 1707-83, Swiss mathematician. The most prolific mathematician who ever lived, he worked at the St. Petersburg Academy of Sciences, Russia (1727-41, 1766-83), and at the Berlin Academy (1741-66). He contributed to areas of both pure and applied mathematics, including calculus, analysis, number theory, topology, algebra, geometry, trigonometry, analytical mechanics, hydrodynamics, and the theory of the moon's motion.

For an incompressible fluid, $\rho = \text{const. w.r.t. both time and position}$. Then the continuity equation reduces to:

incompressible fluid: $\nabla \cdot \mathbf{v} = 0,$ (9)

To see that we now have as many equations as unknowns, note that the unknowns in (8) and (9) are

unknowns: 4 scalars \mathbf{v} and p

which represents the 3 scalar components of \mathbf{v} plus p , for a total of 4 scalar unknowns. To evaluate these unknowns, we have equations (8) and (9):

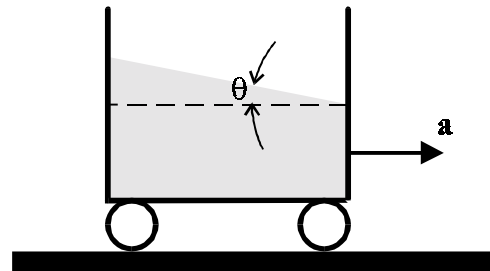
equations: 4 scalars Euler + continuity

but Euler's equation (8) is a vector equation, which can be expanded into 3 independent scalar equations. When added to continuity (a scalar equation), we obtain a total of 4 independent scalar equations, the same number as of scalar unknowns. Thus we are now in position to begin solving problems involving fluid flow. We will call (8) and (9) "Euler's equations of motion for incompressible fluids."

EXAMPLE. Water in a partially filled tank undergoes uniform* acceleration \mathbf{a} in the horizontal plane. Find the angle θ of inclination of the water's surface with respect to the horizontal plane.

Solution. The key to solving this problem is to recognize that, regardless of the angle of inclination, the pressure is equal to 1 atm everywhere on the free surface. Then the pressure gradient ∇p must be normal to this plane. If we can find the orientation of ∇p we will have the orientation of the free surface. Recall Euler's equation of motion for an ideal fluid:

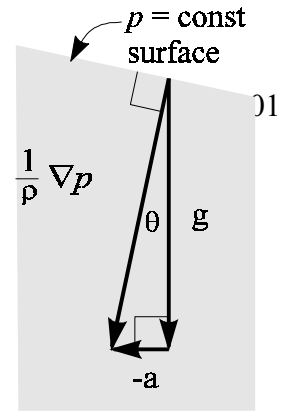
$$\underbrace{\frac{D\mathbf{v}}{Dt}}_{\mathbf{a}} = \mathbf{g} - \frac{1}{\rho} \nabla p$$



* By "uniform acceleration" I mean the same acceleration is experienced at each point and at each time. For a fluid, uniformity at each position occurs only in the steady state after a transient which is nonuniform.

$D\mathbf{v}/Dt$ is just the acceleration of the fluid in a stationary reference frame. At steady state, all of the fluid will undergo the same uniform acceleration as the tank; so $D\mathbf{v}/Dt$ is just \mathbf{a} . Solving for the gradient, we have

$$\frac{1}{\rho} \nabla p = \mathbf{g} - \mathbf{a}$$



Using vector addition in the drawing at right, we can see that the angle of inclination of the free surface (relative to the horizon) is just

$$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

We were lucky in the previous example, because we knew the left-hand side of (8), so instead of 4 scalar unknowns, we only had one: p . The solution was relatively easy. In the more general problem, the left-hand side of (8) is an unknown nonlinear partial differential equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{g} - \frac{1}{\rho} \nabla p \quad (10)$$

In this form, we have expanded $D\mathbf{v}/Dt$ using the relationship between material derivative and partial derivative (see page 15). Now we have 4 scalar unknowns: the three scalar components of \mathbf{v} and pressure: v_x , v_y , v_z and p . Coupled with the continuity equation (for an incompressible fluid)

$$\nabla \cdot \mathbf{v} = 0$$

Euler's equation also gives us 4 scalar equations. One important class of solutions has the form $\mathbf{v} = \nabla \phi$, which is called "potential flow." In the next section, we discuss how this form comes about and identify which physical problems have this form.

KELVIN'S THEOREM

An important precursor to the theory of potential flow is the principle of **conservation of circulation**. Before stating this principle, let me define a quantity which Landau & Lifshitz* call the **velocity circulation**:

* L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Vol. 6 of a Course of Theoretical Physics), Pergamon, New York, 1959.

$$\Gamma \equiv \oint_C \mathbf{v} \cdot d\mathbf{r}$$

for any **closed** contour. Recall that we showed on page 30 that this contour integral is associated with the average angular momentum of fluid elements located on the surface whose edge is C . Kelvin[♦] showed that this velocity circulation is conserved:

$$\frac{d\Gamma}{dt} = 0$$

for any set of **material** points forming a closed contour in an ideal fluid. This result is called **Kelvin's Theorem**.

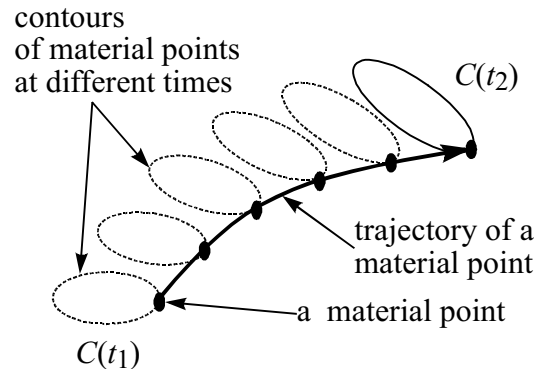
Partial proof:♥ Since the contour C is composed of material points, the time derivative of this contour integral is like the material derivative:

$$\frac{d\Gamma}{dt} = \frac{D}{Dt} \oint_C \mathbf{v} \cdot d\mathbf{r} \stackrel{?}{=} \oint_C \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{r}$$

Since we are always integrating over the same set of material points, a boundary term does not arise when we interchange integration and differentiation operators, although the set of spatial points is time-dependent: $C = C(t)$. Of course, we have not rigorously shown this step to be valid, thus we only claim the proof is “partial.” Next, we substitute Euler’s equation and write each term as the gradient:

$$\frac{d\Gamma}{dt} = \oint_C \left(\mathbf{g} - \frac{1}{\rho} \nabla p \right) \cdot d\mathbf{r} = \oint_C \left(-\nabla \phi_g - \nabla \frac{p}{\rho} \right) \cdot d\mathbf{r} = - \oint_C \nabla \left(\phi_g + \frac{p}{\rho} \right) \cdot d\mathbf{r} = 0$$

In the second equality above, we have introduced the potential energy per unit mass, ϕ_g . Recall that gravity is a conservative force field (see page 32). For an object of constant mass (e.g. a



♦ Lord Kelvin (William Thomson), 1st Baron, 1824-1907, British mathematician and physicist; b. Ireland. He was professor (1846-99) of natural philosophy at the Univ. of Glasgow. His work in thermodynamics coordinating the various existing theories of heat established the law of the conservation of energy as proposed by James Joule. He discovered what is now called the “Thomson effect” in thermoelectricity and introduced the Kelvin scale, or absolute scale, of temperature. His work on the transmission of messages by undersea cables made him a leading authority in this field.

♥ For a more rigorous proof, see Batchelor p269. For a more intuitive proof, see L&L, p15.

brick), Theorem III guarantees that a scalar field $\phi(\mathbf{r})$ exists such that $\mathbf{F}_g = m\mathbf{g} = -\nabla\phi$. For an object of constant mass, we could divide both sides of this equation by m and bring m inside the gradient operator: $\mathbf{g} = -\nabla(\phi/m)$. Similarly, a differential volume element dV would have a differential mass dm and a differential potential energy $d\phi$, such that $\mathbf{g} = -\nabla(d\phi/dm)$. So generally we can write:

$$\mathbf{g} = -\nabla\phi_g \quad (11)$$

where

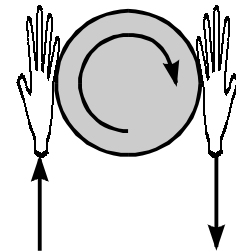
$$\phi_g = \frac{\text{potential energy}}{\text{mass}}.$$

The next-to-last expression (in the equation for $d\Gamma/dt$) must vanish, because it has the form $\nabla s \cdot d\mathbf{r} = ds$: integrating any total differential around a closed contour yields zero. Thus

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \text{const. w.r.t. time}$$

Keep in mind that this applies only if C is composed of material points and only for ideal flow. Since C is composed of material points (which in general move with different velocities), the contour may change shape or move. Given the meaning of this contour integral (Γ = angular momentum), this result implies that (in the absence of friction) angular momentum of fluid is a constant. In general, we change the angular momentum of some object by applying a torque. So this result (i.e. Kelvin's theorem) means that (in the absence of friction) there is no way to apply a torque to ideal fluid elements.

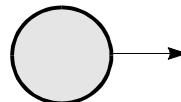
If you think about it, this makes sense: the usual way to apply a torque (with our hands to a cylinder, say) is to hold the cylinder between our hands and then move our hands in opposite directions, as shown in the sketch at right. We thus rely on friction between our hands and the cylinder to exert the torque. If the cylinder were greased and our hands slipped over its surface, we would not be able to apply the torque. This is the essence of what Kelvin's theorem is saying.



IRROTATIONAL FLOW OF AN INCOMPRESSIBLE FLUID

As an example, consider towing a submerged object through an "ideal fluid" which is otherwise stagnant.

$$\mathbf{v}(\mathbf{r}, t=0) = \mathbf{0}$$



submerged object
moving through
stagnant fluid

Consider an arbitrary closed contour in the fluid far from the disturbance caused by the motion of the submerged object. The contour integral vanishes since \mathbf{v} vanishes:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 0 \text{ for } t=0 \quad \text{since } \mathbf{v} = \mathbf{0}$$

for every such contour C . Now at some later time the submerged object moves into the vicinity of C which causes \mathbf{v} to be nonzero. Despite this, Kelvin's Theorem still requires

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 0 \text{ for all } t \quad \text{although } \mathbf{v} \neq \mathbf{0}$$

This is also true for every closed contour in the region (since every contour initially had zero value for this integral).

Applying Theorem III: $\nabla \times \mathbf{v} = \mathbf{0}$ for all \mathbf{r}, t (12)

which is sometimes called the *persistence of irrotationality*. Also from Theorem III, we know $\phi(\mathbf{r}, t)$ exists such that:

$$\mathbf{v} = \nabla \phi \quad (13)$$

where ϕ is called the *velocity potential*.

Significance: Knowing that the solution has the form given by (13) allows us to decouple the four scalar equations represented by (8) and (9):

Substituting (13) into (9) yields Laplace's equation in the velocity potential:

(13) into (9): $\nabla^2 \phi = 0$

Instead of 4 equations in 4 unknowns, we now have a single equation which can be solved for ϕ and $\mathbf{v} = \nabla \phi$, without any coupling to Euler's equation (8). Although Euler was the first to suggest this approach, this is called *Laplace's Equation** after the French mathematician who solved this equation in so many cases.

Knowing the velocity profile \mathbf{v} , we can now determine the pressure profile p from Euler's equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{g} - \frac{1}{\rho} \nabla p \quad (14)$$

* Pierre Simon de Laplace (1749-1827), French mathematician and astronomer, noted for his theory of a nebular origin of the solar system and his investigations into gravity and the stability of planetary motion.

We will now integrate this vector equation to obtain a single scalar equation for the pressure profile. Each term in (14) can be expressed as a gradient of something. For example, we've already seen in (11) that:

$$\mathbf{g} = -\nabla\phi_g$$

Similarly, if $\rho = \text{const}$, then:

$$\frac{1}{\rho} \nabla p \stackrel{\rho = \text{const}}{=} \nabla \left(\frac{p}{\rho} \right) \quad (15)$$

For potential flow, the unsteady term becomes

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial}{\partial t} (\nabla \phi) = \nabla \left(\frac{\partial \phi}{\partial t} \right) \quad (16)$$

Finally, for the convective term, we can apply identity A.3:♦

$$\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla v^2 - \underbrace{\mathbf{v} \times (\nabla \times \mathbf{v})}_{\mathbf{0}} \quad (17)$$

where

$$\mathbf{v} \cdot \mathbf{v} = v^2$$

where $v = |\mathbf{v}|$. In our particular case, the second term vanishes because $\nabla \times \mathbf{v} = \mathbf{0}$ for potential flow. Substituting (11) and (15) through (17) into (14):

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi_g + \frac{p}{\rho} \right) = \mathbf{0}$$

which is called ***Bernoulli's Equation***.♣ It implies that

$$\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi_g + \frac{p}{\rho} = \text{const w.r.t. } \mathbf{r} \quad (18)$$

♦ “Identity A.3” in this equation refers to one of the mathematical identities summarized on the handout titled “Useful Identities in Vector Notation”.

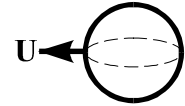
♣ Daniel Bernoulli (1700-1782, Swiss) has often been called the first mathematical physicist; the teacher of Leonhard Euler. His greatest work was his *Hydrodynamica* (1738), which included the principle now known as Bernoulli's principle and anticipated the law of conservation of energy and the kinetic-molecular theory of gases developed a century later. He also made important contributions to probability theory, astronomy, and the theory of differential equations.

is spatially uniform, but it might depend on time. Once the velocity profile is obtained (by solving Laplace's equation), both ϕ and v are known, leaving p as the only unknown.

Potential Flow Around a Sphere

To solve a typical problem involving potential flow, we would first solve Laplace's equation to obtain the velocity profile and then we can evaluate the pressure profile using Bernoulli's equation. Let's illustrate the procedure using an example:

EXAMPLE: Find the velocity and pressure profiles for potential flow caused by a sphere of radius R moving through a stagnant fluid with velocity \mathbf{U} .



Solution: If the fluid behaves ideally, it undergoes potential flow and the velocity profile must satisfy Laplace's equation:

P.D.E.:
$$\nabla^2 \phi = 0$$

Boundary conditions can be formulated by recognizing that fluid far from the sphere is unperturbed:

b.c. #1:
$$\mathbf{v} = \mathbf{0} \text{ far from sphere}$$

while fluid near the sphere cannot penetrate the sphere. To express this mathematically, recall our "bucket-and-stopwatch" method for defining fluid velocity (see page 13). Modifying it slightly to account for the movement of the surface element at velocity \mathbf{U} , the flowrate across a surface element of area da is given by:

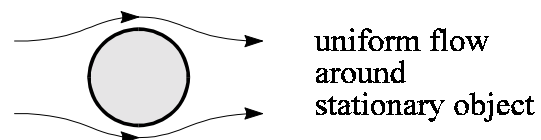
$$dq = \mathbf{n} \cdot (\mathbf{v} - \mathbf{U}) da = 0$$

For an impenetrable sphere, the flowrate must vanish

b.c. #2:
$$\mathbf{n} \cdot (\mathbf{v} - \mathbf{U}) = 0 \text{ on sphere}$$

To solve this problem, we adopt a new reference frame in which the origin moves along with the center of the sphere. It turns out that the PDE does not change upon this shift in reference frame for velocity: the new velocity potential must also satisfy Laplace's equation:

Let reference frame move with object:



P.D.E.: $\nabla^2 \phi = 0$

However, the boundary conditions are changed. In this moving coordinate system, the sphere appears to be stationary and the fluid at infinity is undergoing **uniform flow**

b.c. #1: $\mathbf{v} \rightarrow -\mathbf{U} \equiv U\mathbf{k} \text{ as } r \rightarrow \infty$ (19)

where \mathbf{U} is the velocity of the sphere in the original stationary reference frame.

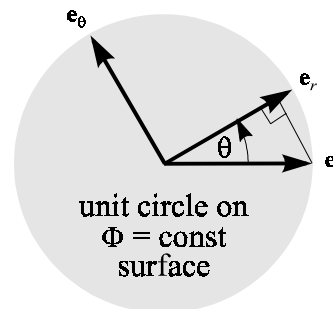
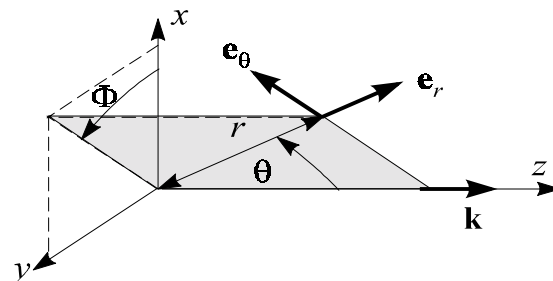
For a stationary sphere, no fluid entering the sphere means $dq = \mathbf{n} \cdot \mathbf{v} da = 0$, or

b.c. #2: $\mathbf{n} \cdot \mathbf{v} = v_r = 0 \text{ at } r=R$

Next we rewrite the b.c.'s in terms of the velocity potential in spherical coordinates. In terms of velocity potential, (19) becomes:

$$\nabla \phi = U\mathbf{k}$$

Now we need to translate \mathbf{k} into the unit vectors in spherical coordinates.*



Referring to the figure above (see page 8), we note that \mathbf{e}_r , \mathbf{e}_θ , and $\mathbf{e}_z = \mathbf{k}$ all lie in the same $\Phi = \text{const}$ plane (shaded region of left-hand figure above). If we shift all three unit vectors to the origin (recall that the origin is not part of the definition of any vector) and re-orient the $\Phi = \text{const}$ plane to coincide with the plane of the page, then we get the figure above at right, from trigonometry of the right triangle it is apparent that

$$\mathbf{k} = (\cos \theta) \mathbf{e}_r - (\sin \theta) \mathbf{e}_\theta$$

* Here Φ is the spherical coordinate, while ϕ is the velocity potential. For more on spherical coordinates, see BS&L p733.

Thus the b.c. can be written as

as $r \rightarrow \infty$:

$$\nabla\phi \rightarrow \underbrace{U \cos\theta}_{\frac{\partial\phi}{\partial r}} \mathbf{e}_r - \underbrace{U \sin\theta}_{\frac{1}{r} \frac{\partial\phi}{\partial\theta}} \mathbf{e}_\theta$$

Equating corresponding components:

$$\frac{\partial\phi}{\partial r} = U \cos\theta \quad \text{and} \quad \frac{1}{r} \frac{\partial\phi}{\partial\theta} = U \sin\theta$$

Integrating either PDE leads to

b.c. #1: $\phi \rightarrow U r \cos\theta + \text{const} \quad \text{as} \quad r \rightarrow \infty$

where we have arbitrarily selected a value of zero for this “const”.^{*} In order to translate b.c. #2, use tables (see [spherical coords](#)[♦]) to express the gradient in spherical coordinates:

$$\begin{aligned} \mathbf{v} &= \nabla\phi \\ v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi &= \frac{\partial\phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \mathbf{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial\phi}{\partial\Phi} \mathbf{e}_\phi \end{aligned}$$

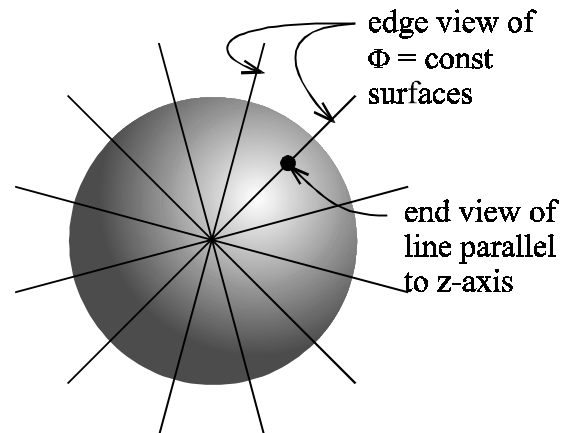
Dotting both sides by the unit vector $\mathbf{n} = \mathbf{e}_r$, then using b.c. #2:

b.c. #2: $v_r = \partial\phi/\partial r = 0 \text{ at } r=R$

We look for a solution which is independent of Φ :

$$\phi = \phi(r, \theta)$$

This implies that the fluid does not have any Φ -component of velocity. In other words, the trajectory of any fluid element remains entirely on a single $\Phi = \text{const}$ surface. The sketch at right shows some edge views of $\Phi = \text{const}$ planes (looking along the z-axis which passes through the center of the sphere).



From p740 of BS&L, we have $\nabla^2\phi$ in spherical coordinates:

^{*} Like most energies in thermodynamics, the reference state for potential is arbitrary and can be chosen solely for convenience.

[♦] http://www.andrew.cmu.edu/course/06-703/Vops_sph.pdf

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0 \quad (20)$$

$$\partial \phi / \partial r = 0 \text{ at } r=R \quad (21)$$

$$\phi = Ur \cos \theta \text{ as } r \rightarrow \infty \quad (22)$$

Although there are systematic procedures, like “separation of variables,” for solving P.D.E.'s which work in many cases, one should first check to see if there is a simple solution. The problem appears formidable, but notice that the trivial solution:

$$\phi = 0 ?$$

satisfies the P.D.E. and the b.c. at $r=R$. Unfortunately, it does not satisfy the b.c. at $r \rightarrow \infty$, so we will have to try another guess. Since the failure occurred with the b.c. at $r \rightarrow \infty$, we look there for the form of our second guess:

$$\phi = A r \cos \theta ?$$

This too satisfies the P.D.E. and the b.c. at $r \rightarrow \infty$ (provided $A=U$) but to satisfy the b.c. at $r=R$, A must be chosen as 0. This tells us that A should have different values at different r 's. So we try a third guess which is slightly more general than the second:

$$\phi(r, \theta) = f(r) \cos \theta \quad (23)$$

Substituting (23) into (20)-(22), we find that $\cos \theta$ cancels out, leaving:

$$r^2 f'' + 2r f' - 2f = 0$$

$$f' = 0 \text{ at } r=R$$

$$f = Ur \text{ as } r \rightarrow \infty$$

Thus we have reduced the problem of solving the P.D.E. to one of solving an O.D.E. We recognize this O.D.E. to be a Cauchy-Euler equation,* which always has at least one solution of the form $f = r^n$. The general solution turns out to be:

$$f(r) = Ar^{-2} + Br$$

Using the b.c.'s, we can evaluate A and B . The particular solution to this problem is:

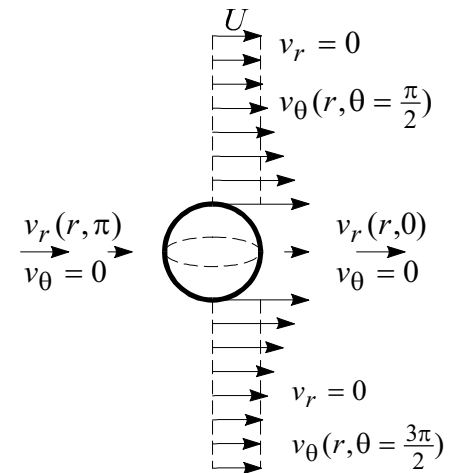
$$f(r) = U \left(r + \frac{1}{2} \frac{R^3}{r^2} \right)$$

Substituting this back into (23) leaves:

$$\phi(r, \theta) = U \left(r + \frac{1}{2} \frac{R^3}{r^2} \right) \cos \theta$$

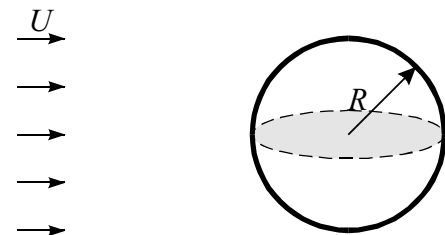
$$v_r = \frac{\partial \phi}{\partial r} = U \left[1 - \left(\frac{R}{r} \right)^3 \right] \cos \theta$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \left(1 + \frac{1}{2} \frac{R^3}{r^3} \right) \sin \theta$$



Notice that for $\theta = \pi$ ($\theta = 0$), $v_\theta = 0$ but v_r decreases (in absolute magnitude) from $-U$ at $r = \infty$ to 0 at $r = R$. At $\theta = \pi/2$, $v_r = 0$ but v_θ increases from U to $(3/2)U$. This increase is necessary to make up for the decrease in flow caused by the sphere blocking part of the flow path.

Having solved for the velocity profile, we can determine the pressure profile from Bernoulli's equation (18). Assuming steady state:



* The general form of an N^{th} -order Cauchy-Euler equation is $\sum_{n=0}^N a_n x^n \frac{d^n y}{dx^n} = 0$. At least one of the N linearly independent solutions has the form $y(x) = Ax^\alpha$. Substituting this form leads to

$$a_0 Ax^\alpha + \sum_{n=1}^N a_n \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)Ax^\alpha = 0$$

Dividing out the common factor Ax^α , we obtain an N^{th} -order polynomial for α . Each distinct root of the polynomial leads to a separate solution. In this example, the roots are $\alpha = -2$ and $+1$.

$$\frac{v^2}{2} + \frac{p}{\rho} + \phi_g = \text{const}$$

independent of position. The "const" can be evaluated using any point where we know both the velocity and the pressure. Suppose the pressure of the undisturbed fluid is known in the reference plane for the gravitational potential ($\phi_g = 0$):

$$\text{at } r \rightarrow \infty, \phi_g = 0: \quad p = p_\infty \text{ and } v^2 = U^2$$

$$\text{Thus} \quad \frac{v^2}{2} + \frac{p}{\rho} + \phi_g = \frac{U^2}{2} + \frac{p_\infty}{\rho}$$

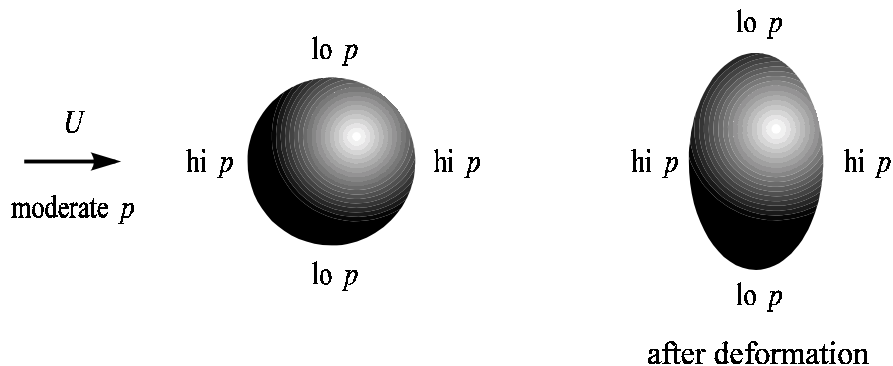
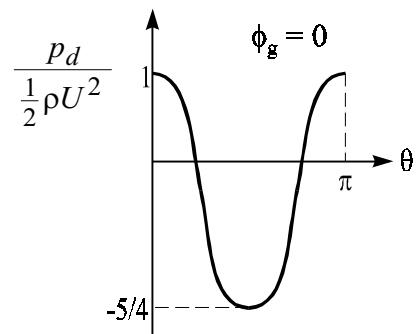
$$\text{or} \quad p(r, \theta) = p_\infty - \rho\phi_g + \frac{1}{2}\rho(U^2 - v^2)$$

$$\text{where} \quad v^2 = \mathbf{v} \cdot \mathbf{v} = [v_r(r, \theta)]^2 + [v_\theta(r, \theta)]^2$$

Substituting the known velocity profile, we obtain the pressure profile. Let's focus on the pressure profile over the surface of the sphere:

$$\text{for } r=R: \quad p(R, \theta) = \underbrace{p_\infty - \rho\phi_g}_{\substack{\text{hydrostatic head} \\ p_h(R, \theta)}} + \underbrace{\frac{1}{8}\rho U^2 (9\cos^2 \theta - 5)}_{\substack{\text{dynamic head} \\ p_d(R, \theta)}}$$

In the sketch at right, we plot the dynamic pressure (dropping the contribution from hydrostatic equilibrium). Note the location of regions having high and low pressure. The sphere is being pushed in at the poles and allowed to expand at the equator. This is why a large bubble rising through stagnant water tends to become distorted from spherical shape. Such bubbles tend to become extended in the horizontal plane and compressed in the vertical direction by the higher pressure.



d'Alembert's Paradox

What is the net force on a rigid sphere owing the pressure profile developed by potential flow around it? The answer turns out to be:

$$\mathbf{F}_p = -\oint_A \mathbf{n} p da = -\oint_A \mathbf{n} (p_h + p_d) da = -\underbrace{\oint_A \mathbf{n} p_h da}_{\rho \mathbf{g} V} - \underbrace{\oint_A \mathbf{n} p_d da}_{\mathbf{0}} = -\rho \mathbf{g} V \quad (24)$$

where V is the volume of the sphere.

Proof: first consider the contribution from dynamic pressure:

$$p_d \equiv \frac{1}{8} \rho U^2 (9 \cos^2 \theta - 5)$$

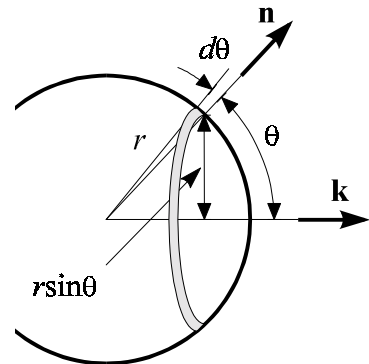
The \mathbf{n} in (24) is a unit vector normal to the surface, pointing outward. In spherical coordinates, with their origin at the center of the sphere, this vector is the unit vector in the r -direction:

$$\mathbf{n} = \mathbf{e}_r(\theta, \phi)$$

which direction depends on location on the surface. Although its length is a constant, its direction varies with position of the sphere; thus \mathbf{n} cannot be treated as a constant. Anticipating that any net force will be parallel to the direction of fluid flow, we dot both sides of (24) by \mathbf{k} :

$$F_p = \mathbf{k} \cdot \mathbf{F}_p = -\int_A \underbrace{\mathbf{k} \cdot \mathbf{n}}_{\cos \theta} p da$$

For the contribution from dynamic pressure, $p_d(r=R, \theta)$ depends solely on θ , so we choose the strip of width $R d\theta$ and length $2\pi R \sin \theta$ as our differential area da . On this strip θ is virtually constant.



$$\begin{aligned} F_{dp} &= -\int_0^\pi \cos \theta \underbrace{p_d(R, \theta)}_{f(\cos \theta)} (2\pi R \sin \theta) (R d\theta) \\ &= 2\pi R^2 \int_0^\pi \cos \theta f(\cos \theta) \underbrace{(-\sin \theta d\theta)}_{d \cos \theta} \\ &\stackrel{x=\cos \theta}{=} 2\pi R^2 \times \frac{1}{8} \rho U^2 \underbrace{\int_1^{-1} x (9x^2 - 5) dx}_0 = 0 \end{aligned}$$

The nonzero contribution from (24) comes from the hydrostatic head. We will leave this calculation as an exercise for the reader (HWK #4, Prob. 1). The net force due to pressure is

$$\mathbf{F}_p = - \underbrace{\int_{\text{sphere}} (p_\infty - \rho\phi_g) \mathbf{n} da}_{-\rho\mathbf{g}V} - \underbrace{\int_{\text{sphere}} \left[\frac{1}{8}\rho U^2 (9\cos^2\theta - 5) \right] \mathbf{n} da}_{\mathbf{0}} = -\rho\mathbf{g}V$$

The net force on the sphere is the sum of its weight and the net pressure force:

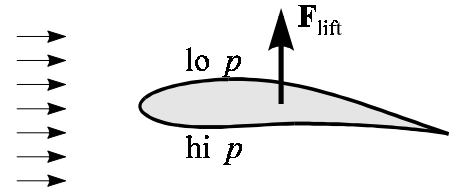
$$\mathbf{F}_p + \mathbf{F}_g = -\rho\mathbf{g}V + \rho_s\mathbf{g}V = (\rho_s - \rho)\mathbf{g}V \neq \mathbf{0}$$

which is the difference in weight of the sphere and the weight of the fluid displaced by it. Because the pressure force is independent of the speed of motion of the sphere through the fluid, the particle will continue to accelerate forever, without ever reaching a “terminal velocity.” Of course, experiments show that falling particles reach a terminal velocity which implies that the gravitational force is balanced by some other force. The other force is fluid drag (friction), which is not predicted by potential flow. This serious discrepancy between the predictions of potential flow theory and experiment is known as:

d'Alembert's ♣ paradox - potential flow predicts no drag but experiments indicate drag.

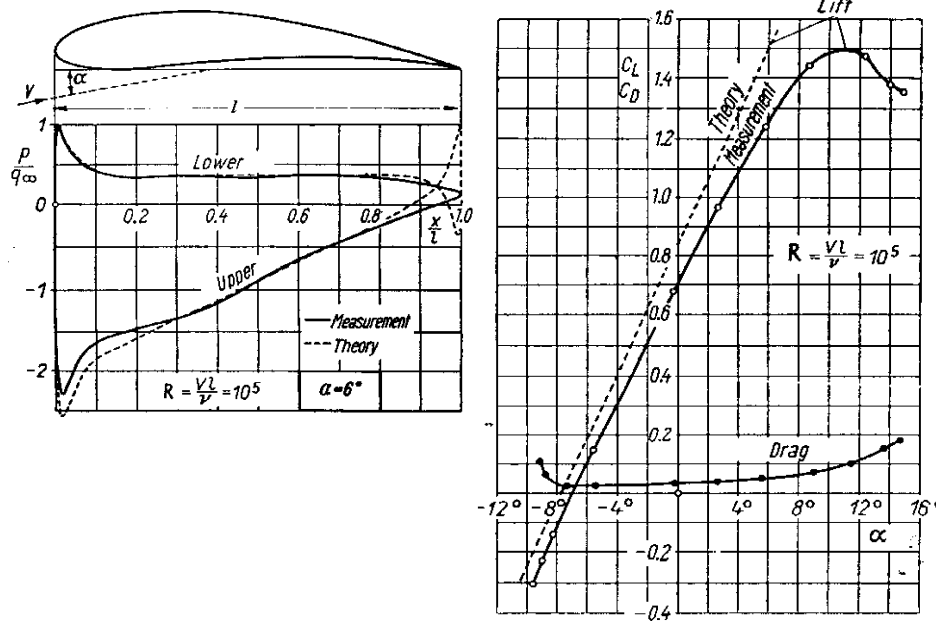
Despite this, potential flow is still useful:

Uses of potential flow – predicts lift (but not drag) on streamline objects moving through stagnant fluid at high Reynolds numbers (but still subsonic, i.e. $v \ll c$).



- correctly predicts $\mathbf{v}(\mathbf{r},t)$ and $p(\mathbf{r},t)$ except very near the surface of the object (i.e. inside boundary layer) and in wake [see pressure profile on airfoil shown below].
- for asymmetric shapes (e.g. airplane wings), it correctly predicts a lift. [See lift force shown below as a function of the angle of attack.]

♣ Jean le Rond d'Alembert (1717–83), French mathematician and philosopher, a leading figure of the Enlightenment. His treatise on dynamics (1743) enunciated d'Alembert's principle, which permitted the reduction of a problem in dynamics to one in statics. He did important work on the mechanics of rigid bodies, the motions of fluids and vibrating strings, and the three-body problem in celestial mechanics.



Figs. 1.12 and 1.13 taken from Schlichting, 6th ed., p22f.

Stream Function

Before we resolve d'Alembert's paradox by adding viscous forces, let's step back for a moment and review what we have accomplished for potential (or irrotational) flow. The mathematical problem might be stated as: find $\mathbf{v}(\mathbf{x}, t)$ such that:

$$\nabla \times \mathbf{v} = \mathbf{0} \quad (25)$$

and

$$\nabla \cdot \mathbf{v} = 0 \quad (26)$$

Eqs. (25) and (26) represent four partial differential equations in the components of the unknown vector field.* Recognizing that the solution is derivable from a potential allows us compress these four equations into one scalar equation in one unknown:

$$\mathbf{v} = \nabla \phi: \quad \nabla^2 \phi = 0$$

which is quite a remarkable trick. The potential is not the only scalar field which a vector field can be expressed in terms of. Velocity can also be expressed in terms of a **stream function**.

* Although it might appear that we have overspecified the problem by specifying both the divergence and curl (which represent four scalar equations in the three components of \mathbf{v}), this turns out not to be true. In general, both the divergence and curl must be specified throughout some region in space before the vector field can be determined in that region.

Potential (ϕ) — a scalar field whose relationship to \mathbf{v} is carefully selected to automatically satisfy irrotationality

$$\mathbf{v} = \nabla\phi \rightarrow \nabla \times \mathbf{v} = \mathbf{0}$$

Whereas the relation between velocity and scalar potential is chosen to automatically satisfy (25), the relationship between velocity and stream function is chosen to automatically satisfy (26):

$$\mathbf{v} = \mathbf{f}(\psi) \rightarrow \nabla \cdot \mathbf{v} = 0$$

Stream Function (ψ) — a scalar field whose relationship to \mathbf{v} is carefully selected to automatically satisfy continuity.

It turns out that it is sufficient to express \mathbf{v} as the curl of another vector. According to Identity C.6, the divergence of the curl of any vector is zero (See HWK #2, Prob. 4d):

$$\mathbf{v} = \nabla \times \mathbf{u} \rightarrow \nabla \cdot \mathbf{v} = 0$$

A vector which can be expressed as the curl of another vector is said to be **solenoidal**. \mathbf{u} is called the **vector potential** of \mathbf{v} . Of course, knowing that $\mathbf{v} = \nabla \times \mathbf{u}$ isn't always of much help because we just trade one unknown vector for another. Fortunately, there are several broad classes of flows for which the form of the vector potential is known:

Two-D Flows

When nothing happens along one of the three directions in R.C.C.S., we have **2-D flow**:

$$\mathbf{v} = v_x(x,y)\mathbf{i} + v_y(x,y)\mathbf{j}$$

or

$$v_z = 0, \partial/\partial z = 0$$

$$\mathbf{v} = \nabla \times [\psi(x,y)\mathbf{e}_z] \stackrel{\text{identity E.3}}{=} \nabla\psi + \underbrace{\psi \nabla \times \mathbf{e}_z}_{\mathbf{0}}$$

For such a flow,*

$$= \left(\frac{\partial\psi}{\partial x} \mathbf{e}_x + \frac{\partial\psi}{\partial y} \mathbf{e}_y \right) \times \mathbf{e}_z = \frac{\partial\psi}{\partial x} \underbrace{\mathbf{e}_x \times \mathbf{e}_z}_{-\mathbf{e}_y} + \frac{\partial\psi}{\partial y} \underbrace{\mathbf{e}_y \times \mathbf{e}_z}_{\mathbf{e}_x} \quad (27)$$

In terms of its scalar components, the velocity is:

* “Identity E.3” in this equation refers to one of the mathematical identities summarized on the handout titled “Useful Identities in Vector Notation”.

$$v_x = \frac{\partial \psi}{\partial y} \quad v_y = -\frac{\partial \psi}{\partial x} \quad v_z = 0 \quad (28)$$

Next we substitute this form for \mathbf{v} into (26):

$$\nabla \cdot \mathbf{v} = \nabla \cdot [\nabla \times (\psi \mathbf{e}_z)] \stackrel{\text{identity C.6}}{=} 0$$

which automatically satisfies continuity (26), for any choice of $\psi(x,y)$. The scalar field $\psi(x,y)$ is called the **stream function**. For irrotational flow, the problem would be to determine $\psi(x,y)$ such that (25) is satisfied:

$$\nabla \times [\nabla \times (\psi \mathbf{e}_z)] = \mathbf{0}$$

We can reduce this to a scalar equation. Using identity E.5 from handout:

$$\nabla \times [\nabla \times (\psi \mathbf{e}_z)] = \nabla [\nabla \cdot (\psi \mathbf{e}_z)] - \nabla^2 (\psi \mathbf{e}_z)$$

but $\nabla \cdot (\psi \mathbf{e}_z) = \partial \psi / \partial z = 0$

and $\nabla^2 (\psi \mathbf{e}_z) = (\nabla^2 \psi) \mathbf{e}_z$

Thus $\nabla \times \mathbf{v} = -(\nabla^2 \psi) \mathbf{e}_z$

So for irrotational flow, the streamfunction must also satisfy Laplace's equation:

$$\nabla \times \mathbf{v} = \mathbf{0}: \quad \nabla^2 \psi = 0$$

Unlike the scalar potential, the streamfunction can be used in all 2-D flows, including those for which the flow is not irrotational. Indeed, we will use the streamfunction to solve Stokes flow of a viscous fluid around a sphere, in which the fluid is not even ideal.

AXISYMMETRIC FLOW (CYLINDRICAL)

Another general class of flows for which a streamfunction exists is axisymmetric flow. In cylindrical coordinates (r, θ, z) , this corresponds to:

$$\mathbf{v} = v_r(r, z) \mathbf{e}_r + v_z(r, z) \mathbf{e}_z$$

or $v_\theta = 0$ and $\partial / \partial \theta = 0$

Then $\nabla \cdot \mathbf{v} = 0$ can be satisfied by seeking \mathbf{v} of the form:

$$\mathbf{v} = \nabla \times [f(r, z) \mathbf{e}_\theta]$$

or

$$\mathbf{v} = \nabla \times \left[\frac{\psi(r, z)}{r} \mathbf{e}_\theta \right]$$

The second expression usually leads to somewhat simpler expressions for $\nabla \times \mathbf{v}$ and is the one used in the table on p131 of BS&L:

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

Computing the curl in cylindrical coordinates and setting it equal to zero leads to the following PDE in ψ (the details are left as an exercise):

$$\nabla \times \mathbf{v} = -\left(\frac{E^2 \psi}{r} \right) \mathbf{e}_\theta = \mathbf{0} \quad \longrightarrow \quad E^2 \psi = 0$$

where

$$E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}$$

Note that $E^2 \psi \neq \nabla^2 \psi$:

$$\nabla^2 \psi(r, z) = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}$$

AXISYMMETRIC FLOW (SPHERICAL)

In spherical coordinates (r, θ, ϕ) , axisymmetric flow means

$$\mathbf{v} = v_r(r, \theta) \mathbf{e}_r + v_\theta(r, \theta) \mathbf{e}_\theta$$

or

$$v_\phi = 0 \text{ and } \partial/\partial \phi = 0$$

where ϕ is the azimuthal angle. Then $\nabla \cdot \mathbf{v} = 0$ can be satisfied by seeking \mathbf{v} of the form:

$$\mathbf{v} = \nabla \times [\psi'(r, \theta) \mathbf{e}_\phi]$$

or

$$\mathbf{v} = \nabla \times \left[\frac{\psi(r, \theta)}{r \sin \theta} \mathbf{e}_\phi \right] = \underbrace{\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}}_{v_r} \mathbf{e}_r - \underbrace{\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}}_{v_\theta} \mathbf{e}_\theta$$

Again, the second expression is the one used in the table on p131 of BS&L (except the signs are reversed). Taking the curl (HWK #4, Prob. 6a):

$$\nabla \times \mathbf{v} = -\frac{\mathbf{e}_\phi}{r \sin \theta} E^2 \psi = \mathbf{0}$$

which requires

$$E^2\psi = 0$$

where

$$E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial\psi}{\partial\theta} \right)$$

Note that $E^2\psi \neq \nabla^2\psi$:

$$\nabla^2\psi(r, \theta) = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r} \frac{\partial\psi}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right)$$

ORTHOGONALITY OF $\psi=\text{CONST}$ AND $\phi=\text{CONST}$

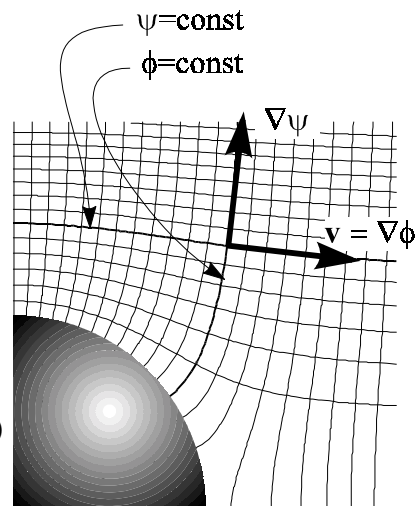
A contour on which $\psi=\text{const}$ is called a **streamline**. A contour on which $\phi=\text{const}$ is called a **iso-potential line**. It turns out that these two contours are orthogonal at any point in the fluid. To see this first note that

$$\mathbf{v} = \nabla\phi$$

From the geometric meaning of gradient (see page 6 of Notes), we know that $\nabla\phi$ and hence \mathbf{v} is normal to a $\phi=\text{const}$ surface (see figure at right). Second, recall that \mathbf{v} can also be written in terms of streamfunction as

$$\text{for R.C.C.S.} \quad \mathbf{v} = \nabla\psi \times \mathbf{k} \quad (29)$$

Recall that the cross product is a vector which is orthogonal to two vectors being multiplied. Thus \mathbf{v} , $\nabla\psi$, and \mathbf{k} are mutually orthogonal. Since \mathbf{v} and $\nabla\phi$ point in the same direction, $\nabla\phi$ and $\nabla\psi$ must also be orthogonal.



Potential Flow Around Sphere. **Lines lie in plane of page, which could be any $\Phi=\text{const. plane}$.**

STREAMLINES, PATHLINES AND STREAKLINES

Streamline - a contour in the fluid whose tangent is everywhere parallel to \mathbf{v} at a given instant of time.

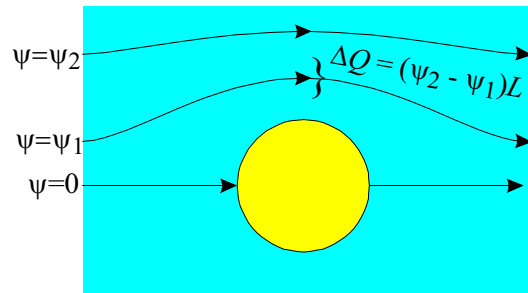
Path Line - trajectory swept out by a fluid element.

Streak Line - a contour on which lie all fluid elements which earlier past through a given point in space (e.g. dye trace)

For steady flows, these three definitions describe the same contour but, more generally, they are different.

PHYSICAL MEANING OF STREAMFUNCTION

The precise meaning of streamfunction is somewhat different for 2-D and axisymmetric flows. Let's focus on 2-D flows normal to a cylinder (not necessarily with circular cross-section, axis corresponds to z -axis in R.C.C.S.). To motivate the somewhat lengthy analysis which follows, we first state the physical meaning. First we observe that material points follow trajectories which can be described as $\psi = \text{const}$. Three such trajectories are shown at right which lie in the xy -plane. When these trajectories are (mathematically) translated along the z -axis a distance L they sweep-out $\psi = \text{const}$ surfaces.

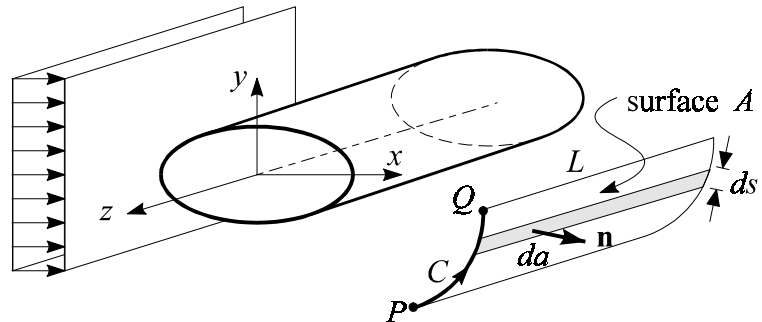


No fluid crosses these surfaces: there are like the walls of a tube. Since no fluid leaves or enters this “tube”, conservation of mass means the mass flowrate must be a constant at any point along the tube. For an incompressible fluid, the volumetric flowrate is also constant. Suppose we denote the volumetric flowrate between any two of these $\psi = \text{const}$ surfaces as ΔQ ; then it turns out that

$$\frac{\Delta Q}{L} = \psi_2 - \psi_1$$

Thus ψ might be interpreted as the volumetric flowrate per unit length between this particular streaming surface and the one corresponding to $\psi = 0$.

Now let's show this. Consider an arbitrary open contour (C) in the xy -plane, cutting across the flow. Next, consider the surface (A) formed by translating this contour a distance L parallel to the z -axis. The volumetric flowrate across A is:



$$Q = \int_A \mathbf{n} \cdot \mathbf{v} da$$

where \mathbf{n} is a unit normal to A . Since nothing changes with z , we choose a short segment of the contour, having length ds and of length L as our differential area element.

$$da = L ds,$$

The flowrate becomes:

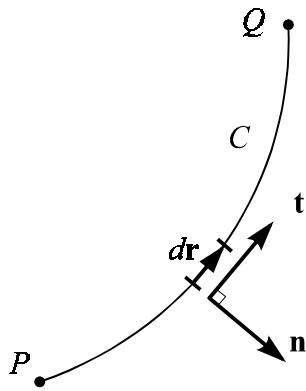
$$\frac{Q}{L} = \int_C \mathbf{n} \cdot \mathbf{v} ds = \int_C \nabla \psi \cdot d\mathbf{r} \quad (30)$$

where \mathbf{n} is now normal to C and lies in the xy -plane. The key to this proof is deriving the second equality in the expression above.

First, \mathbf{v} can be expressed in terms of the gradient of the streamfunction using (27):

$$\mathbf{v} = \nabla\psi \times \mathbf{k} \quad (27)$$

Both \mathbf{v} and $\nabla\psi$ lie in the plane of the page, whereas \mathbf{k} is perpendicular to this plane (points out of page). Post-crossing $\nabla\psi$ by the unit vector \mathbf{k} does not change the magnitude but rotates $\nabla\psi$ by 90° clockwise. If instead, we pre-crossed by \mathbf{k} we would rotate $\nabla\psi$ by 90° counter-clockwise. In either case, the cross product of \mathbf{k} and $\nabla\psi$ is a vector lying in the plane of the page and of the same magnitude as $\nabla\psi$.



The other term in the integrand of (30) is $\mathbf{n} ds$, where ds is the magnitude of a differential displacement along the contour, which we will call $d\mathbf{r}$:

$$ds = |d\mathbf{r}|$$

Since \mathbf{n} is a unit vector $\mathbf{n} ds$ has the same magnitude as $d\mathbf{r}$ but is rotated by 90° . Both $\mathbf{n} ds$ and $d\mathbf{r}$ lie in the plane of the page. Just as in (27), we can rotate one vector into the other by crossing with \mathbf{k} :

$$d\mathbf{r} \times \mathbf{k} = \mathbf{n} ds \quad (31)$$

Substituting (27) and (31) into (30) yields

$$\mathbf{n} \cdot \mathbf{v} ds = \mathbf{v} \cdot \mathbf{n} ds = (\nabla\psi \times \mathbf{k}) \cdot (d\mathbf{r} \times \mathbf{k}) = \nabla\psi \cdot d\mathbf{r} = d\psi \quad (32)$$

The 3rd equality above says that the dot product of the two rotated vectors is the same as the dot product of the two vectors without rotation (since they are both rotated by the same amount). (32) into (30) yields:

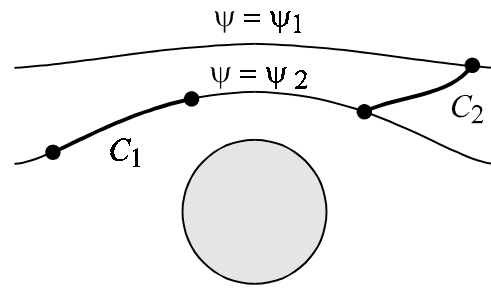
$$\frac{Q}{L} = \int_{C_{P \rightarrow Q}} \mathbf{n} \cdot \mathbf{v} ds = \int_{C_{P \rightarrow Q}} d\psi = \psi(Q) - \psi(P)$$

To extract the physical meaning of this results, consider two contours, denoted as C_1 and C_2 in the figure at right. Notice that if C_1 coincides with a streamline, the velocity is parallel to the contour at every point any no fluid crosses it:

then: $Q/L = 0$

and $\psi(Q) = \psi(P) = \psi_2$

Thus $\psi = \text{const. along a streamline}$



On the other hand, if the contour cuts across two streamlines (see contour C_2 in figure at right), then the difference in value of ψ corresponding to two different streamlines is just the volumetric flowrate of fluid held between the two streamlines (per unit length in the z -direction):

$$\Delta\psi = \Delta Q/L$$

INCOMPRESSIBLE FLUIDS

By “incompressible fluid” we are usually referring to the assumption that the fluid’s density is not a significant function of time or of position. In other words,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

can be replaced by

$$\nabla \cdot \mathbf{v} = 0$$

For steady flows, $\partial \rho / \partial t = 0$ already and the main further requirement is that density gradients be negligible

$$\nabla \cdot (\rho \mathbf{v}) \stackrel{\text{Identity C.1}}{=} \rho(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho \approx \rho(\nabla \cdot \mathbf{v})$$

Since flow causes the pressure to change, we might expect the fluid density to change — at least for gases. As we shall see shortly, gases as well as liquids can be treated as incompressible for some kinds of flow problems. Conversely, in other flow problems, neither gas nor liquid can be treated as incompressible. So what is the real criteria?

For an ideal fluid (i.e. no viscous dissipation to cause ∇T), density variations come about primarily because of pressure variations. For an isentropic expansion, the compressibility of the fluid turns out to be:

$$\left(\frac{\partial \rho}{\partial p} \right)_S = \frac{1}{c^2}$$

where

c = speed of sound in the fluid

Thus changes in density caused by changes in pressure can be estimated as

$$\Delta \rho \approx \frac{1}{c^2} \Delta p \quad (33)$$

According to Bernoulli's equation, pressure changes for steady flow are related to velocity changes:

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{const.} \quad \text{or} \quad \Delta p = -\frac{1}{2} \rho \Delta v^2 \quad (34)$$

(34) into (33):

$$\Delta \rho \approx -\frac{\rho}{2} \frac{\Delta v^2}{c^2}$$

The largest change in density corresponds to the largest change in v^2 , which is $v_{\max}^2 - 0$:

$$\left(\left| \frac{\Delta \rho}{\rho} \right| \right)_{\max} = \frac{1}{2} \left(\frac{v_{\max}}{c} \right)^2$$

If the fraction change in density is small enough, then it can be neglected:

Criteria 1:

$$v_{\max} \ll c$$

for air at sea level:

$$c = 342 \text{ m/s} = 700 \text{ mph}$$

for distilled water at 25°C:

$$c = 1500 \text{ m/s} = 3400 \text{ mph}$$

For unsteady flows, a second criteria must be met:

Criteria 2:

$$\tau \gg \frac{l}{c}$$

where

τ = time over which significant changes in v occur

l = distance over which changes in v occur

l/c = time for sound to propagate a distance l

For steady flow $\tau = \infty$ and Criteria 2 is always satisfied.

Any fluid can be considered incompressible if both criteria are met.

Viscous Fluids

To resolve d'Alembert's paradox, we need to introduce another force into our force balance. This force can be thought of as resulting from friction between the fluid elements. Friction gives rise to **viscous heating** which represent an irreversible conversion of mechanical energy into heat. Indeed, friction is the main difference between an ideal fluid and a real fluid.

friction \rightarrow irreversible deformation \rightarrow *nonideal* flow

TENSORIAL NATURE OF SURFACE FORCES

Friction is a surface force like hydrostatic pressure, but unlike pressure, friction is not isotropic:

isotropic - independent of orientation (direction)

We say that pressure is isotropic because the magnitude of the pressure force is independent of the orientation of the surface on which it acts. Recall:

$$d\mathbf{F}_p = -p d\mathbf{a}$$

$$|d\mathbf{F}_p| = p da$$

which is independent of \mathbf{n} (orientation). Viscous friction does not have this property:

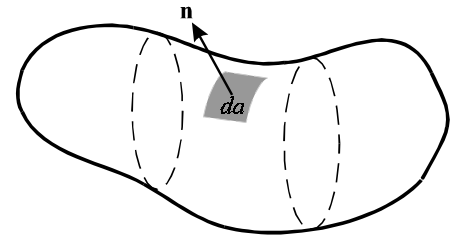
$$|d\mathbf{F}_f| \text{ depends on } \mathbf{n}$$

but its magnitude is proportional to da :

$$|d\mathbf{F}_f| \propto da$$

Thus it makes sense to talk about the force per unit area, which we usually call **pressure**. In the more general case in which the magnitude of the force per unit area might depend on the orientation of the surface, we use the name **stress**. In general, all surface forces can be lumped together and written as

$$d\mathbf{F}_{surf} \equiv \mathbf{t}(\mathbf{n}) da$$

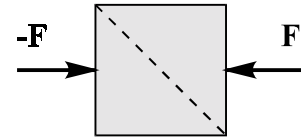


where $\mathbf{t}(\mathbf{n})$ denotes the surface force per unit area acting on the body through the surface element whose orientation is given by the unit outward normal \mathbf{n} . We call this quantity the **stress vector**.^{*} In what follows, I will try to convince you that the effect of orientation of the surface can be expressed as

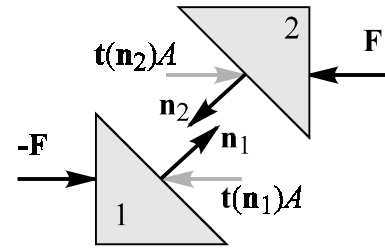
$$\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot \underline{\mathbf{T}}$$

where $\underline{\mathbf{T}}$ is called the **stress tensor**. While $\underline{\mathbf{T}}$ depends on position and possibly time, it does not depend on the orientation of the differential surface element da , which is given by \mathbf{n} . $\underline{\mathbf{T}}$ is sometimes call the “state of stress” of the fluid.

First, let’s try to understand better what we mean when we say a material experiences some “stress”. Consider a block of some material which bears some externally applied *equilibrium* load (see figure at right). By “equilibrium” I mean there is no net force and no net torque applied to the body: thus it is at **mechanical equilibrium** and has no tendency to accelerate. The material might be a fluid or a solid, but what we are about to say is easier to imagine if we think of material as a solid block.



Now consider some mathematical surface inside the material (indicated by the dotted line in the figure above). What are the forces exerted across this mathematical surface? To answer this question, suppose we actually separated the block into two pieces by physically cutting along this surface without changing the loading. Block “1” now experiences an unbalanced load $-\mathbf{F}$ while block “2” experiences an unbalanced load $+\mathbf{F}$. Once separated, both blocks would tend to accelerate in opposite directions.



Why don’t the two halves accelerate when connected? Apparently, half “2” must have exerted a force on half “1” which we denote as $\mathbf{t}(\mathbf{n}_1)A$, where A is the area of the cut face, and which equals

$$\mathbf{t}(\mathbf{n}_1)A = +\mathbf{F}$$

while half “1” exerted an equal but opposite force on half “2”

$$\mathbf{t}(\mathbf{n}_2)A = -\mathbf{F} = -\mathbf{t}(\mathbf{n}_1)A \quad (35)$$

When expressed per unit area, this internal force between the two halves (when they are physically connected) is what we mean by the “stress” experienced by the material under load. Given that $\mathbf{n}_1 = -\mathbf{n}_2$, (35) tells us that

^{*}see Whitaker, *Introduction to Fluid Mechanics*, Chapt. 4.

$$\mathbf{t}(\mathbf{n}_2) = -\mathbf{t}(-\mathbf{n}_2)$$

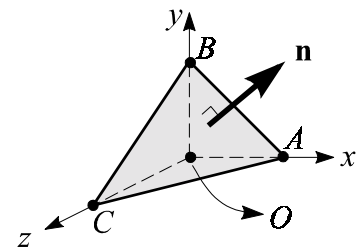
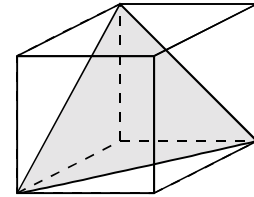
More generally, for any mathematically surface having orientation \mathbf{n} , we can write $\mathbf{t}(\mathbf{n}) = -\mathbf{t}(-\mathbf{n})$

or
$$\mathbf{t}(-\mathbf{n}) = -\mathbf{t}(\mathbf{n}) \quad (36)$$

In essence, this is just a statement of Newton's Third Law (for every action, there is an equal but opposite reaction).

Now let's generalize to three dimensions. Suppose we know the distribution of stress (i.e. the "loading") on all six faces of a block of material. Furthermore, suppose this loading is an equilibrium loading (no net force and no net torque on the block). Let's try to calculate the stress on a mathematical surface cutting through the material at an arbitrary angle. Let the orientation of the mathematical surface be given by the unit normal \mathbf{n} .

Problem: given the surface stress on mutually perpendicular planes [i.e. given $\mathbf{t}(\mathbf{i})$ for $x=\text{const}$ plane, $\mathbf{t}(\mathbf{j})$ for $y=\text{const}$ plane, and $\mathbf{t}(\mathbf{k})$ for $z=\text{const}$ plane], calculate the surface stress on a plane of arbitrary orientation specified by the unit normal \mathbf{n} .



Given: $\mathbf{t}(\mathbf{i})$, $\mathbf{t}(\mathbf{j})$, and $\mathbf{t}(\mathbf{k})$

Find: $\mathbf{t}(\mathbf{n})$

Solution: We choose the tetrahedron $ABCO$ as our "system." For the surface forces to be balanced*

$$\oint_A \mathbf{t}(\mathbf{n}) da = \mathbf{0}$$

We evaluate this surface integral by subdividing the surface A into the four faces of the tetrahedron: planes ABC , BCO , AOC , and ABO . For each surface, we need to evaluate the outward normal \mathbf{n} , the stress vector \mathbf{t} and the surface area A . The results are tabulated below:

*More generally, in our force balance, we should include body forces and inertia as well as surfaces forces. If we let $V \rightarrow 0$ then the volume integrals for body forces and inertia vanish more rapidly than surface forces. Thus for the surface A enclosing a tiny volume V , we must still require $\oint_A \mathbf{t} da = \mathbf{0}$ for the surface of every differential volume element.

Plane	Outward Normal	Area	Stress Vector
ABC	\mathbf{n}	A	$\mathbf{t}(\mathbf{n})$
BCO	$-\mathbf{i}$	$\mathbf{n} \cdot \mathbf{i} A$	$\mathbf{t}(-\mathbf{i})$
AOC	$-\mathbf{j}$	$\mathbf{n} \cdot \mathbf{j} A$	$\mathbf{t}(-\mathbf{j})$
ABO	$-\mathbf{k}$	$\mathbf{n} \cdot \mathbf{k} A$	$\mathbf{t}(-\mathbf{k})$

Applying the Mean Value Theorem to our surface integral:

$$\oint_A \mathbf{t} da = \mathbf{0} = \left[\langle \mathbf{t}(\mathbf{n}) \rangle + (\mathbf{n} \cdot \mathbf{i}) \langle \mathbf{t}(-\mathbf{i}) \rangle + (\mathbf{n} \cdot \mathbf{j}) \langle \mathbf{t}(-\mathbf{j}) \rangle + (\mathbf{n} \cdot \mathbf{k}) \langle \mathbf{t}(-\mathbf{k}) \rangle \right] A$$

Dividing by A cancels the A out of our expression for the integral. Taking the limit as all the dimensions of the block vanish (i.e. as $A \rightarrow 0$) allows us to replace the unknown averages with their limit, which is the value of the vector at the point that the tetrahedron collapses about.

$$\mathbf{t}(\mathbf{n}) + (\mathbf{n} \cdot \mathbf{i}) \mathbf{t}(-\mathbf{i}) + (\mathbf{n} \cdot \mathbf{j}) \mathbf{t}(-\mathbf{j}) + (\mathbf{n} \cdot \mathbf{k}) \mathbf{t}(-\mathbf{k}) = \mathbf{0}$$

or
$$\mathbf{t}(\mathbf{n}) = -(\mathbf{n} \cdot \mathbf{i}) \mathbf{t}(-\mathbf{i}) - (\mathbf{n} \cdot \mathbf{j}) \mathbf{t}(-\mathbf{j}) - (\mathbf{n} \cdot \mathbf{k}) \mathbf{t}(-\mathbf{k}) \quad (37)$$

Next we apply (36). Of course, we can replace \mathbf{n} in (36) with \mathbf{i} or \mathbf{j} or \mathbf{k} . Using (36) in (37):

$$\mathbf{t}(\mathbf{n}) = (\mathbf{n} \cdot \mathbf{i}) \mathbf{t}(\mathbf{i}) + (\mathbf{n} \cdot \mathbf{j}) \mathbf{t}(\mathbf{j}) + (\mathbf{n} \cdot \mathbf{k}) \mathbf{t}(\mathbf{k})$$

Recalling the definition of dyadic product, we could re-write this expression as

$$\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{i} \mathbf{t}(\mathbf{i}) + \mathbf{n} \cdot \mathbf{j} \mathbf{t}(\mathbf{j}) + \mathbf{n} \cdot \mathbf{k} \mathbf{t}(\mathbf{k})$$

$$\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot [\mathbf{i} \mathbf{t}(\mathbf{i}) + \mathbf{j} \mathbf{t}(\mathbf{j}) + \mathbf{k} \mathbf{t}(\mathbf{k})] = \mathbf{n} \cdot \underline{\underline{\mathbf{T}}}$$

The sum (inside square brackets) of these three dyadic products is some second-rank tensor, which is independent of \mathbf{n} . We denote this tensor by $\underline{\underline{\mathbf{T}}}$.

Significance: to calculate the surface force on some differential surface area da having orientation \mathbf{n} , we just multiply this expression for the stress by the area:

$$d\mathbf{F}_{surf} = \mathbf{n} \cdot \underline{\underline{\mathbf{T}}} da \quad (38)$$

where $\underline{\underline{\mathbf{T}}}$ is called the **stress tensor** and $d\mathbf{F}_{surf}$ represents the net surface force (all contributions) acting **on** the body whose **outward** normal is \mathbf{n} . While $\underline{\underline{\mathbf{T}}}$ does not depend on \mathbf{n} , it might depend on position inside a solid which is nonuniformly stressed; thus

$$\underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}(\mathbf{r})$$

represents the **state of stress** of the fluid. In terms of the components of the tensor, we say that

$T_{ij} = j^{\text{th}}$ component of the force acting on a $r_i = \text{const}$ surface

To make this more understandable, it might help to express $\underline{\mathbf{T}}$ in terms of other variables in a familiar problem, say an ideal fluid.

For an ideal fluid:

$$\underline{\mathbf{T}} = -p\underline{\mathbf{I}}$$

To show that this is correct, we will compute the surface force and show that it reduces to the familiar force due to pressure:

$$d\mathbf{F}_{surf} = \mathbf{n} \cdot \underline{\mathbf{T}} da = \mathbf{n} \cdot (-p\underline{\mathbf{I}}) da = -p(\mathbf{n} \cdot \underline{\mathbf{I}}) da = -p \mathbf{n} da = d\mathbf{F}_p$$

GENERALIZATION OF EULER'S EQUATION

Recall that Euler's equation was derived by applying Newton's Second Law ($F=ma$) to any fluid element. To generalize this result to include friction, we replace the "pressure force" by the surface force.

Instead of:

$$\int_V \rho \frac{D\mathbf{v}}{Dt} dV = \mathbf{F}_g + \mathbf{F}_p = \int_V \rho \mathbf{g} dV - \int_A \mathbf{n} p da$$

we have:

$$\int_V \rho \frac{D\mathbf{v}}{Dt} dV = \mathbf{F}_g + \mathbf{F}_{surf} = \int_V \rho \mathbf{g} dV + \int_A \mathbf{n} \cdot \underline{\mathbf{T}} da$$

Applying the Divergence Theorem and combining the three volume integrals:

$$\int_V \left(\rho \frac{D\mathbf{v}}{Dt} - \rho \mathbf{g} - \nabla \cdot \underline{\mathbf{T}} \right) dV = \mathbf{0}$$

Since this must result for all choices of V in some region, the integrand must vanish at every point in that region, or:

$$\frac{D\mathbf{v}}{Dt} = \mathbf{g} + \frac{1}{\rho} \nabla \cdot \underline{\mathbf{T}} \quad (39)$$

which is a more generalized version of *Euler's Equation*.

For an ideal fluid:

$$\underline{\mathbf{T}} = -p\underline{\mathbf{I}}$$

and

$$\nabla \cdot \underline{\mathbf{T}} = -\nabla \cdot (p\underline{\mathbf{I}}) \stackrel{\text{identity C.11}}{=} -\nabla p$$

For a proof of identity C.11, see Hwk #3, Prob. 4. For a real fluid, there is an additional contribution to $\underline{\underline{\mathbf{T}}}$ from friction:

For a real fluid:
$$\underline{\underline{\mathbf{T}}} = -p\underline{\underline{\mathbf{I}}} + \underline{\underline{\boldsymbol{\tau}}} \quad (40)$$

where $\underline{\underline{\boldsymbol{\tau}}}$ is called the *viscous stress* or the *deviatoric stress* (since it represents the deviation from ideal). Note that:

CAUTION: $\underline{\underline{\boldsymbol{\tau}}}$ in these notes (and in Whitaker) \rightarrow $-\underline{\underline{\boldsymbol{\tau}}}$ in BS&L

By convention, τ_{ij} in these notes is positive for *tensile* stresses (which result from stretching a solid rod) whereas τ_{ij} in BS&L is positive for *compressive* stresses (which result from putting a fluid under pressure). Although BS&L's notation might make more sense for fluids (which usually do not experience tensile stresses), we will use the other convention because it is more commonly used in continuum mechanics: in particular, this is the convention used by Whitaker.

When written in terms of scalar components, the above equation between $\underline{\underline{\mathbf{T}}}$ and $\underline{\underline{\boldsymbol{\tau}}}$ represents 9 equations in 10 unknowns (the nine components of $\underline{\underline{\boldsymbol{\tau}}}$ plus p). Clearly p and $\underline{\underline{\boldsymbol{\tau}}}$ cannot yet be uniquely determined, given the state of stress $\underline{\underline{\mathbf{T}}}$. One additional relationship is required. For a real fluid, we somewhat arbitrarily define p as the average of the three diagonal components of $\underline{\underline{\mathbf{T}}}$:

$$p \equiv -\frac{1}{3}(\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}}) \quad (41)$$

where
$$\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}} = \left(\sum_i \sum_j T_{ij} \mathbf{e}_i \mathbf{e}_j \right) : \left(\sum_k \mathbf{e}_k \mathbf{e}_k \right) = \sum_i \sum_j \sum_k T_{ij} \underbrace{(\mathbf{e}_i \cdot \mathbf{e}_k)}_{\delta_{ik}} \underbrace{(\mathbf{e}_j \cdot \mathbf{e}_k)}_{\delta_{jk}} = \sum_k T_{kk}$$

or, in Cartesian coords.:
$$\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}} = T_{xx} + T_{yy} + T_{zz}$$

In any coordinate system, $\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}}$ is called the *trace* of $\underline{\underline{\mathbf{T}}}$. Then p can be thought of as the isotropic contribution to stress (that part of the normal stress which acts equally in all directions) while $\underline{\underline{\boldsymbol{\tau}}}$ represents the remainder, or the nonisotropic part. (40) might be regarded as decomposing $\underline{\underline{\boldsymbol{\tau}}}$ into an isotropic part and a nonisotropic part, which is a common thing to do with tensors.

The choice of p made in (41) also represents the *thermodynamic pressure*; i.e. the hydrodynamic p is now the same as the p appearing in the thermodynamic *equation of state*:

$$p = p(p, T)$$

To summarize, we decompose the state of stress into two contributions: an isotropic pressure:

$$p = -\frac{1}{3}(\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}})$$

and a deviatoric stress:
$$\underline{\underline{\boldsymbol{\tau}}} = \underline{\underline{\mathbf{T}}} - (1/3)(\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}$$

Taking the divergence of (40):
$$\nabla \cdot \underline{\underline{\mathbf{T}}} = -\nabla p + \nabla \cdot \underline{\underline{\boldsymbol{\tau}}}$$

and substituting it into Euler's Eq. (39):

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + \underbrace{\nabla \cdot \boldsymbol{\tau}}_{\text{new for real fluids}}$$

MOMENTUM FLUX

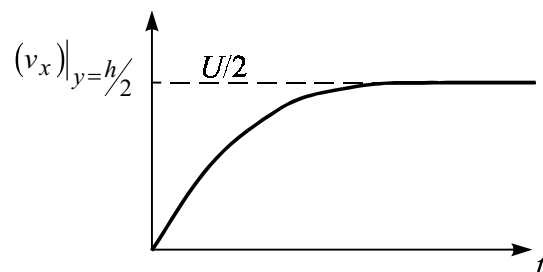
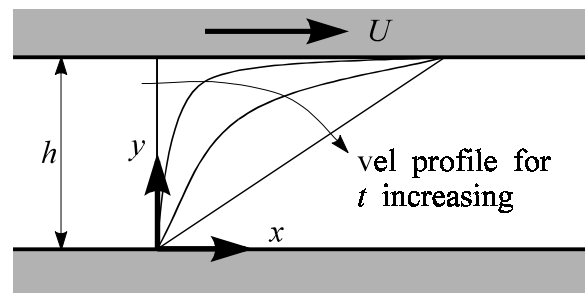
Up to this point we have spoken of $\underline{\mathbf{T}}$ as the stress on the fluid. $\underline{\mathbf{T}}$ can also be thought of as **momentum flux**. To convince yourself that momentum is being transported (like heat and mass), consider the problem of **unsteady simple shear flow**. At time $t=0$, an initially quiescent fluid confined between infinite parallel plates is disturbed by imposing motion on the upper plate. The velocity profile gradually develops into linear shear flow.

Note that, like all other fluid elements, the fluid element at $y=h/2$ undergoes acceleration:

$$\partial v_x / \partial t > 0 \text{ at } y=h/2$$

In other words, the fluid element is gaining momentum. How can it acquire momentum? Answer: momentum is transported to $y=h/2$ from above through friction between fluid elements. There are two directions associated with transport of momentum:

- 1) direction of the momentum being transported (in this example, x -momentum is transported)
- 2) direction in which the momentum is transported (y)



For this reason momentum flux must be a 2nd rank tensor. It turns out that:

$$-\underline{\mathbf{T}} = \text{diffusive flux of momentum}^*$$

$$\rho \mathbf{v} \mathbf{v} = \text{convective flux of momentum}$$

* "Diffusive" means it results from random collisions between molecules.

In multicomponent systems, the flux of any species i due to convection is written as the product of concentration and velocity:

$$\text{conv. flux of species } i = c_i \mathbf{v} [=] \text{ mol-cm}^{-2}\text{-s}^{-1} [=] \text{ rate/area}$$

where

$$c_i [=] \text{ mol/cm}^3$$

More generally, by “flux,” I mean:

flux — that tensor (of whatever rank), which when **pre-dotted** by $\mathbf{n}da$, gives the rate of transport through the surface element having area da in the direction of its unit normal \mathbf{n} .

For example, the flux of fluid mass by convection is

$$\rho \mathbf{v} [=] \text{ g-cm}^{-2}\text{-s}^{-1} [=] \text{ rate/area}$$

Proof: pre-dotting by $\mathbf{n}da$ we obtain:

$$(\mathbf{n}da) \cdot (\rho \mathbf{v}) = \rho \mathbf{n} \cdot \mathbf{v} da = \rho(dq) [=] \text{ g/s}$$

which represents the mass flowrate across da . Indeed the convective flux of total fluid mass also has the form of concentration times velocity since

$$\rho = \text{concentration of mass} [=] \text{ mass/vol.}$$

It might come as less of a surprise that $\rho \mathbf{v} \mathbf{v}$ is the convective flux of momentum if you realize that

$$\rho \mathbf{v} = \text{conc. of momentum} [=] \text{ momentum/vol.}$$

After all, momentum is mass times velocity. So

$$(\rho \mathbf{v}) \mathbf{v} = \rho \mathbf{v} \mathbf{v} = \text{conv. flux of momentum}$$

Example: Apply conservation of linear momentum to an arbitrary fluid system. Thus prove that $\underline{\mathbf{T}}$ and $\rho \mathbf{v} \mathbf{v}$ are momentum fluxes, as claimed.

Solution: We choose a system that has fixed boundaries in the laboratory reference frame: in other words, $V \neq V(t)$. Referring to the discussion above in which $\rho \mathbf{v}$ is the concentration of momentum, $\rho \mathbf{v} \mathbf{v}$ is the momentum flux by convection, and $-\underline{\mathbf{T}}$ is the momentum flux by diffusion, conservation of momentum requires

$$\underbrace{\frac{d}{dt} \int_V \rho \mathbf{v} dV}_{\text{accumulation}} = \underbrace{-\oint_A \mathbf{n} \cdot \rho \mathbf{v} \mathbf{v} da}_{\text{in by convection}} \quad \underbrace{-\oint_A \mathbf{n} \cdot (-\underline{\mathbf{T}}) da}_{\text{in by diffusion}} \quad + \underbrace{\int_V \rho \mathbf{g} dV}_{\text{in by external forces}}$$

Applying the divergence theorem, combining the resulting volume integrals, and invoking the result for arbitrary V , we would obtain

$$\underbrace{\frac{\partial}{\partial t}(\rho \mathbf{v})}_{\frac{\partial \rho}{\partial t} \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t}} + \underbrace{\nabla \cdot (\rho \mathbf{v} \mathbf{v})}_{[\nabla \cdot (\rho \mathbf{v})] \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}} - \nabla \cdot \underline{\underline{\mathbf{T}}} - \rho \mathbf{g} = \mathbf{0}$$

Expanding the partial time-derivative of the product and expanding the divergence of the dyadic product using identity C.8

$$\underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right]}_0 \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \underline{\underline{\mathbf{T}}} - \rho \mathbf{g} = \mathbf{0}$$

Finally, we recognize the factor inside square brackets must vanish according to the general continuity equation. After this term is dropped, the remainder is Euler's equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot \underline{\underline{\mathbf{T}}} + \rho \mathbf{g}$$

Comment: In writing the statement of conservation of momentum, we have a term representing transport of momentum into the system by the action of external forces. Clearly, an object falling from rest in a gravitational field acquires momentum through the action of gravity. Does this acquisition represent transport to the body from outside, or does it represent a "generation" term? If it is spontaneous momentum generation, then momentum is not conserved.

With a little reflection, we can convince ourselves that the action of gravity is "transport" and not "generation." Some of the earth's momentum is being transported to the falling object. When the object eventually collides with the earth and comes to rest again, that momentum is transported back to the earth. The total momentum of the universe has not changed: momentum is conserved.

Actually, the term we call "diffusion of momentum" also arises from the action of an external force: this time it's the action of "surface" forces, rather than "body" forces.

RESPONSE OF ELASTIC SOLIDS TO NORMAL (UNIAXIAL) STRESS

By generalizing, the force balance plus the mass balance now contain more unknowns than equations:

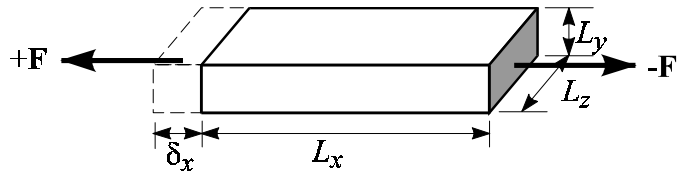
$$\underline{\underline{\mathbf{T}}}, \mathbf{v}, \text{ and } p \rightarrow 9+3+1 = 13 \text{ unknowns}$$

$$\text{Euler} + \text{Continuity} \rightarrow 3+1 = 4 \text{ equations}$$

Missing is the **constitutive equation** which is an empirical description of how the fluid or solid material responds to stress. Obtaining this relationship is an important objective of that field of science known as continuum mechanics.

Let's start by considering solids, whose response is more familiar. Suppose I try to stretch a rectangular bar by applying a tensile force, \mathbf{F} .

We will attempt to describe the response of the bar under conditions of mechanical equilibrium. To have the bar stretch instead of accelerating as a result of the force, I must apply an equal but opposite force to the other end. Let the x -axis be aligned with the direction of the applied force:



$$F_x = |\mathbf{F}|$$

At equilibrium, the length of the bar will increase by an amount δ_x . Hooke's law tell us that:

$$\delta_x \propto \frac{F_x L_x}{L_y L_z}$$

but

$$\frac{F_x}{L_y L_z} = T_{xx}$$

is the applied stress. The two subscripts on stress denote the two directions associated with it: the first subscript denotes the orientation of the surface the force is applied to ($x=\text{const}$) while the second denotes the direction of the applied force (x).

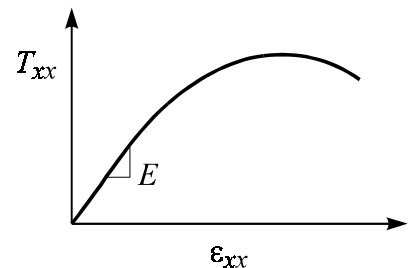
Since the deformation is proportional to the original length of the bar, it makes sense to define the deformation per unit length, which is called:

strain:

$$\epsilon_{xx} = \frac{\delta_x}{L_x}$$

Later we will see that strain is a second-rank tensor. For now, we will interpret the order of the subscripts as follows: the first subscript gives surface (in the above case, an $x=\text{const}$ surface), while the second subscript gives the direction of the deformation (in the above case, the x -direction).

Experiments show that the strain increases with the applied stress as shown in the figure at right. At low stress levels, the strain is



directly proportional to the applied stress: Hooke's law* for *purely elastic solids* is:

$$T_{xx} = E\epsilon_{xx}$$

$$E [=] \text{ force/area}$$

where E is called the **Young's Modulus**. It turns out that deformation also occurs in the y and z directions:

$$\epsilon_{yy} = \epsilon_{zz} = -\nu\epsilon_{xx}$$

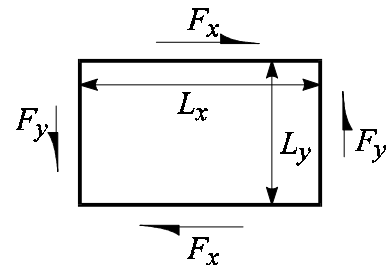
where ν is called **Poisson's ratio**. For most materials

$$0.28 < \nu < 0.33$$

RESPONSE OF ELASTIC SOLIDS TO SHEAR STRESS

Instead of a normal force, suppose I apply a shear force on the upper face. To keep the object from translating, I must apply an equal but opposite force on the lower face.

This generates a force couple or torque, which will cause the body to spin. To prevent a steady increase in rotation speed, I must apply an equal but opposite torque. Recalling that torque is force times lever arm:



$$F_x L_y = F_y L_x$$

Dividing by $L_x L_y L_z$:

$$\frac{F_x}{L_x L_z} = \frac{F_y}{L_y L_z}$$

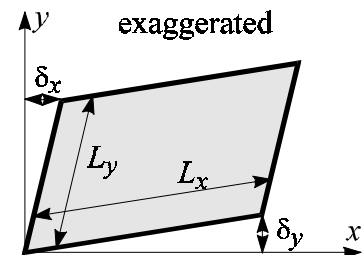
or

$$T_{yx} = T_{xy}$$

where T_{yx} = x -component of force/area acting on y =const surface. Note that this implies that the stress tensor is **symmetric**. This turns out to be true for virtually all loadings.*

This loading produces a shear deformation in which

$$\delta_x \propto L_y$$



* Robert Hooke (1635-1703). English experimental physicist. Hooke's law first stated in 1660.

* For a proof based on the assumption of local equilibrium, see W:4.3

and

$$\delta_y \propto L_x$$

The corresponding definition of **shear strain** is:

$$\frac{1}{2} \left(\frac{\delta_x}{L_y} + \frac{\delta_y}{L_x} \right) = \varepsilon_{xy} = \varepsilon_{yx}$$

Since we applied a symmetric stress, by applying forces in both the x and y directions, we average the strains in the x and y directions to obtain a symmetric strain. Moreover, this definition yields a strain which is invariant to rotation of the xy axes. As with normal stress, shear stress produces a shear strain in direct proportion to the stress:

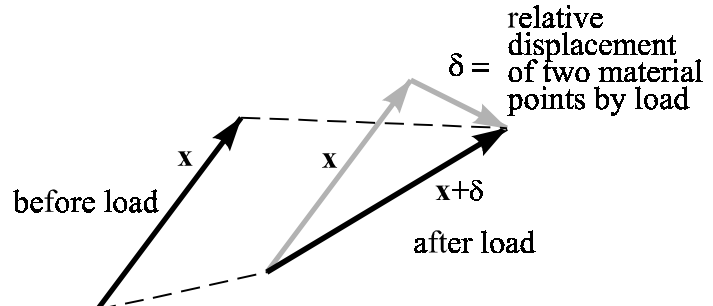
$$T_{xy} = 2G\varepsilon_{xy} \quad (\text{pure shear})$$

where G is call the **modulus of elasticity for pure strain**. Although the units are the same, the value of $2G$ is different from that of E .

GENERALIZED HOOKE'S LAW

Now let's try to generalize to some arbitrary loading which might involve both shear and normal stresses.

Consider two arbitrary material points in the material. Let \mathbf{x} denote the position of the second material point relative to the first, before the load is applied. After the load is applied, both material points move as shown by the dashed lines in the figure at right. After the load the position of the second material point relative to the first becomes $\mathbf{x} + \delta$, where



$$\delta(\mathbf{x}) = \text{relative displacement}$$

of two material points initially located at \mathbf{x} . The resulting strain can be generalized as:

$$\underline{\underline{\varepsilon}} = \frac{1}{2} \left[\nabla \underline{\underline{\delta}} + (\nabla \underline{\underline{\delta}})^t \right]$$

or

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \delta_j}{\partial x_i} + \frac{\partial \delta_i}{\partial x_j} \right)$$

Note that this general definition of strain reduces to earlier expression for strain in the cases of pure normal or pure shear stress:

uniaxial normal stress: $\epsilon_{xx} = \frac{1}{2} \left(\frac{\partial \delta_x}{\partial x} + \frac{\partial \delta_x}{\partial x} \right) = \frac{\delta_x}{L_x}$ and $T_{xx} = E \epsilon_{xx}$

pure shear: $\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial \delta_y}{\partial x} + \frac{\partial \delta_x}{\partial y} \right) = \frac{1}{2} \left(\frac{\delta_y}{L_x} + \frac{\delta_x}{L_y} \right)$ and $T_{xy} = 2G \epsilon_{xy}$

If the strain components are all small, we might reasonably suppose that Hooke's law generalizes into a linear relationship between any component of strain and the nine components of stress (or vice versa):

$$T_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \epsilon_{kl} \quad (42)$$

There are then nine coefficients for each component of stress, making a total of 81 possible coefficients. But just by requiring:

- symmetry of $\underline{\underline{\mathbf{T}}}$ ($T_{ij}=T_{ji}$)
- isotropy of material (e.g. same Young's modulus applies to uniaxial stress in x -direction as for y and z)

it can be shown that the number of independent constants is reduced from 81 to only 2. It is customary to express the stress tensor as the sum of two tensors: one isotropic $[(\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}]$ and the remainder $[\underline{\underline{\epsilon}} - (\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}]$. Denote the two independent constants as k_1 and k_2 and use them as weighting factors in the sum:

$$\underline{\underline{\mathbf{T}}} = k_1 \underbrace{\left[\underline{\underline{\epsilon}} - (\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}} \right]}_{\text{remainder}} + k_2 \underbrace{(\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}}_{\text{isotropic}}$$

This can be rearranged $\underline{\underline{\mathbf{T}}} = k_1 \underline{\underline{\epsilon}} + (k_2 - k_1)(\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}$

Sometimes the constants are chosen as the two coefficients in this new form

$$\underline{\underline{\mathbf{T}}} = 2\eta \underline{\underline{\epsilon}} + \lambda (\underline{\underline{\epsilon}}:\underline{\underline{\mathbf{I}}})\underline{\underline{\mathbf{I}}}$$

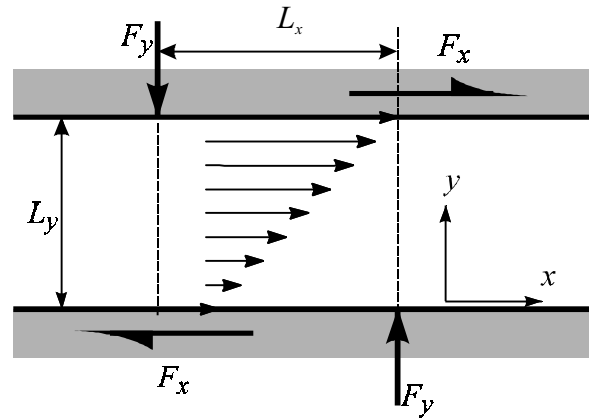
which represents the generalization of Hooke's law of elasticity. Thus of G , E and ν , only two are linearly independent properties of the material; using the above relationship we can show (HWK Set #5, Prob. 1) that

$$\nu = \frac{E}{2G} - 1$$

RESPONSE OF A VISCOUS FLUID TO PURE SHEAR

Suppose I have a thin layer of fluid held between parallel plates and I apply a force F_x to the upper plate. To cause deformation, rather than simple translation of the system, I must apply an equal but opposite force to the lower plate.

As before, this produces a force couple or torque. To keep the system from rotating, I must apply an opposing torque. Once again this equilibrium state corresponds to a symmetric stress tensor. From our experience, we know that eventually the upper plate will slide past the lower plate at some steady relative speed U_x .



response:
$$U_x \propto \frac{F_x L_y}{L_x L_z}$$

This speed represents a *rate of deformation*:

$$U_x = \frac{d\delta_x}{dt}$$

and the speed per unit thickness represents a *rate of strain*:

$$\frac{U_x}{L_y} = \frac{d\delta_x/dt}{L_y} = \frac{d}{dt} \left(\frac{\delta_x}{L_y} \right) = \frac{d\epsilon_{yx}}{dt}$$

Finally, the applied force divided by the area over which it's applied represents a shear stress:

$$\frac{F_x}{L_x L_z} = T_{yx} = \tau_{yx} = \tau_{xy}$$

Rewriting the proportionality as an equality, we have:

$$\tau_{yx} = \mu \frac{d\epsilon_{yx}}{dt}$$

which is called *Newton's Law of Viscosity* (1686). Alternatively, BSL point out that

$$\frac{U_x}{L_y} = \frac{dv_x}{dy} \quad \text{and} \quad (\tau_{xy})_{\text{BSL}} = -(\tau_{xy})_{\text{notes}} = -\mu \frac{dv_x}{dy}$$

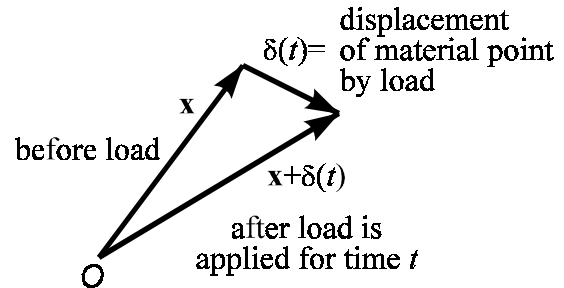
GENERALIZED NEWTON'S LAW OF VISCOSITY

This generalization of Newton's law of viscosity to arbitrary loading parallels that of Hooke's Law with the strain tensor replaced by the rate of strain. The main difference is that now the deformation is a function of time:

$$\delta(\mathbf{x}, t) = \text{displacement of material pt.} \\ = \text{deformation}$$

The trajectory of a material point initially located at $\mathbf{x}(0)$ is given by:

$$\mathbf{x}[\mathbf{x}(0), t] = \mathbf{x}(0) + \delta[\mathbf{x}(0), t]$$



Keeping the material point fixed (i.e. $\mathbf{x}(0)$ is constant) while we take the time derivative is the same as taking the material derivative of position. Thus the rate of deformation of fluid elements is:

$$\frac{d\delta}{dt} = \frac{D\mathbf{x}}{Dt} = \mathbf{v} \quad (\text{rate of deformation})$$

given by the fluid velocity. Just like we define the gradient of the deformation to be the strain:

$$\underline{\underline{\varepsilon}} = \frac{1}{2} \left[\nabla \underline{\delta} + (\nabla \underline{\delta})^t \right], \quad (\text{strain})$$

the gradient of the rate of deformation must be the **rate of strain**:

$$\underline{\underline{\mathbf{d}}} = \frac{D\underline{\underline{\varepsilon}}}{Dt} = \frac{1}{2} \frac{D}{Dt} \left[\nabla \underline{\delta} + (\nabla \underline{\delta})^t \right] = \frac{1}{2} \left[\nabla \frac{D\delta}{Dt} + \left(\nabla \frac{D\delta}{Dt} \right)^t \right] = \frac{1}{2} \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^t \right] (\text{rate of strain})$$

Newton's law of viscosity is a linear relationship between the stress and the rate of strain. Generalizing that linear relationship again leads to a relationship like (42), except that the strain tensor ε_{kl} is replaced by the rate of strain tensor d_{kl} . Once again, in order for the stress tensor $\underline{\underline{\mathbf{T}}}$ or $\underline{\underline{\boldsymbol{\tau}}}$ to be symmetric and for the material to be isotropic, only two of the 81 coefficients C_{ijkl} are independent. We use the two independent coefficients to multiply the isotropic part of $\underline{\underline{\boldsymbol{\tau}}}$ and its remainder:

$$\underline{\underline{\boldsymbol{\tau}}} = 2\mu \underline{\underline{\mathbf{d}}} + \left(\kappa - \frac{2}{3}\mu \right) (\underline{\underline{\mathbf{d}}} : \underline{\underline{\mathbf{I}}}) \underline{\underline{\mathbf{I}}}$$

where μ is the usual viscosity and κ is called the *second coefficient of viscosity*. $\underline{\underline{\mathbf{d}}}:\underline{\underline{\mathbf{I}}}$ has a special significance, which we will now point out. As we showed on page 70, $\underline{\underline{\mathbf{d}}}:\underline{\underline{\mathbf{I}}}$ is the trace of $\underline{\underline{\mathbf{d}}}$:

$$\underline{\underline{\mathbf{d}}}:\underline{\underline{\mathbf{I}}} = d_{11} + d_{22} + d_{33}$$

Writing in terms of the velocity \mathbf{v} :

$$d_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

$$d_{11} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_1} \right) = \frac{\partial v_1}{\partial x_1}$$

so

$$\underline{\underline{\mathbf{d}}}:\underline{\underline{\mathbf{I}}} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \nabla \cdot \mathbf{v}$$

So the trace of the rate of deformation tensor is just the divergence of the velocity. *Newton's law of viscosity* becomes:

$$\underline{\underline{\boldsymbol{\tau}}} = 2\mu \underline{\underline{\mathbf{d}}} + \left(\kappa - \frac{2}{3}\mu \right) (\nabla \cdot \mathbf{v}) \underline{\underline{\mathbf{I}}}$$

For an incompressible fluid: $\nabla \cdot \mathbf{v} = 0$

leaving

$$\underline{\underline{\boldsymbol{\tau}}} = 2\mu \underline{\underline{\mathbf{d}}} = \mu \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^t \right] \quad (\text{incompressible})$$

NAVIER-STOKES EQUATION

Once again, let's go back to Euler's equation, generalized to account for the tensorial nature of viscous friction:

$$\rho D\mathbf{v}/Dt = \rho \mathbf{g} - \nabla p + \nabla \cdot \underline{\underline{\boldsymbol{\tau}}}$$

For an incompressible fluid, $\nabla \cdot \mathbf{v} = 0$ and

$$\underline{\underline{\boldsymbol{\tau}}} = 2\mu \underline{\underline{\mathbf{d}}} = \mu [\nabla \mathbf{v} + (\nabla \mathbf{v})^t]$$

$$\nabla \cdot \underline{\underline{\boldsymbol{\tau}}} = \mu [\nabla \cdot \nabla \mathbf{v} + \nabla \cdot (\nabla \mathbf{v})^t]$$

but using identity C.10:

$$\nabla \cdot (\nabla \mathbf{v})^t = \nabla (\nabla \cdot \mathbf{v}) = \mathbf{0}$$

leaving

$$\nabla \cdot \underline{\underline{\boldsymbol{\tau}}} = \mu \nabla \cdot \nabla \mathbf{v} = \mu \nabla^2 \mathbf{v}$$

Euler's equation becomes: $\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{v}$ ($\rho, \mu = \text{const}$)

which is known as the **Navier-Stokes Equation** (1822).^{*} Now we have as many equations as unknowns:

$\mathbf{v}, p \rightarrow 4$ unknowns

N-SE, Continuity $\rightarrow 4$ eqns

BOUNDARY CONDITIONS

But to successfully model a flow problem, we need more than a sufficient number of differential equations. We also need boundary conditions.

A typical boundary is the interface between two immiscible phases -- either two fluids or a fluid and a solid. One such boundary condition which can generally be applied is the **no slip condition**:

$$\mathbf{v}^I = \mathbf{v}^{II}$$

For example, in the problem of uniform flow around a stationary solid sphere, this requires:

at $r=a$: $\mathbf{v}_f = \mathbf{v}_s = \mathbf{0}$

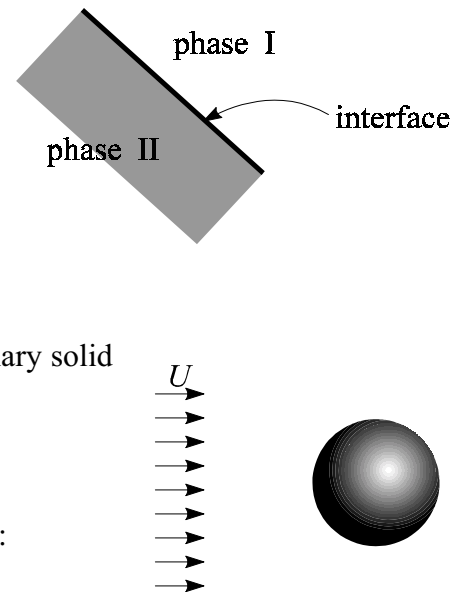
which means that there is no flow of fluid across the boundary:

at $r=a$: $v_r = 0$

We also assumed this boundary condition when we solved the potential flow problem. But “no slip” also means:

at $r=a$: $v_\theta = 0$

Note that our potential flow solution did not satisfy this second equation:



^{*} Sir George Gabriel Stokes (1819-1903): British (Irish born) mathematician and physicist, known for his study of hydrodynamics. Lucasian professor of mathematics at Cambridge University 1849-1903 (longest-serving Lucasian professor); president of Royal Society (1885-1890).

for pot'l flow:

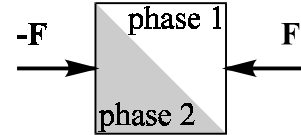
$$v_r(a, \theta) = 0$$

$$v_\theta(a, \theta) = -(3/2)U \sin \theta \neq 0$$

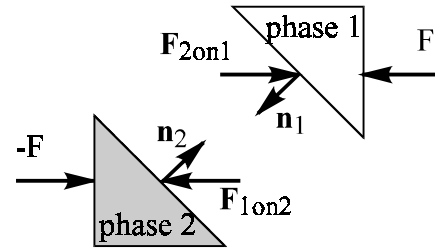
In addition to d'Alembert's paradox, potential flow fails to satisfy the no-slip condition.*

For a fluid-solid interface, in which the velocity of the solid phase is known, the no-slip condition is sufficient. But in the case of fluid-fluid interface, the velocity of the second fluid is usually unknown. Then no-slip just relates one unknown to another.

A second boundary condition can be obtained by considering the stresses acting on the material on either side of the interface. Suppose we were to apply a loading as on page 66 to a two-phase region straddling the interface, as suggested in the figure at right. Note that the loading is balanced: that is there is no net force on the system.



If we were to split the system into two parts along the interface, each of the two halves would tend to accelerate. This suggests that, when the two phases are in contact, each exerts an “internal” force on the other, as shown in the figure at right. These forces are equal but opposite:



$$\mathbf{F}_{1on2} = -\mathbf{F}_{2on1} = \mathbf{F}$$

which might be thought of as “Newton’s Third Law”: for every action, there is an equal but opposite reaction. Using (38) (on page 68) to express the forces in terms of the stress tensor:

$$\mathbf{n}_2 \cdot \underline{\underline{\mathbf{T}}}_1 A = -\mathbf{n}_1 \cdot \underline{\underline{\mathbf{T}}}_2 A = \mathbf{F}$$

where A is the area of the interface and where \mathbf{n}_i is the normal to the interface pointing out of phase i . However, from the geometry, we can deduce that

$$\mathbf{n}_1 = -\mathbf{n}_2 \equiv \mathbf{n}$$

Newton’s Third Law becomes:

$$\mathbf{n} \cdot \underline{\underline{\mathbf{T}}}_1 = \mathbf{n} \cdot \underline{\underline{\mathbf{T}}}_2 \quad (43)$$

When the interface is highly curved (e.g. a small oil droplet in water), then surface tension can produce a discontinuity in the normal components of the above force, which has not been

* Although "no slip" is usually applicable, there are at least two situations where no slip might not be applicable: 1) when the mean-free path of gas molecules is comparable to the geometric dimension, and 2) when a liquid does not wet the solid.

included in the above analysis [see L&L, Chapt. 7 or Hunter, Vol I., p237f]. The more general form of this boundary condition is

$$\mathbf{n} \cdot (\mathbf{T}_1 - \mathbf{T}_2) = \gamma(\nabla_s \cdot \mathbf{n})\mathbf{n} \quad (44)$$

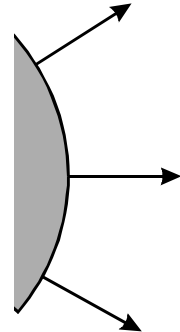
where γ is another property of the fluid called the *surface tension* and

$$\nabla_s \equiv (\mathbf{I} - \mathbf{nn}) \cdot \nabla$$

is the surface component of the ∇ operator. We will have more to say about this near the end of the course. For now, we will neglect surface tension effects.

$$\nabla_s \cdot \mathbf{n} = \text{curvature of surface [=] m}^{-1}$$

For a flat surface, \mathbf{n} is independent of position along the surface so that $\nabla_s \cdot \mathbf{n} = 0$ and (44) reduces to (43).



Exact Solutions of N-S Equations

Exact solution of the Navier-Stokes equations presents a formidable mathematical problem. By “exact” I mean:

exact — neither viscous nor inertial terms are neglected (i.e. approximated by zero, as opposed to being identically zero)

One difficulty is the non-linear inertial term. Most of the powerful mathematical techniques (such as eigenfunction expansions, used in “separation of variables”) only work when the P.D.E. is linear.

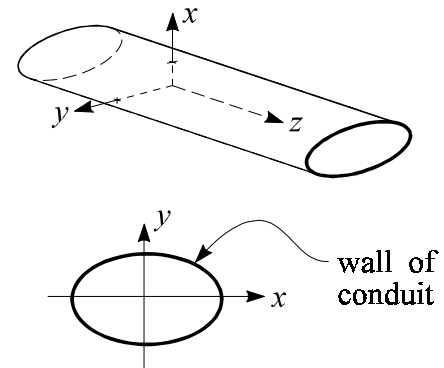
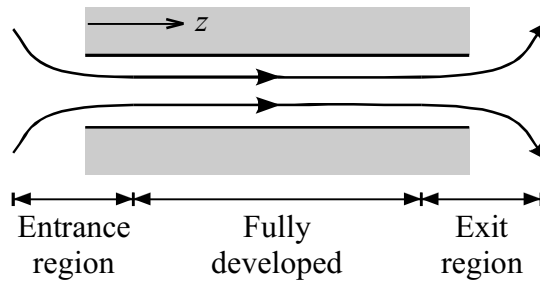
Of course, if we can neglect the higher-order viscous terms, then we can cope with the non-linearity using a velocity potential, as we did earlier in solving problems in potential flow. However, viscous terms are seldom completely negligible and leaving them in the equation makes the problem much more difficult by increasing the order of the P.D.E. Nevertheless, it is possible to find exact solutions in certain cases, usually when the inertial terms vanish in some natural way. We will now examine a few of these problems having exact solutions.

PROBLEMS WITH ZERO INERTIA

First, let's consider problems in which the fluids elements travel along straight streamlines at constant velocity. Then their acceleration vanishes identically.

Flow in Long Straight Conduit of Uniform Cross Section

Suppose we have pressure-driven flow in a long straight conduit whose cross section does not vary along the flow. In mathematical terms, the conduit is a cylinder of arbitrary cross-section. Define Cartesian coordinates such that the axis of the cylinder corresponds to the z -axis. In a very long pipe, we expect that v_z will depend on z (as well as x and y) near the entrance and exit of the pipe.



In particular, near the entrance, we say the velocity profile is “developing”; i.e. evolving with z . Outside the entrance and exit regions v_z will be independent of z . This situation is called **fully developed flow**. For fully developed, steady flow:

$$v_z = v_z(x, y)$$

$$v_x = v_y = 0$$

Note that this automatically satisfies the Continuity Equation:

$$\nabla \cdot \mathbf{v} = \partial v_z / \partial z = 0$$

For this form of solution, it turns out that the nonlinear inertial terms automatically vanish:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{0}$$

To convince yourself of this, consider steady flow. Since the velocity of a fluid element is constant along a straight streamline:

for steady flow:
$$\mathbf{v} \cdot \nabla \mathbf{v} = D\mathbf{v}/Dt = \mathbf{a} = \mathbf{0}$$

In other words, fluids elements are not accelerating. Thus inertial forces are identically zero in steady pipe flows. Strictly speaking, Reynolds number should not be thought of the ratio of inertial to viscous forces in this problem, since inertia is zero for laminar flow although the Reynolds number is not zero.

For a steady flow, the N-S equations are:

$$\mathbf{0} = \mu \nabla^2 \mathbf{v} - \nabla p + \rho \mathbf{g}$$

Note that for a vertical pipe:*

$$p = p(z)$$

but more generally for a horizontal or inclined pipe:

$$p = p(x, y, z)$$

where the dependence on position in the cross section arises from the contribution to pressure from the hydrostatic head (i.e. from \mathbf{g}). For this and some other problems, it's helpful to decompose the total pressure into contributions from gravity (i.e. hydrostatic pressure, p_h) and from flow (called the **dynamic pressure**, P)

$$\begin{aligned} p &= p_h + P \\ \nabla p &= \underbrace{\nabla p_h}_{\rho \mathbf{g}} + \nabla P \end{aligned}$$

From our earlier analysis of hydrostatic equilibrium (see page 39), we know that $\nabla p_h = \rho \mathbf{g}$.

Note that

$$\nabla p - \rho \mathbf{g} = \nabla P$$

Next, we substitute this into the N-S Equation. Expanding them in component form:

$$\begin{aligned} x\text{-component:} & \quad 0 = -\frac{\partial P}{\partial x} \\ y\text{-component:} & \quad 0 = -\frac{\partial P}{\partial y} \end{aligned} \left. \vphantom{\begin{aligned} x\text{-component:} \\ y\text{-component:} \end{aligned}} \right\} P = P(z)$$

$$z\text{-component:} \quad 0 = \mu \nabla^2 v_z - \frac{dP}{dz} \tag{45}$$

where we have written the last term as the *total* derivative instead of the *partial* derivative because the first two equations require that P be a function of z alone. We can immediately deduce that P must be a linear function of z

$$\underbrace{\mu \nabla^2 v_z}_{g(x,y)} = \underbrace{\frac{dP}{dz}}_{f(z)}$$

since the dP/dz is independent of x and y , while the velocity profile is independent of z , so (45) requires that the two functions of different variables be equal:

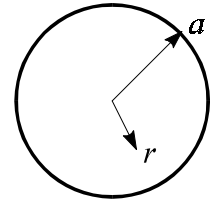
$$g(x, y) = f(z) = \text{const. w.r.t. } x, y, z$$

* In the absence of gravity, $\partial p/\partial x$ and $\partial p/\partial y$ must vanish: see the x - and y -components of (45).

which can only be true if both functions equal the same constant: thus $P(z)$ must be a linear function. For steady flow in a circular conduit of radius a , the “no-slip” b.c. requires

at $r = a$:

$$v_z = 0 \quad (46)$$



Since neither the b.c. nor the differential equation contain any dependence on θ , we expect the solution to be axisymmetric about the z -axis:

$$v_z = v_z(r)$$

(45) becomes:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{1}{\mu} \frac{dP}{dz} = \text{const}$$

The general solution of this equation is:

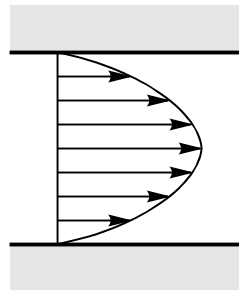
$$v_z(r) = \frac{1}{4\mu} \frac{dP}{dz} r^2 + c_1 \ln r + c_2$$

Requiring the solution to be bounded at the center of the tube (as $r \rightarrow 0$) forces us to choose $c_1 = 0$ while the remaining constant can be chosen to satisfy the no-slip condition (46). The particular solution is the familiar parabolic velocity profile:

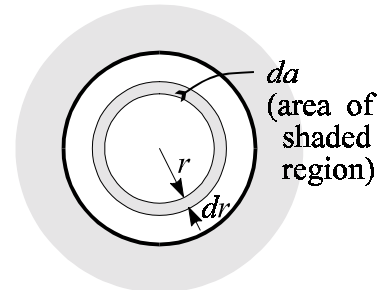
$$v_z(r) = \frac{1}{4\mu} \frac{dP}{dz} (r^2 - a^2)$$

The volumetric flowrate through the conduit is computed from:

$$Q = \int_A \mathbf{n} \cdot \mathbf{v} \, da = \int_0^a v_z(r) (2\pi r) dr$$



side view



end view

or

$$Q = \frac{\pi a^4}{8\mu} \left(-\frac{dP}{dz} \right)$$

which is called **Poiseuille's Formula**. This formula has been derived for a number of different cross sections. In general

$$Q = \frac{kA^2}{\mu} \left(-\frac{dP}{dz} \right)$$

where

A = cross-sectional area of duct

and where k is some constant which depends on the shape of the duct; e.g.

circle: $k = 1/8\pi = 0.0398$

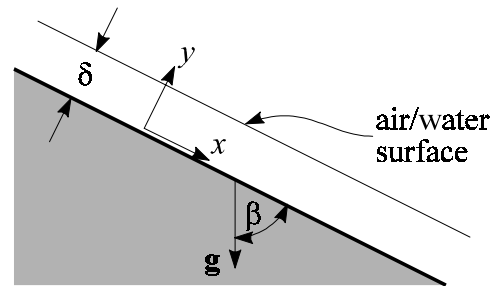
square: $k = 0.0351$

ellipse:
$$\frac{\varepsilon}{4\pi(1 + \varepsilon^2)}$$

where $\varepsilon = b/a \leq 1$ is the ratio of the minor to major axis.

Flow of Thin Film Down Inclined Plane

Suppose we have fluid overflowing some reservoir and down an inclined plane surface. Although there might be some entrance or exit effects (at the upstream and downstream ends of the plane), if the plane is sufficiently long compared to these regions, then what we see in experiments is a region in which the fluid flows downward as a film of uniform thickness. Let's try to analyze this central region in which the film is uniform.



Let the x -axis be oriented parallel to the inclined plane in the direction of flow and let the y -axis be perpendicular to the plane. One of the boundary conditions will be “no-slip” at the solid surface:

at $y=0$: $v_x = 0$

At the very least, there must be flow in the x -direction: otherwise there could be no viscous force to balance gravity. In addition, the x -component must vary with y : v_x must vanish at $y=0$ (no slip) and be nonzero for $y>0$. The simplest form of solution which is consistent with those constraints is:

$$\mathbf{v} = v_x(y)\mathbf{e}_x$$

Note that this form automatically satisfies the requirement of continuity for an incompressible fluid:

$$\nabla \cdot \mathbf{v} = \partial v_x / \partial x = 0$$

Substituting this velocity field into the Navier-Stokes equations:

x : $0 = -\partial p / \partial x + \mu d^2 v_x / dy^2 + \rho g_x$ (47)

y : $0 = -\partial p / \partial y + \rho g_y$ (48)

z : $0 = -\partial p / \partial z$

where

$$g_x = \mathbf{g} \cdot \mathbf{e}_x = g \cos \beta$$

and

$$g_y = \mathbf{g} \cdot \mathbf{e}_y = -g \sin \beta$$

Integrating (48) with respect to y :

$$p(x, y) = -\rho g y \sin \beta + c(x) \quad (49)$$

where the integration constant might depend on x , but cannot depend on z (according to z component above) or y . Now let's turn to the boundary conditions. **Continuity of stress** across the interface yields:

$$\mathbf{n} \cdot \underline{\underline{T}}_{liq} = \mathbf{n} \cdot \underline{\underline{T}}_{gas} = \mathbf{n} \cdot (-p_{gas} \mathbf{I} + \underline{\underline{\tau}}_{gas})$$

Now the viscosity of air is about 0.001 times that of water. Then it is reasonable to treat the air as an inviscid fluid: i.e. neglect $\underline{\underline{\tau}}_{gas}$. This leaves

$$\mathbf{n} \cdot \underline{\underline{T}}_{liq} \approx -\mathbf{n} p_{gas} = -\mathbf{n} p_{atm} \quad (50)$$

where $p_{atm} = 1 \text{ atm}$. Now \mathbf{e}_y is the unit normal to the interface in this problem

$$\mathbf{e}_y \cdot \underline{\underline{T}}_{liq} = \mathbf{e}_y \cdot (-p \mathbf{I} + \underline{\underline{\tau}}) = -p \mathbf{e}_y + \tau_{yx} \mathbf{e}_x + \tau_{yy} \mathbf{e}_y + \tau_{yz} \mathbf{e}_z$$

When $v_x = v_x(y)$, the only nonzero components of the deviatoric stress tensor are $\tau_{xy} = \tau_{yx}$.* Dropping the other terms, (50) becomes:

$$-p \mathbf{e}_y + \tau_{yx} \mathbf{e}_x = -\mathbf{e}_y p_{atm}$$

Equating separate components:

$$x\text{-component:} \quad \tau_{yx}(x, y=\delta) = 0 \quad (51)$$

$$y\text{-component:} \quad p(x, y=\delta) = p_{atm}$$

With this result, we can evaluate the integration constant in (49):

$$p(x, y=\delta) = -\rho g \delta \sin \beta + c(x) = p_{atm}$$

$$\text{Thus} \quad c(x) = p_{atm} + \rho g \delta \sin \beta$$

* For a Newtonian fluid, the 9 scalar components of the stress tensor are expressed in terms of the derivatives of the velocity field on p145 of Whitaker. These expressions (except for a change in sign) can be found on p88 of BSL or at the website for our course —

(49) becomes: $p(x,y) = p_{atm} + \rho g(\delta - y)\sin\beta$

Thus $\partial p / \partial x = 0$ and (47) becomes:

$$\mu d^2 v_x / dy^2 = -\rho g \cos\beta$$

No-slip at the wall requires:

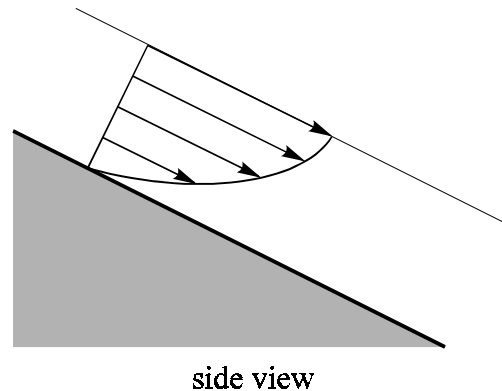
at $y=0$: $v_x = 0$

whereas if the stress in (51) is evaluated using Newton's law of viscosity, we also require:

at $y=\delta$: $\tau_{yx} = \mu dv_x / dy = 0$

Using these two boundary conditions, the velocity profile can be uniquely determined:

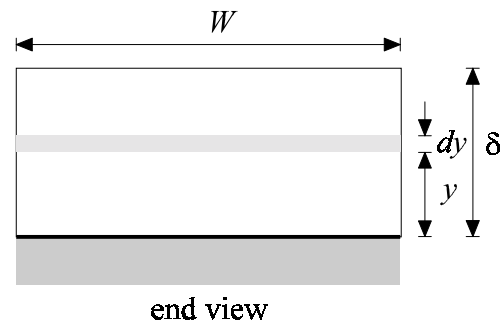
$$v_x(y) = \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right]$$



By integrating over a plane perpendicular to the flow, we can evaluate the flowrate:

$$Q = \int_A \mathbf{n} \cdot \mathbf{v} \, da = \int_0^W \int_0^\delta v_x(y) \, dy \, dz$$

$$Q = \frac{\rho g W \delta^3 \cos\beta}{3\mu}$$

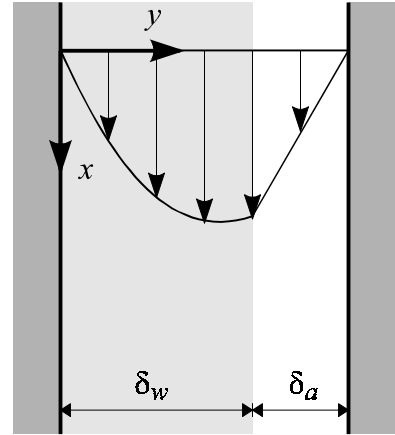


where W is the width of the plane.

Time Out: The main novelty of this problem is the treatment of the free surface. We treated the air as if it was inviscid, although it has some viscosity. How important is the drag imposed by the air? This is the subject of Hwk #6, Prob. 2

To answer this question, let's consider a vertical film of water in contact with a vertical film of air, as shown in the sketch at right. Let's re-solve the problem and see how large δ_a must be for a given δ_w before we can neglect the effect of the air. For fully developed flow, the velocity and pressure profiles should have the form:

$$\begin{aligned}v_x &= v_x(y) \\v_y &= v_z = 0 \\p(x, y, z) &= 1 \text{ atm}\end{aligned}$$



NSE_x becomes:

for water:
$$0 = \mu_w \frac{d^2 v_x^w}{dy^2} + \rho_w g$$

for air:
$$0 = \mu_a \frac{d^2 v_x^a}{dy^2}$$

where we have taken $\rho_a = 0$. Applying “no slip” at each of the three interfaces

at $y=0$:
$$v_x^w(0) = 0$$

at $y=\delta_w$:
$$v_x^w(\delta_w) = v_x^a(\delta_w) \equiv U$$

at $y=\delta_w + \delta_a$:
$$v_x^a(\delta_w + \delta_a) = 0$$

The particular solutions to the two ODE's are

$$v_x^w(y) = \frac{\rho_w g}{2\mu_w} (\delta_w - y)y + U \frac{y}{\delta_w} \quad (52)$$

$$v_x^a(y) = U \frac{\delta_w + \delta_a - y}{\delta_a} \quad (53)$$

The interfacial speed U is unknown, but is chosen to match the shear stress at the interface:

at $y=\delta_w$:
$$\tau_{yx}^w(\delta_w) = \tau_{yx}^a(\delta_w)$$

For Newtonian fluids, the stresses can be related to velocity profile:

$$\mu_w \left. \frac{dv_x^w}{dy} \right|_{y=\delta_w} = \mu_a \left. \frac{dv_x^a}{dy} \right|_{y=\delta_w}$$

Using the velocity profiles of (52) and (53), this stress matching yields:

$$-\frac{1}{2}\rho_w g \delta_w + \mu_w \frac{U}{\delta_w} = \mu_a \frac{U}{\delta_a}$$

Solving for U :

$$U = \frac{1}{2} \frac{\rho_w g \delta_w}{\frac{\mu_w}{\delta_w} + \frac{\mu_a}{\delta_a}}$$

The flowrate of the water is

$$\frac{Q}{W} = \int_0^{\delta_w} v_x^w(y) dy = \frac{\rho_w g \delta_w^3}{12\mu_w} + \frac{1}{2} U \delta_w = \frac{\rho_w g \delta_w^3}{12\mu_w} \frac{4 \frac{\mu_w}{\delta_w} + \frac{\mu_a}{\delta_a}}{\frac{\mu_w}{\delta_w} + \frac{\mu_a}{\delta_a}} \quad (54)$$

If the air film is very thick, the flowrate becomes

$$\frac{Q_\infty}{W} \equiv \lim_{\delta_a \rightarrow \infty} \{Q\} = \frac{\rho_w g \delta_w^3}{3\mu_w} \quad (55)$$

which is the same expression we had in the Notes when the viscosity of air was completely ignored. Dividing (54) by (55):

$$\frac{Q}{Q_\infty} = \frac{1}{4} \frac{4 \frac{\mu_w}{\delta_w} + \frac{\mu_a}{\delta_a}}{\frac{\mu_w}{\delta_w} + \frac{\mu_a}{\delta_a}} = \frac{1}{4} \frac{4 \frac{\mu_w}{\delta_w} + \frac{\delta_w}{\delta_a}}{\frac{\mu_w}{\delta_w} + \frac{\delta_w}{\delta_a}}$$

When the viscosity of water is 1000 times larger than air (i.e. $\mu_w = 1000 \mu_a$), this gives

$$\frac{Q}{Q_\infty} = \frac{1}{4} \frac{4000 + \frac{\delta_w}{\delta_a}}{1000 + \frac{\delta_w}{\delta_a}}$$

To reduce the flowrate by 1% means $\frac{Q}{Q_\infty} = 0.99$, for which the air film thickness must be

$$\delta_a = 0.074 \delta_w$$

So even if the air is stilled by a nearby boundary, the drag of the air on the free surface of the water will be negligible (provided the boundary is not too close). In the absence of a rigid boundary in the air, negligible error is made by treating the air as inviscid.

Time In!

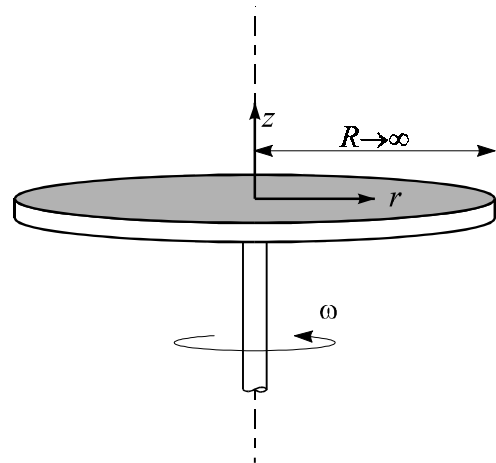
PROBLEMS WITH NON-ZERO INERTIA

L&L list only three flow problems in which both including viscous and inertial terms are important and in which exact solutions are known:

- 1) rotating disk
- 2) converging (or diverging) flow between nonparallel planes
- 3) the submerged jet

*Rotating Disk**

We will now examine the solution of the first problem — the rotating disk — because it is used as a model system for transport experiments. An infinite plane disk immersed in a viscous fluid rotates uniformly about its axis. Determine the motion of the fluid caused by this motion of the disk. This problem was first solved by von Karmen (1921) using cylindrical coordinates with the z -axis coinciding with the axis of rotation.



Define:
$$\zeta = z \sqrt{\frac{\omega}{\nu}}$$

where $\nu = \mu/\rho$ [=] cm²/s

is called the *kinematic viscosity*.

Then:
$$v_r(r, z) = r\omega F(\zeta)$$

$$v_\theta(r, z) = r\omega G(\zeta)$$

* See S:p93 (6th Ed); L&L:p79.

$$v_z(z) = \sqrt{\nu\omega} H(\zeta)$$

$$p(z) = \mu\omega P(\zeta)$$

Continuity and N-S become:

$$2F + H' = 0$$

$$r: F^2 + F'H - G^2 - F'' = 0$$

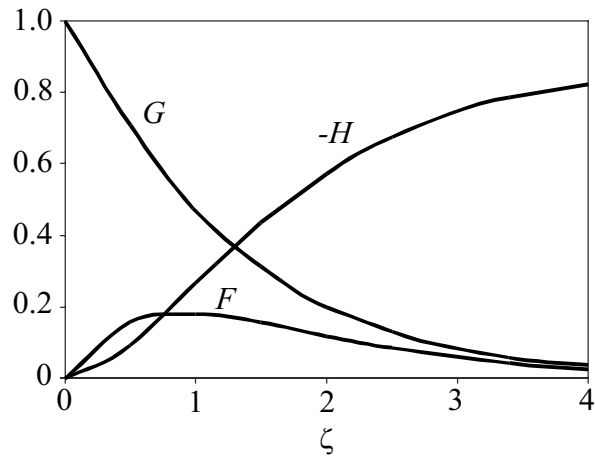
$$\theta: 2FG + HG' - G'' = 0$$

$$z: P' + HH' - H'' = 0$$

where the prime (') denotes differentiation with respect to ζ . Boundary conditions take the form:

$$z=0: F=H=P=0, G=1$$

$$z=\infty: F=G=0$$



An important property of this solution is that the z -component of velocity depends only on z . For this reason, convective heat and mass transfer problems becomes one-dimensional.

This is perhaps the only problem in which there is flow normal to a wall where the mass-transfer coefficient can be determined analytically. For this reason, the rotating disk is a favorite tool of researchers in mass transfer.

Creeping Flow Approximation

CONE-AND-PLATE VISCOMETER

The cone-and-plate viscometer consists of a flat plate, on which is placed a pool of liquid whose viscosity is to be measured, and an inverted cone which is lowered into the liquid until its apex just touches the plate. The cone is then rotated at some angular velocity, Ω , and the torque required to turn the cone or to keep the plate stationary is measured.

Spherical coordinates are most convenient to describe this problem, since the surface of the cone and of the plate can be defined as $\theta = \text{const}$ surfaces:

surface of cone: $\theta = \alpha$

surface of plate: $\theta = \pi/2$

The cone is undergoing solid-body rotation (see HWK #4, Prob. 3):

for $\theta \leq \alpha$: $\mathbf{v}(\mathbf{r}) = \underline{\Omega} \times \mathbf{r}$

In spherical coordinates, the position vector is

$$\mathbf{r} = r \mathbf{e}_r(\theta, \phi)$$

while the angular velocity is (see figure and p49):

$$\underline{\Omega} = \Omega \mathbf{e}_z = \Omega[(\cos\theta)\mathbf{e}_r - (\sin\theta)\mathbf{e}_\theta] \quad (56)$$

Substituting into the expression for solid body rotation:

$$\theta < \alpha: \quad \mathbf{v} = \underline{\Omega} \times \mathbf{r} = r\Omega \sin\theta \mathbf{e}_\phi$$

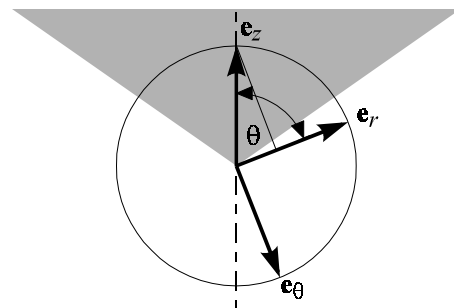
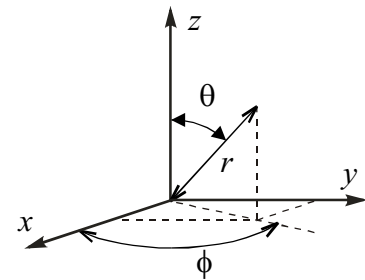
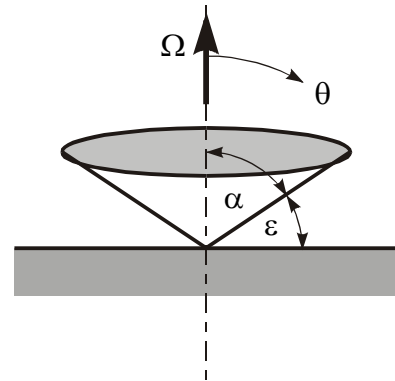
The principle direction of flow is the ϕ -direction. “No-slip” requires:

$$\text{at } \theta = \alpha: \quad v_\phi = r\Omega \sin\alpha$$

$$v_r = v_\theta = 0$$

$$\text{at } \theta = \pi/2: \quad v_\phi = v_r = v_\theta = 0$$

The simplest velocity profile which is consistent with these boundary conditions is:



$$v_\phi = v_\phi(r, \theta)$$

$$v_r = v_\theta = 0$$

Next, I'd like to argue that the pressure profile can be expected to be independent of ϕ .

$$p = p(r, \theta)$$

Since there exists only a ϕ -component of velocity, fluid streamlines will turn out to be circles (the contour corresponding to $r=\text{const}$ and $\theta=\text{const}$). The circle corresponds to $0 \leq \phi \leq 2\pi$.

By analogy with the last problem, you might guess (incorrectly) that pressure must decrease along the direction of flow. However, in steady flow p cannot continuously decrease with ϕ for all ϕ . At the very least, pressure must be periodic in ϕ ; in other words, $p(r, \theta, \phi) = p(r, \theta, \phi + 2\pi)$. So any decreases in pressure over part of the cycle will have to be balanced by increases over the remaining part. Why should the pressure be higher at some points along the streamline than at other? There is no geometrical asymmetry with respect to ϕ and no reason to expect any ϕ -dependence in the pressure.

The velocity profile automatically satisfies continuity:

$$\nabla \cdot \mathbf{v} = (r \sin \theta)^{-1} \partial v_\phi / \partial \phi = 0$$

Ignoring gravity, the Navier-Stokes equations in spherical coordinates become:♣

$$r: \quad -\rho v_\phi^2 / r = -\partial p / \partial r \quad (57)$$

$$\theta: \quad -\rho v_\phi^2 \cot \theta / r = -r^{-1} \partial p / \partial \theta \quad (58)$$

$$\phi: \quad 0 = \mu \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\phi}{\partial \theta} \right) - \frac{v_\phi}{r^2 \sin^2 \theta} \right\} \quad (59)$$

Notice that the pressure and velocity fields have been separated. We can first solve the ϕ -component for the velocity profile and then substitute the result into the r and θ -components to solve for the pressure profile. Based on the boundary conditions, we might try a solution of the form:

$$v_\phi(r, \theta) = r f(\theta)$$

When this is substituted into the ϕ -equation above, the r -dependence cancels out, leaving a second order ordinary differential equation in $f(\theta)$. The solution leads to:

♣ http://www.andrew.cmu.edu/course/06-703/NSE_sph.pdf

$$v_\phi = r\Omega \sin \alpha \frac{g(\theta)}{g(\alpha)} \quad (60)$$

where

$$g(\theta) = \cot \theta + \frac{1}{2} \ln \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) \sin \theta$$

This function is plotted in the figure at right. Notice that in the center (i.e. for θ near $\pi/2$) the function is nearly linear. A more careful asymptotic analysis would reveal that

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \{g(\theta)\} = 2\varepsilon + O(\varepsilon^5) \quad (61)$$

where

$$\varepsilon \equiv \frac{\pi}{2} - \theta$$

Notice that this is a linear function of θ as expected from the plot above.

Velocity Profile for Shallow Cones. Most cone-and-plate viscometers are designed with cone angles near to $\pi/2$. The reason for this will be apparent in a moment. Note the arc lengths $r\varepsilon$ and $r\varepsilon_1$ on the figure at right. For shallow cones (i.e. as $\alpha \rightarrow \pi/2$), $r\varepsilon$ asymptotically becomes the vertical distance from some arbitrary point (r, θ) in the fluid to the plate,

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \{r\varepsilon\} = z$$

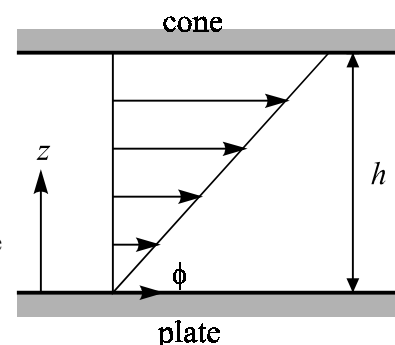
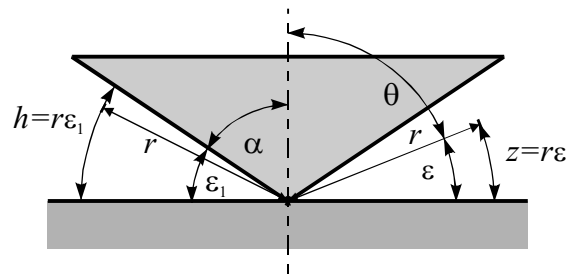
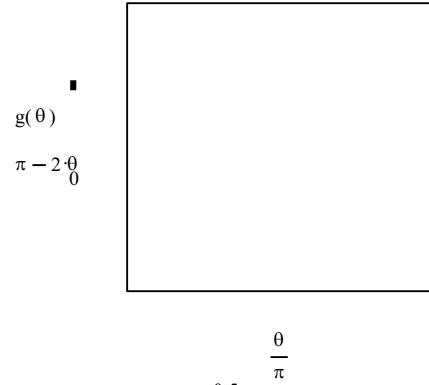
while $r\varepsilon_1$ becomes the vertical distance between the plate at the cone:

$$\lim_{\theta_1 \rightarrow \frac{\pi}{2}} \{r\varepsilon_1\} = h(r)$$

So in this limit we can use (61) to replace

as $\alpha \rightarrow \pi/2$:

$$g(\theta) \rightarrow 2\varepsilon = 2 \frac{z}{r}$$



and $g(\alpha) \rightarrow 2\varepsilon_1 = 2\frac{h}{r}$ and $\sin \alpha \rightarrow 1$

Then (60) simplifies to linear shear flow (at least locally)

for $\alpha \rightarrow \pi/2$:
$$v_\phi = r\Omega \frac{z}{h}$$

Notice that the rate of strain is independent of position:

for $\alpha \rightarrow \pi/2$:
$$\frac{\partial v_\phi}{\partial z} = \frac{r\Omega}{h(r)} = \frac{r\Omega}{r\varepsilon_1} = \frac{\Omega}{\varepsilon_1} = \text{const.}$$

This is an important advantage for a rheometer, since all the fluid experiences the same strain rate. Later, we will show that stress is also spatially uniform for shallow cones.

Torque. To use this device as a viscometer, we need to interpret torque measurements. So let's try to evaluate the torque for a given velocity profile. Recall that torque is force times lever arm:

$$T = (r \sin \alpha) F$$

Using vector notation:

$$\underline{T} = \mathbf{r} \times \mathbf{F}$$

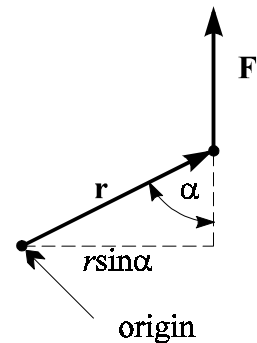
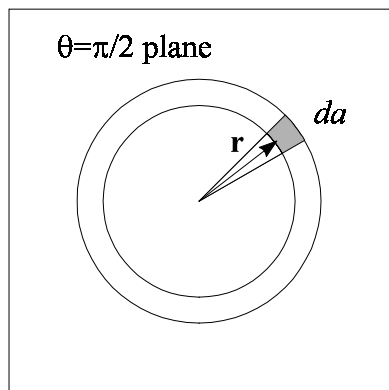
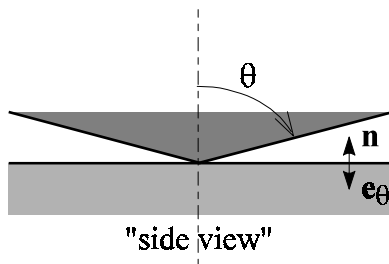
In our problem, the force \mathbf{F} is not concentrated at one point, but instead is distributed over the surface of the plate. Let's consider the contribution to force and torque from some differential element of surface having area da .

Once the velocity profile is known, we can evaluate the stress field from Newton's law of viscosity. Given the stress field, we can calculate the force on any differential solid surface element of area da from:

$$d\mathbf{F} = \mathbf{n} \cdot \underline{\underline{T}} da$$

where \mathbf{n} is a unit normal pointing *out* of the body $d\mathbf{F}$ acts *on*. Similarly, we calculated the contribution to the torque by crossing this force with the local lever arm:

$$d\underline{\underline{T}} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times (\mathbf{n} \cdot \underline{\underline{T}} da)$$



Let's calculate the torque exerted by the fluid acting on the stationary plate. Then we want to choose \mathbf{n} to point out of the plate or

$$\mathbf{n} = -\mathbf{e}_\theta$$

The net force exerted on the lower surface is:

$$\mathbf{F} = \int_A d\mathbf{F} = -\int_A \mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}} da$$

Similarly, the torque exerted on the lower surface is:

$$\underline{\underline{\mathcal{T}}} = \int_A \mathbf{r} \times d\mathbf{F} = \int_A \mathbf{r} \times (-\mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}}) da \quad (62)$$

First let's consider:

$$\mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}} = T_{\theta r} \mathbf{e}_r + T_{\theta\theta} \mathbf{e}_\theta + T_{\theta\phi} \mathbf{e}_\phi$$

To evaluate the torque, we need to first cross this vector with the lever arm, which is the position vector. In spherical coordinates, the position vector is:

$$\mathbf{r}(r, \theta, \phi) = r \mathbf{e}_r(\theta, \phi)$$

$$\begin{aligned} \mathbf{r} \times (\mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}}) &= r T_{\theta r} (\mathbf{e}_r \times \mathbf{e}_r) + r T_{\theta\theta} (\mathbf{e}_r \times \mathbf{e}_\theta) + r T_{\theta\phi} (\mathbf{e}_r \times \mathbf{e}_\phi) \\ &= r T_{\theta\theta} \mathbf{e}_\phi - r T_{\theta\phi} \mathbf{e}_\theta \end{aligned}$$

Anticipating that the torque vector $\underline{\underline{\mathbf{T}}}$ will have the same direction as the angular velocity $\underline{\underline{\Omega}}$, we dot both sides by \mathbf{e}_z which equals $-\mathbf{e}_\theta$ on the plate ($\theta = \pi/2$):

$$\frac{dT_z}{da} = -\mathbf{e}_\theta \cdot [\mathbf{r} \times (-\mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}})] = \mathbf{e}_\theta \cdot [\mathbf{r} \times (\mathbf{e}_\theta \cdot \underline{\underline{\mathbf{T}}})] = \mathbf{e}_\theta \cdot [r T_{\theta\theta} \mathbf{e}_\phi - r T_{\theta\phi} \mathbf{e}_\theta] = -r T_{\theta\phi}$$

Now we substitute Newton's law of viscosity:^{*}

$$T_{\theta\phi} = \tau_{\theta\phi} = \frac{\mu}{r} \left[\sin \theta \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{\sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] \quad (63)$$

What's left is to substitute our velocity profile from (60) into (63), which yields

$$T_{\theta\phi} = \frac{C}{\sin^2 \theta} \quad \text{where} \quad C \equiv -2\mu\Omega \frac{\sin \alpha}{g(\alpha)} \quad (64)$$

^{*} http://www.andrew.cmu.edu/course/06-703/NLV_sph.pdf

and where α is the cone angle. On the surface of the flat plate, $\theta = \pi/2$ and $\sin\theta = 1$. $T_{\theta\phi}$ (evaluated at $\theta=\pi/2$) will be independent of ϕ ; then we can choose da to be a ring of radius r and thickness dr , leaving:

$$T_z = \int_0^R (-r T_{\theta\phi})(2\pi r dr)$$

$$\underline{\mathcal{T}} = \frac{2}{3} \pi \mu R^3 \underline{\Omega} \frac{\sin \alpha}{g(\alpha)} \quad (65)$$

where R is the radius of the region of the plate which is wetted by the fluid.

Shear Stress and Torque for Shallow Cones. If the cone angle α is very shallow (i.e. $\alpha \rightarrow \pi/2$) then $T_{\theta\phi}$ will be practically independent of position: from (64):

$$\lim_{\alpha \rightarrow \theta \rightarrow \frac{\pi}{2}} T_{\theta\phi}(\theta) = C = \frac{-2\mu\Omega}{2\varepsilon}$$

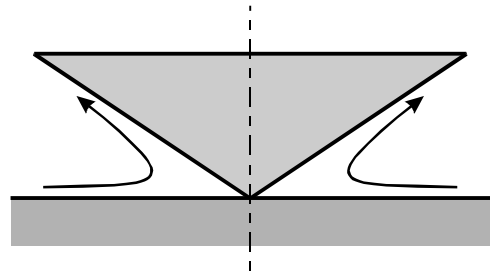
for any θ . This means the entire fluid experiences the same stress. This has a number of important advantages for rheological studies making the cone-and-plate viscometer one of the important *viscometric flows*. Substituting (61) into (65)

$$\lim_{\varepsilon \rightarrow 0} [\underline{\mathcal{T}}] = \frac{2}{3} \pi \mu R^3 \frac{\Omega}{\varepsilon}$$

Comment on Solution. There is a problem with our solution to NSE: when (60) is substituted back into (57) and (58), there is no single function $p(r, \theta)$ which will satisfy them. In other words, (60) is not an exact solution. It turns out that (60) is a reasonably good approximation if Ω is not too large. The exact solution has the form:

$$\mathbf{v} = r\Omega \sin \alpha \frac{g(\theta)}{g(\alpha)} \mathbf{e}_\phi + O(\Omega^2)$$

where $O(\Omega^2)$ means that this term vanishes like Ω^2 as $\Omega \rightarrow 0$. The centripetal force on fluid elements undergoing a circular orbit causes those fluid elements to be “thrown outward” in the $+r$ -direction. Since fluid elements near the rotating cone are rotating faster than fluid elements near the stationary plate, we have outflow near the cone supplied by inflow near the plate. The resulting r - and θ -components of velocity are called *secondary flow*, whereas the original ϕ -component is called the *primary flow*.



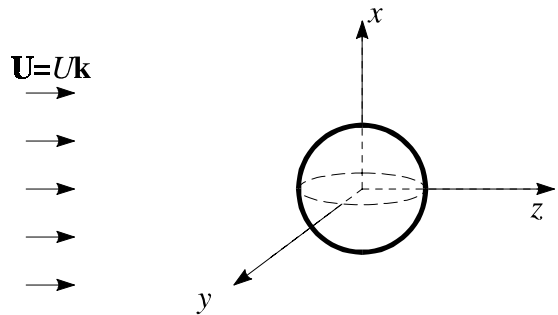
Ω^2 (the secondary flow) vanishes faster than Ω (the primary flow), so for small Ω , the leading term is approximately correct. This is called the *creeping-flow approximation*.

CREEPING FLOW AROUND A SPHERE ($Re \rightarrow 0$)

Let's return to the problem of flow around a sphere (or motion of a sphere through a stagnant fluid). For boundary conditions, we impose “no slip” on the surface of the sphere and far from the sphere the flow is undisturbed:

$$\text{at } r=a: \quad \mathbf{v} = \mathbf{0}$$

$$\text{as } r \rightarrow \infty: \quad \mathbf{v} \rightarrow \mathbf{U}$$



Here an exact solution to the Navier-Stokes equations is not possible. Of course, the vector equation can be converted into a scalar equations using the stream function, but that yields a 4th order nonlinear P.D.E. Although this could be solved numerically, considerable simplification can be obtained if either the viscous terms or the inertial terms can be neglected — even if they are not identically zero. One limiting case is **creeping flow** which corresponds to the limit in which the Reynolds number is small (i.e. $Re \rightarrow 0$). In this limit the inertial terms in the Navier Stokes equations can usually be neglected.

Scaling

To show that inertial terms are negligible, let's try to estimate the order of magnitude of viscous and inertial terms for uniform flow at speed U over a sphere of radius R .

$$\underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\text{inertia}} = -\nabla p + \underbrace{\mu \nabla^2 \mathbf{v}}_{\text{viscous}}$$

We will use a technique called **scaling** (akin to dimensional analysis). We start by listing all the parameters in the problem. In this problem, the parameters are

parameters: U, R, ρ and μ

A characteristic value for each term in the equations of motion is then written as a product of these parameters raised to some power:

$$\text{each term} \propto U^a R^b \rho^c \mu^d$$

For example, throughout most of the region, the fluid velocity is undisturbed:

$$|\mathbf{v}| \approx U$$

where the symbol “ \approx ” should be read as “scales like”. In “scaling” we ignore any position dependence as well as any numerical coefficients, so $|\mathbf{v}|$ scales as U , although $|\mathbf{v}|$ might be significantly less than U near the surface of the sphere. From the boundary conditions, \mathbf{v} changes from $\mathbf{0}$ at $r=R$ to \mathbf{U} at $r=\infty$. To estimate the magnitude of the gradient $\nabla \mathbf{v}$ we need to estimate

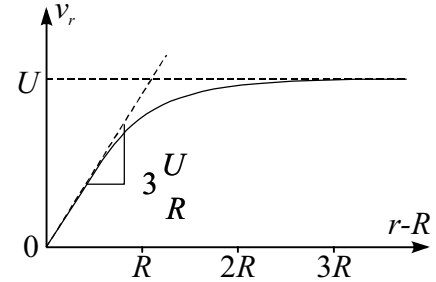
over what distance most of this change occurs. In the solution of the potential flow problem, the velocity profile along the rear stagnation line is

$$v_r(r, \theta = 0) = U \left[1 - \left(\frac{R}{r} \right)^3 \right] \cos \theta$$

We expect something similar for Stokes flow. Two of the nine scalar components of $\nabla \mathbf{v}$ are

$$\frac{\partial v_r}{\partial r} = 3U \frac{R^3}{r^4} \cos \theta \approx \frac{U}{R}$$

$$\frac{1}{r} \frac{\partial v_r}{\partial \theta} = -U \left(\frac{1}{r} - \frac{R^3}{r^4} \right) \sin \theta \approx \frac{U}{R}$$



We again ignore the position dependence: we scale r as R and treat $\cos \theta$ and $\sin \theta$ as “1”. Then both components above scale as U/R as do other components of $\nabla \mathbf{v}$ associated with derivatives of v_θ . So

$$|\nabla \mathbf{v}| \approx \frac{\Delta v}{\Delta r} \approx \frac{U}{R}$$

Likewise the velocity gradient can be expected to decay from a maximum value of U/R near the sphere surface to zero in the bulk; this change occurs over a distance on the order of R .

$$|\nabla^2 \mathbf{v}| \approx \frac{\Delta |\nabla \mathbf{v}|}{\Delta r} \approx \frac{U/R}{R} = \frac{U}{R^2}$$

With these estimates, we can further estimate the magnitude of viscous and inertial forces:

$$\text{inertia} = \rho |\mathbf{v} \cdot \nabla \mathbf{v}| \approx \rho(U) \left(\frac{U}{R} \right) = \frac{\rho U^2}{R}$$

$$\text{viscous} = \mu |\nabla^2 \mathbf{v}| \approx \mu \frac{U}{R^2}$$

$$\frac{\text{inertia}}{\text{viscous}} \approx \frac{\rho U^2 / R}{\mu U / R^2} = \frac{\rho U R}{\mu} = Re$$

where Re is the **Reynolds number**. So as $Re \rightarrow 0$, we should be able to neglect inertial forces

for $Re \ll 1$:

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{v} \quad (66)$$

$$\nabla \cdot \mathbf{v} = 0$$

at $r=R$:

$$\mathbf{v} = \mathbf{0}$$

as $r \rightarrow \infty$:

$$\mathbf{v} \rightarrow \mathbf{U}, p \rightarrow p_\infty$$

Eq. (66), called **Stokes Equation**, is common to all low-Reynolds number problems; it's not specific to the sphere. A common trick to reduce the number of unknowns is to take the curl of both sides of the equation:

$$\mathbf{0} = -\nabla \times \nabla p + \mu \nabla \times \nabla^2 \mathbf{v} \quad (67)$$

But

$$\nabla \times \nabla p = \mathbf{0}$$

Moreover, using identity F.1,

$$\nabla^2 \mathbf{v} = \nabla \underbrace{(\nabla \cdot \mathbf{v})}_0 - \nabla \times (\nabla \times \mathbf{v}) = -\text{curl}^2 \mathbf{v}$$

for incompressible flow. Then (67) becomes:

$$\text{curl}^3 \mathbf{v} = \mathbf{0} \quad (68)$$

Velocity Profile

We might try to seek a solution having the form of potential flow:

$$\mathbf{v} = \nabla \phi \quad (69)$$

since

$$\nabla \times \mathbf{v} = \nabla \times \nabla \phi = \mathbf{0}$$

(68) is automatically satisfied. But we know that the potential flow solution for the sphere does not satisfy the no slip boundary condition. On the other hand, the boundary conditions are axisymmetric:

$$v_\phi = 0, \partial/\partial\phi = 0$$

so we might seek a solution using the stream function:

$$\mathbf{v} = \nabla \times \left[\frac{\psi(r, \theta)}{r \sin \theta} \mathbf{e}_\phi \right]$$

Computing the curl in spherical coordinates using the tables

(http://www.andrew.cmu.edu/course/06-703/Vops_sph.pdf) (also see HWK #5, prob. 1):

$$\text{curl} \left(\frac{\mathbf{e}_\phi \psi}{r \sin \theta} \right) = \mathbf{v} = \underbrace{\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}}_{v_r} \mathbf{e}_r - \underbrace{\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}}_{v_\theta} \mathbf{e}_\theta$$

Taking the curl a second time, using the same tables:

$$\text{curl}^2 \left(\frac{\mathbf{e}_\phi \psi}{r \sin \theta} \right) = \nabla \times \mathbf{v} = \dots = \frac{\mathbf{e}_\phi}{r \sin \theta} (-E^2 \psi) \quad (70)$$

where E^2 is a partial differential operator given by:

$$E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right)$$

Since we have to take the curl twice more, it might look like we have a lot of algebra to look forward to. But, as it turns out, the rest is easy. We need to evaluate:

$$\text{curl}^3 \mathbf{v} = \text{curl}^2 (\nabla \times \mathbf{v}) = \text{curl}^2 \left\{ \frac{\mathbf{e}_\phi (-E^2 \psi)}{r \sin \theta} \right\}$$

where the second equality is (70). The argument of this final curl-squared has exactly the same form as that of the left-hand side of Eq. (70), except that the scalar ψ , which is a function of r, θ , is replaced by $-E^2 \psi$, which is also a scalar function of r, θ . So all I have to do is to replace ψ by $-E^2 \psi$ on the right-hand side of (70):

$$\text{curl}^3 \mathbf{v} = \text{curl}^2 \left\{ \frac{\mathbf{e}_\phi (-E^2 \psi)}{r \sin \theta} \right\} = \frac{\mathbf{e}_\phi}{r \sin \theta} \left[-E^2 (-E^2 \psi) \right] = \frac{\mathbf{e}_\phi}{r \sin \theta} E^2 (E^2 \psi)$$

To satisfy (68), which represents the curl of the Navier-Stokes equation, we choose the streamfunction to satisfy:

$$E^2(E^2 \psi) = 0$$

Translating the boundary conditions at $r=R$ in terms of stream function:

$$v_r=0: \quad \partial \psi / \partial \theta = 0 \quad \text{at } r=R$$

$$v_\theta=0: \quad \partial \psi / \partial r = 0$$

Translating the boundary conditions at $r \rightarrow \infty$ in terms of stream function (see HWK #5, Prob. 1a):

$$\text{as } r \rightarrow \infty: \quad \psi \rightarrow (1/2) U r^2 \sin^2 \theta \quad (71)$$

The trivial solution $\psi = 0$ satisfies the P.D.E. and the b.c. at $r=R$, but not the b.c. at $r \rightarrow \infty$. Based on the form of this nonhomogeneous b.c., we guess the solution has the following form:

Try:
$$\psi(r, \theta) = f(r) \sin^2 \theta$$

$$E^2 \psi = E^2 [f(r) \sin^2 \theta] = \dots = (f'' - 2r^{-2}f) \sin^2 \theta = g(r) \sin^2 \theta$$

where:
$$g(r) = f'' - 2r^{-2}f$$

Then:
$$E^2(E^2 \psi) = (g'' - 2r^{-2}g) \sin^2 \theta$$

The Navier-Stokes equation become

$$E^2(E^2 \psi) = 0: \quad g'' - 2r^{-2}g = 0 \quad (72)$$

$$\text{def'n of } g: \quad f'' - 2r^{-2}f = g(r) \quad (73)$$

$$\text{b.c. 's:} \quad f \rightarrow (1/2)Ur^2 \text{ as } r \rightarrow \infty \quad (74)$$

$$f = f' = 0 \text{ at } r = R \quad (75)$$

These two coupled second-order O.D.E.'s can be combined to produce a single fourth-order O.D.E. in $f(r)$. The result is a Cauchy-Euler equation whose general solution is (see footnote on page 52):

$$f(r) = c_1 r^{-1} + c_2 r + c_3 r^2 + c_4 r^4$$

$$\text{Applying b.c. (74):} \quad c_4 = 0$$

$$c_3 = (1/2)U$$

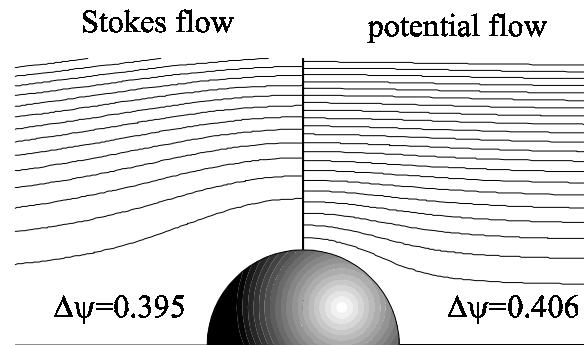
As r becomes large, terms which are proportional to higher power of r dominate those of lower power. To have the third term win over the fourth, requires us to kill the fourth term by setting its coefficient to zero. The remaining constants are evaluated in a straightforward manner. The result is:

$$\psi(r, \theta) = UR^2 \left[\frac{1}{4} \frac{R}{r} - \frac{3}{4} \frac{r}{R} + \frac{1}{2} \left(\frac{r}{R} \right)^2 \right] \sin^2 \theta \quad (76)$$

$$v_r(r, \theta) = U \left[1 - \frac{3}{2} \frac{R}{r} + \frac{1}{2} \left(\frac{R}{r} \right)^3 \right] \cos \theta$$

$$v_\theta(r, \theta) = -U \left[1 - \frac{3}{4} \frac{R}{r} - \frac{1}{4} \left(\frac{R}{r} \right)^3 \right] \sin \theta$$

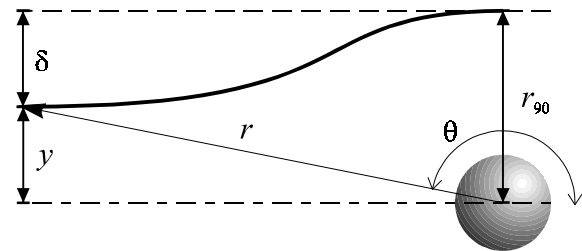
The figure at right compares the streamlines for Stokes flow with those for potential flow. The streamlines correspond to values of the streamfunction which are uniformly spaced at about the same interval for both profiles.



Displacement of Distant Streamlines

From the figure above, you can see that streamlines in Stokes flow are displaced away from the sphere – even those streamlines which are quite distant – especially compared with potential flow, in which the distant streamlines become straight. It turns out that *all* streamlines are displaced away from the sphere in Stokes flow.

Consider the streamline in the figure at right which corresponds to $\psi = \psi_0$. Far upstream, the coordinates of the streamline correspond to $r \rightarrow \infty$ and $\theta \rightarrow \pi$ such that $r \sin \theta \rightarrow \text{const}$, which we will denote as y (see Hwk #5, Prob. 1a). The relationship between y and ψ_0 can be deduced from (71):



$$\lim_{\substack{r \rightarrow \infty \\ \theta \rightarrow \pi}} \psi(r, \theta) \rightarrow \frac{1}{2} U \underbrace{r^2 \sin^2 \theta}_{y^2}$$

$$\psi_0 = \frac{1}{2} U y^2 \quad (77)$$

At the equatorial plane $\theta = \pi/2$, the r coordinate of the streamline must satisfy (76):

$$\psi(r_{90}, \frac{\pi}{2}) = \psi_0 = UR^2 \left[\frac{1}{4} \frac{R}{r_{90}} - \frac{3}{4} \frac{r_{90}}{R} + \frac{1}{2} \left(\frac{r_{90}}{R} \right)^2 \right] \quad (78)$$

Eliminating ψ_0 between (77) and (78):

$$y^2 = r_{90}^2 - \frac{3}{2} R r_{90} + \frac{1}{2} \frac{R^3}{r_{90}}$$

or

$$y = r_{90} \sqrt{1 - \frac{3}{2} \left(\frac{R}{r_{90}} \right) + \frac{1}{2} \left(\frac{R}{r_{90}} \right)^3}$$

For distant streamlines, r_{90} will be very large compared to R , so that $R/r_{90} \ll 1$. Then the square-root in the expression above is unity plus a small correction, which can be estimated as the first term in a Taylor series expansion of the square-root function: $\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon + \dots$

$$y = r_{90} \left[1 - \frac{3}{4} \left(\frac{R}{r_{90}} \right) + O\left(\frac{R}{r_{90}} \right)^2 \right] \approx r_{90} - \frac{3}{4} R$$

where the “ $O(R/r_{90})^2$ ” means that these terms decay to zero like $(R/r_{90})^2$ as $R/r_{90} \rightarrow 0$. When R/r_{90} is sufficiently small, we can neglect these terms compared to the others. This is how the second (approximate) equality above was obtained.

Finally

$$\lim_{r_{90} \rightarrow \infty} \{r_{90} - y\} = \delta = \frac{3}{4} R$$

Thus, even streamlines infinitely far away are displaced a distance equal to $\frac{3}{4} R$.

Pressure Profile

Once the velocity profile is known, the pressure can be determined from

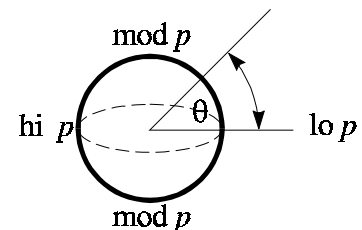
$$\nabla p = \mu \nabla^2 \mathbf{v}$$

Substituting \mathbf{v} :

$$\begin{cases} \frac{\partial p}{\partial r} = 3\mu U R r^{-3} \cos \theta \\ \frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{3}{2} \mu U R r^{-3} \sin \theta \end{cases}$$

and then integrating with $p \rightarrow p_\infty$ at $r \rightarrow \infty$ as the b.c. yields:

$$p(r, \theta) = p_\infty - \frac{3}{2} \mu U R \frac{\cos \theta}{r^2}$$



This profile corresponds to a higher pressure on the upstream side of the sphere ($\theta = \pi$) than on the downstream side ($\theta = 0$).

the downstream side ($\theta=0$). Thus it appears as if a drag will be produced by this solution.

$$-\int_A p \mathbf{n} da = \dots = 2\pi\mu RU \mathbf{k}$$

is called the **form drag** which arises from normal stresses. This however is not the total drag. More generally, we expect the net force exerted on the particle by the fluid is:

$$\mathbf{F} = \oint_A \mathbf{n} \cdot \underline{\underline{\mathbf{T}}} da = \underbrace{-\oint_A p \mathbf{n} da}_{2\pi\mu RU \mathbf{k} \text{ (form drag)}} + \underbrace{\oint_A \mathbf{n} \cdot \underline{\underline{\boldsymbol{\tau}}} da}_{4\pi\mu RU \mathbf{k} \text{ (skin friction)}} = 6\pi\mu RU \mathbf{k}$$

where A is the surface of the sphere $r=R$

$$\mathbf{n} = \mathbf{e}_r$$

From symmetry, we expect that this force will have only a z -component (the direction of bulk motion); so let's concentrate on finding that component:

$$F_z = \int_A \mathbf{k} \cdot (\mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}}) da \quad (79)$$

Substituting

$$\underline{\underline{\mathbf{T}}} = -p \underline{\underline{\mathbf{I}}} + \underline{\underline{\boldsymbol{\tau}}}$$

$$\mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}} = -p(\mathbf{e}_r \cdot \underline{\underline{\mathbf{I}}}) + \mathbf{e}_r \cdot \underline{\underline{\boldsymbol{\tau}}}$$

$$= -p\mathbf{e}_r + \mathbf{e}_r \cdot \underline{\underline{\boldsymbol{\tau}}}$$

$$\mathbf{e}_r \cdot \underline{\underline{\boldsymbol{\tau}}} = \tau_{rr}\mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta + \tau_{r\phi}\mathbf{e}_\phi$$

For the determined velocity profile, Newton's law of viscosity in spherical coordinates (W:146) yields $\tau_{r\phi}=0$ for all r and $\tau_{rr}=0$ at $r=R$:

at $r=R$:

$$\mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}} = -p\mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta$$

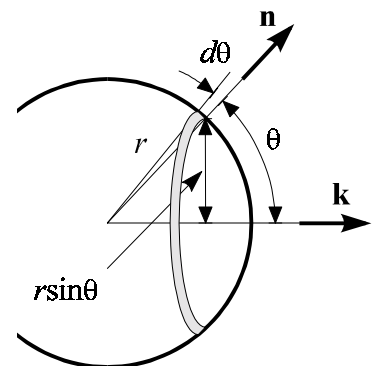
$$\mathbf{k} \cdot (\mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}}) = -p(\mathbf{k} \cdot \mathbf{e}_r) + \tau_{r\theta}(\mathbf{k} \cdot \mathbf{e}_\theta)$$

Using $\mathbf{k} = \mathbf{e}_z$ from (56): $\mathbf{k} \cdot (\mathbf{e}_r \cdot \underline{\underline{\mathbf{T}}}) = -p\cos\theta - \tau_{r\theta}\sin\theta$

Finally, we can integrate using an azimuthal strip of width $Rd\theta$ and of radius $R\sin\theta$:

$$da = 2\pi(R\sin\theta)(Rd\theta)$$

(79) becomes:



$$F_z = 2\pi R^2 \int_0^\pi \left[-p(R, \theta) \sin \theta \cos \theta - \tau_{r\theta}(R, \theta) \sin^2 \theta \right] d\theta$$

Finally $\tau_{r\theta}$ is expressed in terms of the velocity profile using Newton's law of viscosity and subsequent integration yields:

$$F_z = 2\pi\mu RU + 4\pi\mu RU$$

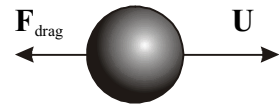
We have already noted that the contribution from the pressure profile is called form drag. The second contribution is called **skin friction**. The total drag force is

$$F_{drag} = 6\pi\mu RU \quad (\text{moving reference frame})$$

which is known as **Stokes Law** (1850).^{*} In the expression above, U is the velocity of the distant fluid flowing around a stationary sphere. If we switch back to the original (stationary) reference frame (see page 48f), Stokes Law becomes

$$F_{drag} = -6\pi\mu RU \quad (\text{stationary reference frame})$$

where now U is the velocity of the sphere moving through otherwise stagnant fluid. Note that the drag force acts in a direction opposing the direction of motion of the sphere.



Experimental results for the drag force around submerged bodies are usually expressed in terms of a dimensionless **drag coefficient**. The quantity that is used to make this drag force dimensionless is

$$\frac{1}{2}\rho U^2 [=] \frac{\text{K.E.}}{\text{vol}} = \frac{\text{force}}{\text{area}}$$

This quantity also can be shown (according to Bernoulli's equation) to represent the Δp required to stop fluid which is flowing at speed U . Multiplying this by the projected area gives the force required to stop the flow, which would otherwise pass through the sphere:

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho U^2 \times \underbrace{\pi R^2}_{\text{projected area}}}$$

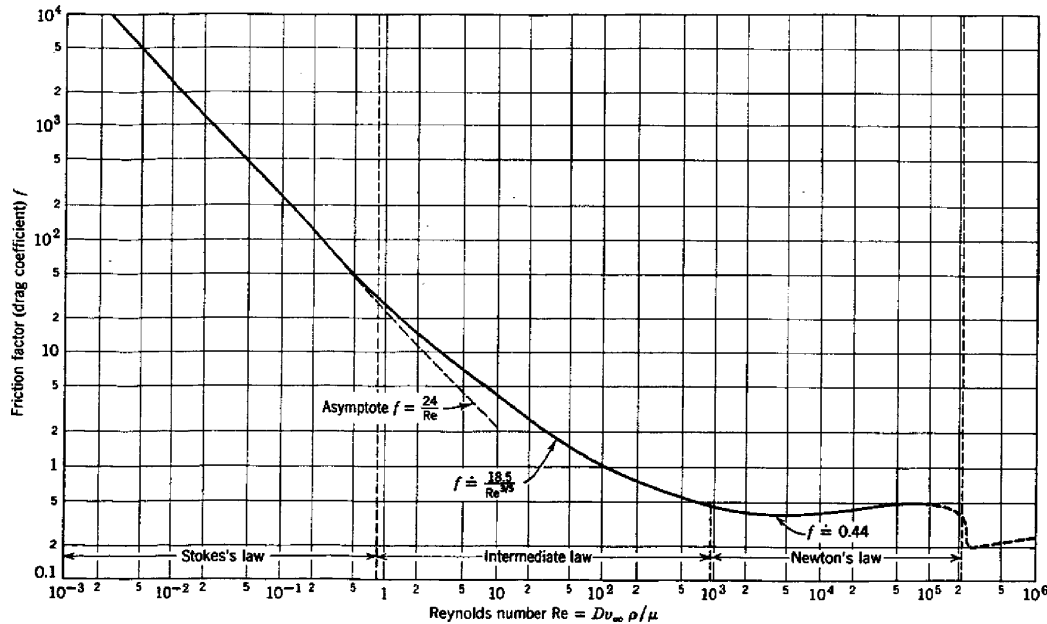
^{*} Sir George G. Stokes (1819-1903), born in Shreen, Ireland, educated at Cambridge; theoretical physicist.

where πR^2 is the projected area of the sphere; in other words, πR^2 is the area of the sphere's shadow cast along the direction of flow. Using this definition, Stokes equation for the drag force on a sphere yields:

$$C_D = \frac{12}{Re}$$

where

$$Re \equiv \frac{\rho UR}{\mu}$$



Comparison with experimental results confirm that this works very well for $Re < 0.1$. The reproduction above (taken from BSL, p192) uses the diameter for Reynolds number, rather than the radius. The “friction factor” on the y-axis differs by a factor of two from the drag coefficient defined above.

CORRECTING FOR INERTIAL TERMS

For larger Re , Stokes law underestimates the drag force. Of course, this is due to the increasing importance of inertia which has been neglected. A number of investigators have attempted to extend the validity of Stokes law by including inertial terms in the analysis. We will now summarize some of these and hint at the difficulties involved.

First, note that as $Re \rightarrow 0$ for flow around a sphere, the NSE approximated by

$$\mu \nabla^2 \mathbf{v} = \nabla p$$

which is called **Stokes equation**. Stokes (1850) solved this equation to obtain:

$$\mathbf{F} = 6\pi\mu R\mathbf{U}.$$

The usual approximation to solving differential equations containing a small parameter (i.e. ϵ) is to perform a ***perturbation expansion***. Basically, the idea is to perturb the small parameter away from zero value by means of a Taylor series.

PERTURBATION EXPANSION

time out: Let's illustrate the main idea behind this powerful mathematical technique by means of a simple example (Example 25.1 from Greenberg). Find the solution $y(x;\epsilon)$ to the following ODE involving a small parameter which we denote as ϵ :

$$y' + y + \epsilon y^2 = 0 \quad (80)$$

subject to the initial condition: $y(0) = \cos \epsilon \quad (81)$

where the independent variable spans the range $0 \leq x < \infty$ and the parameter ϵ is small: $0 \leq \epsilon \ll 1$. Note that the solution of this problem is almost trivial in the special case of $\epsilon=0$ (first-order linear ODE)

$$y(x;0) = e^{-x}$$

but obtaining a solution when $\epsilon \neq 0$ is more challenging (because the ODE becomes nonlinear). The general idea behind a perturbation expansion is to “perturb” the easily obtained solution away from $\epsilon=0$ by seeking a solution in the form of a Taylor series expansion about $\epsilon=0$:

$$y(x;\epsilon) = \underbrace{y(x;0)}_{y_0(x)} + \underbrace{\frac{\partial y(x;\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0}}_{y_1(x)} \epsilon + \frac{1}{2!} \underbrace{\frac{\partial^2 y(x;\epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=0}}_{y_2(x)} \epsilon^2 + \dots$$

Of course, this assumes that $y(x,\epsilon)$ is “analytic” about $\epsilon=0$.^{*} It remains to be seen if the solution has this property or not. Renaming the unknown coefficients (i.e. the partial derivatives), we look for a solution having the form

$$y(x,\epsilon) = y_0(x) + y_1(x)\epsilon + y_2(x)\epsilon^2 + \dots \quad (82)$$

Note that the coefficients $y_0, y_1, y_2 \dots$ are not functions of the parameter ϵ . Substituting (82) into (80):

$$(y'_0 + y'_1\epsilon + y'_2\epsilon^2 + \dots) + (y_0 + y_1\epsilon + y_2\epsilon^2 + \dots) + \epsilon(y_0 + y_1\epsilon + y_2\epsilon^2 + \dots)^2 = 0 \quad (83)$$

^{*} “Analytic” means that these partial derivatives exist and that the series converges to $y(x,\epsilon)$.

Next we expand the squared sum by distributing each term of the first series over each term in the second series:

$$(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots)^2 = (y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots)(y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots)$$

$$\begin{array}{r} y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots \\ y_0 + y_1\varepsilon + y_2\varepsilon^2 + \dots \\ \hline y_0^2 + y_0y_1\varepsilon + y_0y_2\varepsilon^2 + \dots \\ y_0y_1\varepsilon + y_1^2\varepsilon^2 + \dots \\ \hline y_0y_2\varepsilon^2 + \dots \\ \hline y_0^2 + 2y_0y_1\varepsilon + (y_1^2 + 2y_0y_2)\varepsilon^2 + \dots \end{array}$$

Substituting this result for the squared sum in to (83) and collecting terms of like power in ε :

Solution continues ... and will be completed in the next revision of the notes.

First, let's write the equation in dimensionless form: we will denote the dimensionless variables using an asterisk:

$$\mathbf{v}^* \equiv \frac{\mathbf{v}}{U} \quad \mathbf{r}^* \equiv \frac{\mathbf{r}}{R} \quad \nabla^* \equiv R\nabla \quad p^* \equiv \frac{p - p_\infty}{\mu U / R} \quad \varepsilon \equiv \text{Re} = \frac{\rho UR}{\mu}$$

The Navier-Stokes equations for steady flow become:

$$\varepsilon \mathbf{v}^* \cdot \nabla^* \mathbf{v}^* = \nabla^{*2} \mathbf{v}^* - \nabla^* p^*$$

We then look for a solution having the form of a Taylor series expansion about $\varepsilon = 0$:

$$\mathbf{v}^*(\mathbf{r}^*, \theta; \varepsilon) = \mathbf{v}_0(\mathbf{r}^*, \theta) + \varepsilon \mathbf{v}_1(\mathbf{r}^*, \theta) + \varepsilon^2 \mathbf{v}_2(\mathbf{r}^*, \theta) + \dots$$

Similarly for the pressure profile. Substituting this infinite series into the Navier-Stokes equation (dropping the *'s)

$$\varepsilon(\mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \dots) \cdot (\nabla \mathbf{v}_0 + \varepsilon \nabla \mathbf{v}_1 + \dots) = (\nabla^2 \mathbf{v}_0 + \varepsilon \nabla^2 \mathbf{v}_1 + \dots) - (\nabla p_0 + \varepsilon \nabla p_1 + \dots)$$

$$\mathbf{0} \varepsilon^0 + (\mathbf{v}_0 \cdot \nabla \mathbf{v}_0) \varepsilon^1 + \dots = (\nabla^2 \mathbf{v}_0 - \nabla p_0) \varepsilon^0 + (\nabla^2 \mathbf{v}_1 - \nabla p_1) \varepsilon^1 + \dots$$

Next we bring all terms to the right-hand side of the equation. We then collect terms of like order in the small parameter (i.e. terms which are multiplied by ε raised to the same power). To

obtain zero for the sum for all values of ε we cannot rely on cancellation of positive and negative terms.* Instead, the coefficient of each ε^n must vanish separately. This leaves the following equations:

$$\varepsilon^0: \quad \mathbf{0} = \nabla^2 \mathbf{v}_0 - \nabla p_0 \quad (84)$$

$$\varepsilon^1: \quad \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = \nabla^2 \mathbf{v}_1 - \nabla p_1 \quad (85)$$

and so on ... Note that (84) is just Stokes equation. The solution for \mathbf{v}_0 can then be substituted into (85) leaving a linear equation to be solved for \mathbf{v}_1 and p_1 .

This is the approach used by Whitehead (1889).♥ In many problems, this procedure works. Unfortunately, the solution for the higher order terms in the current problem cannot satisfy the boundary conditions. This result is known as *Whitehead's Paradox*.♦ Another method must be used.

As an alternative, *Oseen* (1910) used an entirely different approach. He approximated the inertial terms and solved:

$$\rho \mathbf{U} \cdot \nabla \mathbf{v} = \mu \nabla^2 \mathbf{v} - \nabla p$$

thus obtaining: $\mathbf{F} = 6\pi\mu R\mathbf{U}[1 + (3/8)Re]$

In principle, one could refine the solution further by substituting the resulting solution for \mathbf{v} in place of \mathbf{U} and then re-solving for an improved \mathbf{v} . In practice, although the Oseen equations are linear, their solution is sufficiently difficult that no second approximations are known.

In 1957, *Proudman & Pearson* obtained the next order correction using a different technique called *matched-asymptotic expansions*. In this technique a different form for the expansion is sought near the sphere (which is called the “inner expansion”):

$$r \approx R: \quad \mathbf{v}^i = \mathbf{v}_0^i(r, \theta) + \mathbf{v}_1^i(r, \theta)(Re) + \mathbf{v}_2^i(r, \theta)(Re)^2 + \dots$$

* If the sum of different functions of ε vanished for one particular value of ε (as a result of cancellation of positive and negative terms), then this same sum of functions evaluated at a different value of ε would not necessarily vanish. The only way we can guarantee that the sum vanishes for every value of ε is to make every term vanish for value of ε . See also footnote on page 27 which concerns integrals rather than sums.

♥ Alfred North Whitehead (1861-1947), English mathematician and philosopher, who collaborated with Bertrand Russell on *Principia Mathematica* (1910-13) and, from the mid-1920s, taught at Harvard University and developed a comprehensive metaphysical theory.

♦ See Van Dyke, p152-3.

and far from the sphere (which is called the “outer expansion”):

$$r \gg R: \quad \mathbf{v}^o = \mathbf{v}_0^o(\rho, \theta) + \mathbf{v}_1^o(\rho, \theta)(Re) + \mathbf{v}_2^o(\rho, \theta)(Re)^2 + \dots$$

where

$$\rho = rRe$$

The inner expansion is identical with Whitehead's which can be made to satisfy the no slip condition at $r=R$ but the result does not have the correct form far from the sphere. Rather, the outer expansion is made to satisfy the boundary condition far from the sphere and then appropriately match with the inner solution to determine the remaining integration constants. The result is:

$$\mathbf{F} = 6\pi\mu R\mathbf{U}[1 + (3/8)Re + (9/40)Re^2 \ln Re + \dots]$$

The term inside square brackets can be thought of as a correction to Stokes equation:

$Re =$	0.01	0.1	1.0
$[...] =$	1.004	1.032	1.375

Comparing the bracketed term with unity gives some idea of the error incurred by neglecting inertia.

FLOW AROUND CYLINDER AS $Re \rightarrow 0$

Now let's look at the analogous problem of uniform flow normal to a cylinder at very low Reynolds number. If we drop the inertial terms in the Navier-Stokes equation, we obtain:

$$\mu \nabla^2 \mathbf{v} = \nabla p$$

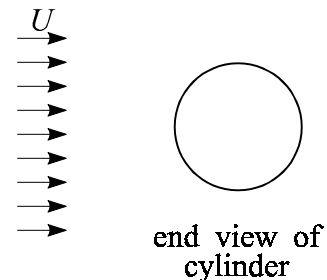
$$\nabla \cdot \mathbf{v} = 0$$

$r \rightarrow \infty:$

$$\mathbf{v} \rightarrow \mathbf{U}$$

$r=R:$

$$\mathbf{v} = \mathbf{0}$$



For flow normal to the cylinder, we expect the velocity profile to correspond to 2-D flow:

$$v_z = 0 \text{ and } \partial/\partial z = 0$$

So that a solution can be found using the stream function:

$$\mathbf{v} = \nabla \times [\psi(r, \theta) \mathbf{e}_z]$$

For slow enough flows, we expect that inertial terms can be neglected, leaving Stokes Equation (66). After the pressure is eliminated (by taking the curl of Stokes equation), we obtain a single equation in the unknown velocity profile :

recall (68): $\text{curl}^3 \mathbf{v} = \mathbf{0} = \text{curl}^4(\psi \mathbf{e}_z) = \nabla^2(\nabla^2 \psi) \mathbf{e}_z$

The boundary conditions are determined in a manner similar to those for a sphere (see Hwk #7, Prob. 1):

$$r \rightarrow \infty: \quad \psi \rightarrow U r \sin \theta$$

$$r=R: \quad \partial \psi / \partial r = \partial \psi / \partial \theta = 0$$

Based on the b.c. at $r \rightarrow \infty$ (which is the only nonhomogeneous part of the problem), the stream function should have the form:

$$\psi(r, \theta) = f(r) \sin \theta$$

Requiring $\nabla^2(\nabla^2 \psi) = 0$

generates an ODE for $f(r)$ whose general solution can be obtained. Unfortunately, none of the particular solutions can satisfy all of the b.c.'s (see HWK #7, Prob. 1). This is known as:

Stokes Paradox (1850) - Stokes equation for uniform flow normal to a cylinder has no solution.

As it turns, the inertial terms dropped by Stokes are not entirely negligible -- no matter how small Re is. **Lamb** (1911)* obtained a solution for the circular cylinder as $Re \rightarrow 0$ using Oseen's approximation:

$$\rho \mathbf{U} \cdot \nabla \mathbf{v} = \mu \nabla^2 \mathbf{v} - \nabla p$$

$$\nabla \cdot \mathbf{v} = 0$$

Lamb's solution for the drag force per unit length of cylinder is:

$$\frac{F}{L} = \frac{4\pi\mu U}{\frac{1}{2} - \gamma - \ln \frac{Re}{4}}$$

where $\gamma = 0.577...$ is Euler's constant. Notice that, unlike Stokes' solution for the sphere, Re appears explicitly in this result. No matter how small Re , this logarithm term in the denominator is never negligible.

* Sir Horace Lamb (1839-1934), English mathematician who contributed to the field of mathematical physics. He wrote *Hydrodynamics* (1895) which was, for many years, the standard work on hydrodynamics. His many papers, principally on applied mathematics, detailed his researches on wave propagation, electrical induction, earthquake tremors, and the theory of tides and waves

Boundary-Layer Approximation

FLOW AROUND CYLINDER AS $Re \rightarrow \infty$

$$\underbrace{\rho \mathbf{v} \cdot \nabla \mathbf{v}}_{\text{inertia}} = -\nabla p + \underbrace{\mu \nabla^2 \mathbf{v}}_{\text{viscous}}$$

We have just seen that restricting attention to the limiting case of very small Reynolds number allows an analytical solution to the Navier-Stokes equation by neglecting or approximating the inertial terms. Similarly, we might expect that, in the opposite limit of very large Reynolds number, we might be able to obtain an approximate solution by neglecting or approximating the viscous terms.

Let's return to the problem of flow normal to a cylinder, but at very large Re . If we just drop the viscous terms from the Navier-Stokes equation, we get Euler's equation for an *ideal fluid*. Recall that a solution which satisfies the differential equation and the boundary condition far from the cylinder is *potential flow*. From HWK #4, Prob. 4a:

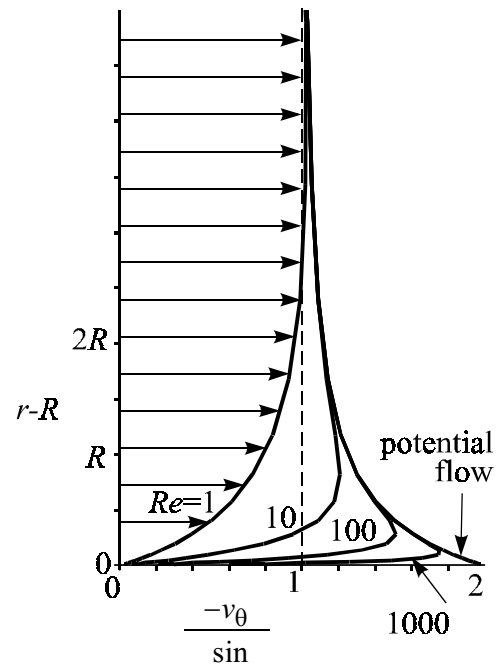
$$v_r(r, \theta) = U \left[1 - \left(\frac{R}{r} \right)^2 \right] \cos \theta$$

$$v_\theta(r, \theta) = -U \left[1 + \left(\frac{R}{r} \right)^2 \right] \sin \theta$$

At $r=R$: $v_r(R, \theta) = 0$

and $v_\theta(R, \theta) = -2U \sin \theta$

which violates the no-slip boundary condition. An exact solution of the NSE's can be obtained numerically. Comparing the solution for $Re \gg 1$ to the potential flow solution (see figure at right), we see that the exact solution follows potential flow everywhere except in a narrow region near the surface of the cylinder where the exact solution turns downward in order to satisfy the no-slip condition.



In the potential flow solution the velocity gradient goes like

outside b.l.:

$$\frac{\partial v_\theta}{\partial r} \approx \frac{U}{R}$$

This is also the behavior of the exact solution outside the boundary layer. By contrast, near the surface (inside the boundary layer) as the Reynolds number increases, the velocity gradient gets steeper. A closer analysis (which we will perform in a few lectures) reveals

inside b.l.:

$$\lim_{\text{Re} \rightarrow \infty} \left\{ \frac{\partial v_\theta}{\partial r} \bigg|_{r=R} \right\} \approx \frac{U}{R} \sqrt{\text{Re}}$$

The thickness of this region in which the two solutions differ decreases as the Re gets larger. This region is known as the:

boundary layer: a very thin region near to a boundary in which the solution has a gradient which is orders of magnitude larger than its characteristic value outside the region.

MATHEMATICAL NATURE OF BOUNDARY LAYERS

Boundary layers arise in solutions of differential equations in which the highest order derivative is multiplied by a small parameter. To illustrate the mathematical singularity which results, consider a simple example:

Example: find asymptotic solution to the following problem as $\varepsilon \rightarrow 0$:

$$\varepsilon y'' + y' + y = 0$$

subject to:

$$y(0) = 0$$

$$y(1) = 1$$

Problem is to find the asymptotic behavior of $y(x)$ as $\varepsilon \rightarrow 0$. This problem was presented Prandtl* (the father of boundary-layer analysis) to his class on fluid mechanics at Goettingen U. during the winter semester of 1931/2.

Solution: For ε sufficient small, you might guess that the first term can be neglected, leaving

$$y' + y = 0$$

whose general solution is:

$$y(x) = A \exp(-x)$$

* Ludwig Prandtl (1875-1953), German physicist who is considered to be the father of aerodynamics. His discovery (1904) of the boundary layer, which adjoins the surface of a body moving in air or water, led to an understanding of skin friction drag and of the way in which streamlining reduces the drag of airplane wings and other moving bodies.

Now A can be chosen to satisfy either of the boundary conditions, but not both. To satisfy $y(0) = 0$, we must choose $A=0$ which does not satisfy $y(1) = 1$:

$$y(0) = 0 \rightarrow A=0 \rightarrow y(x) = 0 \text{ for all } x \rightarrow y(1) = 0 \neq 1$$

On the other hand, if we choose $A=e$ to satisfy $y(1) = 1$, then we cannot satisfy $y(0) = 0$:

$$y(1) = 1 \rightarrow A=e \rightarrow y(x) = \exp(1-x) \text{ for all } x \rightarrow y(0) = e \neq 0$$

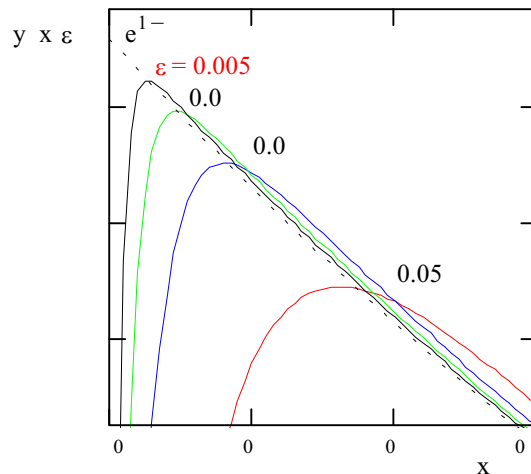
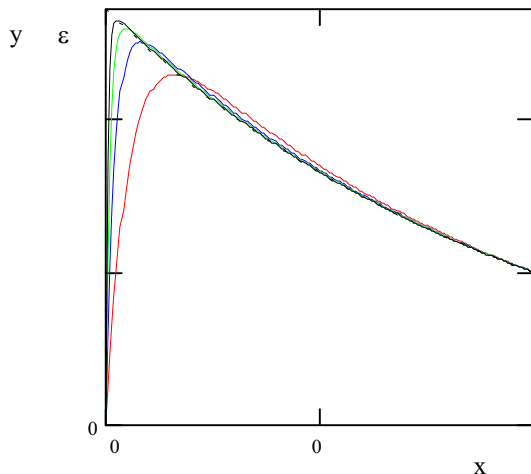
The reason we can't satisfy both boundary conditions with this approximation is that, by neglecting the first term, the order of the differential equation reduced from 2 to 1. With a first order O.D.E., we can only satisfy one boundary condition. Thus $\varepsilon=0$ is *singularly different* from ε being arbitrarily small, but not identically zero.

$$\varepsilon=0 \rightarrow \text{O.D.E. is 1st order}$$

$$\varepsilon \rightarrow 0 \rightarrow \text{O.D.E. is 2nd order}$$

Now the exact solution to this linear O.D.E. with constant coefficients is easily determined:

$$y(x; \varepsilon) = \exp\left(\frac{1-x}{2\varepsilon}\right) \frac{\sinh\left(\frac{x}{2\varepsilon} \sqrt{1-4\varepsilon}\right)}{\sinh\left(\frac{1}{2\varepsilon} \sqrt{1-4\varepsilon}\right)}$$



Comparing the exact solution to that obtained by neglecting the term containing the small parameter for the case of $\varepsilon=0.05$, we see that the approximation is good except near $x=0$. For smaller ε , the region in which the exact and approximate solutions differ shrinks. To see what's happening to cause this problem, let's take advantage of knowing the exact solution and deduce its asymptotic behavior inside the boundary layer. In particular, let's look at the initial slope:

$$y'(0;\varepsilon) = \frac{\sqrt{1-4\varepsilon}}{2\varepsilon} \frac{\exp\left(\frac{1}{2\varepsilon}\right)}{\sinh\left(\frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}\right)}$$

Now consider the limiting behavior as $\varepsilon \rightarrow 0$.

$$\sqrt{1-4\varepsilon} \rightarrow 1$$

and

$$1/2\varepsilon \rightarrow \infty$$

$$\lim_{\varepsilon \rightarrow 0} [y'(0;\varepsilon)] = \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{2\varepsilon} \frac{\exp\left(\frac{1}{2\varepsilon}\right)}{\sinh\left(\frac{1}{2\varepsilon}\right)} \right] \rightarrow \frac{\infty}{\infty}$$

Clearly this limit is highly indeterminant as both \exp and \sinh “blow up” fairly quickly as their arguments become large. This indeterminacy cannot be resolved with the help of L’Hôpital’s rule because, no matter how many times you differentiate \exp or \sinh , they still “blow up.” The indeterminacy can be resolved instead by comparing the asymptotic behavior of \sinh with that of \exp :

Recall the definition of \sinh :

$$\sinh z = \frac{e^z - e^{-z}}{2} \approx \frac{1}{2} e^z \quad \text{as } z \rightarrow +\infty$$

so

$$\frac{\exp(z)}{\sinh(z)} \rightarrow 2 \quad \text{as } z \rightarrow +\infty$$

As $z \rightarrow +\infty$, the second term of this definition becomes negligible compared to the first. Thus the ratio of \exp to \sinh approaches 2 and our expression above becomes:

$$y'(0;\varepsilon \rightarrow 0) \approx 1/\varepsilon \rightarrow \infty$$

So the derivative of our function at this boundary is very strongly dependent on the value of the small parameter. In particular, the derivative is not bounded in the limit $\varepsilon \rightarrow 0$. This singularity is the essential nature of any boundary layer. In the above analysis, \exp/\sinh was bounded although both functions become unbounded as their arguments becomes large. We say that the singularity of these two functions in this limit is **of the same order** and we write this as:

$$\sinh(z) = O(e^z) \quad \text{as } z \rightarrow \infty$$

which is called the **Big "Oh"** notation (Bachman-Landau^{*}). More generally, when we say that

$$f(x) = O[g(x)] \text{ as } x \rightarrow a$$

we mean

$$f(x)/g(x) \text{ is bounded as } x \rightarrow a$$

Likewise, we could say that:

$$y'(0, \varepsilon) = O(\varepsilon^{-1}) \text{ as } \varepsilon \rightarrow 0$$

We could also show that

$$y''(0, \varepsilon) = O(\varepsilon^{-2}) \text{ as } \varepsilon \rightarrow 0$$

which means that the second derivative blows up even faster than the first. This notation gives us a convenient way of describing the strength of a pole.

at $x=0$:

$$\underbrace{\varepsilon y''}_{O(\varepsilon^{-1})} + \underbrace{y'}_{O(\varepsilon^{-1})} + \underbrace{y}_0 = 0 \quad (86)$$

Thus at the boundary, the first term of the O.D.E. is the same order of magnitude as the second term. So no matter how small ε becomes, the first term can never be neglected. This is the root of the problem. One way to obtain the correct order in ε of the solution using a Taylor series expansion would be to transform the independent variable:

Let

$$X \equiv x/\varepsilon \quad \text{and} \quad Y(X; \varepsilon) \equiv y(x; \varepsilon)$$

Then

$$y' = \frac{dy}{dx} = \frac{dY}{dX} \underbrace{\frac{dX}{dx}}_{\varepsilon^{-1}} = \varepsilon^{-1} \frac{dY}{dX}$$

and

$$y'' = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\varepsilon^{-1} \frac{dY}{dX} \right) = \frac{d}{dX} \left(\varepsilon^{-1} \frac{dY}{dX} \right) \underbrace{\frac{dX}{dx}}_{\varepsilon^{-1}} = \varepsilon^{-2} \frac{d^2 Y}{dX^2}$$

Note that if dY/dX and $d^2 Y/dX^2$ are $O(\varepsilon^0)$ as $\varepsilon \rightarrow 0$, we will obtain the correct order for y' and y'' . This is the basic idea behind “stretch transformation” which is part of the “inner expansion” which will be presented in the second half of the next section.

^{*} This is the German mathematician Edmund Landau, not the Russian physicist Lev Davidovich Landau.

MATCHED-ASYMPTOTIC EXPANSIONS*

Matched-asymptotic expansions is a very general technique for coping with singularities like boundary layers. MAE is one type of *singular perturbation expansion*. Greenberg describes it as “one of the most important advances in applied mathematics in this century.” Although the beginnings go back to the 19th century and such names as Lindstedt and Poincaré,^{*} it was not until the 1960's that singular perturbation techniques became part of the standard analytical tools of engineers, scientists and mathematicians. Since boundary layers frequently arise in transport phenomena, let's apply this technique to solve the simple problem posed by Prandtl.

EXAMPLE: Use Matched Asymptotic Expansions to find the asymptotic behavior of the solution $y(x;\varepsilon)$ to the following problem as $\varepsilon \rightarrow 0$.

$$\varepsilon y'' + y' + y = 0 \quad (87)$$

subject to:

$$y(0) = 0$$

$$y(1) = 1$$

Solution: Following Prandtl (1905), we divide the domain into two regions:

inner region: $0 \leq x \leq \delta, \quad y = y^i$ satisfies inner b.c.

outer region: $\delta \leq x \leq 1 \quad y = y^o$ satisfies outer b.c.

where δ is the thickness of the boundary layer located near $x=0$. Within each region, we seek a solution which is a Taylor series expansion of the function $y(x,\varepsilon)$ about $\varepsilon=0$:

$$y(x;\varepsilon) = \underbrace{y(x;0)}_{y_0(x)} + \underbrace{\frac{\partial y(x;\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon=0}}_{y_1(x)} \varepsilon + \underbrace{\frac{1}{2!} \frac{\partial^2 y(x;\varepsilon)}{\partial \varepsilon^2} \Big|_{\varepsilon=0}}_{y_2(x)} \varepsilon^2 + \dots$$

- **The outer problem.** The solution outside the boundary layer can often be found as a *regular perturbation expansion*:

$$y^o(x,\varepsilon) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + \dots \quad (88)$$

* The example problem introduced above was solved using MAE's by Greenberg, p508ff.

^{*}(Jules) Henri Poincaré (1854-1912), French mathematician, theoretical astronomer, and philosopher of science who influenced cosmogony, relativity, and topology and was a gifted interpreter of science to a wide public

which is just a Taylor series expansion about $\varepsilon=0$. (88) into (87):

$$\varepsilon(y_0'' + \varepsilon y_1'' + \dots) + (y_0' + \varepsilon y_1' + \dots) + (y_0 + \varepsilon y_1 + \dots) = 0$$

Collecting terms of like power in ε :

$$(y_0' + y_0)\varepsilon^0 + (y_0'' + y_1' + y_1)\varepsilon^1 + \dots = 0$$

In order for this sum to vanish for all values of ε , each coefficient must separately vanish:

$$\varepsilon^0: \quad y_0' + y_0 = 0 \quad (89)$$

$$\varepsilon^1: \quad y_1' + y_1 = -y_0'' \quad (90)$$

and similarly for higher order terms. Note that we have succeeded in obtaining a set of O.D.E.'s for the set of coefficient functions whose solution can be easily uncoupled. If we start with (89), $y_0(x)$ can be determined so that it is known when we solve (90). The outer solution must be subject to the outer boundary condition:

$$y(1) = 1$$

Expanding this in a Taylor series about $\varepsilon=0$:

$$y_0(1) + y_1(1)\varepsilon + y_2(1)\varepsilon^2 + \dots = (1)\varepsilon^0 + (0)\varepsilon^1 + (0)\varepsilon^2 + \dots$$

$$\text{Thus} \quad y_0(1) = 1 \quad (91)$$

$$y_1(1) = 0$$

and so on. The solution to (89) subject to (91) is:

$$y_0(x) = \exp(1-x) \quad (92)$$

We could now substitute (92) into (90) and obtain the solution for $y_1(x)$. Similarly, we could obtain $y_2(x)$, $y_3(x)$ and so on. If we stop after the leading term, the outer solution becomes:

$$y^o = y_0(x) + O(\varepsilon)$$

$$y^o(x, \varepsilon) = \exp(1-x) + O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0 \quad (93)$$

- **The inner problem.** To cope with the boundary-layer, we transform the independent variable using a *stretch transformation*, whose general form is

$$X = \frac{x - x_0}{\varepsilon^n} \quad (n > 0)$$

where x_0 is the location of the boundary at which the boundary layer arises. In this transformation the distance from the boundary ($x - x_0$) is magnified ("stretched") by an amount

depending on the small parameter. In our problem, the boundary layer arises at $x_0=0$, so the transformation becomes

$$X = \frac{x}{\varepsilon^n}$$

Rewriting (87) in terms of the new variables X and $Y(X) \equiv y(x)$:

$$y' = \frac{dy}{dx} = \frac{dY}{dX} \frac{dX}{dx} = \varepsilon^{-n} \dot{Y}$$

Similarly

$$y'' = \varepsilon^{-2n} \ddot{Y}$$

where

$$\dot{Y} \text{ denotes } dY/dX$$

$$\text{Substituting into (87):} \quad \varepsilon^{1-2n} \ddot{Y} + \varepsilon^{-n} \dot{Y} + Y = 0 \quad (94)$$

The purpose of the stretch transformation is to make Y, \dot{Y}, \ddot{Y} all $O(\varepsilon^0)$:

$$\text{as } \varepsilon \rightarrow 0 \text{ (} X=\text{const):} \quad Y, \dot{Y}, \ddot{Y} = O(\varepsilon^0)$$

whereas

$$y, y', y'' = O(\varepsilon^0), O(\varepsilon^{-n}), O(\varepsilon^{-2n})$$

Now we are in a position to choose a value for n . Recall from the exact solution [see (86)] that the second-derivative term is not negligible inside the boundary layer. This is generally the most important consideration in choosing n : we select the value of the parameter n such that we do not lose the term containing the highest order derivative (afterall, dropping the highest order derivative is what lead to the outer expansion, which we have shown fails in the inner region). To keep the highest order derivative,

- This term must be lowest order* in ε (otherwise it will be swamped by a lower order term)
- This term must **not** be the only term which has this order (if this is the only term of that order, O.D.E. requires it to vanish identically)

As a first attempt, we might try to make all terms of the same order:

$$1-2n = -n = 0$$

which is impossible for any single value of n . Next, we might try balancing first and third:

*The “order” of a term with respect to a small parameter ε (as opposed to the order of the derivative) refers to the exponent (power) to which ε is raised. For example, we say that a term which tends to vanish like ε^n as $\varepsilon \rightarrow 0$ is “of order n .”

$$1-2n = 0 \text{ or } n=1/2$$

Orders (i.e. the exponent of ε) of the three terms then are:

$$0, -1/2, 0$$

This is no good, because highest order derivative (i.e. the first term) is not lowest order in ε . Finally we try to balance first and second term:

$$1-2n = -n, \text{ or } n=1$$

Now the orders of each term is: $-1, -1, 0$

which is OK. Using $n=1$, (94) becomes (after multiplying by ε):

$$\ddot{Y} + \dot{Y} + \varepsilon Y = 0 \quad (95)$$

Instead of (88), we seek a solution inside the boundary layer which has the following form:

$$\text{for } x \leq \delta: \quad y^i(x, \varepsilon) \equiv Y(X, \varepsilon) = Y_0(X) + Y_1(X)\varepsilon + Y_2(X)\varepsilon^2 + \dots \quad (96)$$

(96) into (95) and collecting term of like order in ε :

$$\varepsilon^0: \quad \ddot{Y}_0 + \dot{Y}_0 = 0$$

whose general solution is: $Y_0(X) = A + B\exp(-X)$

Applying the inner boundary condition: $Y_0(0) = 0$

we can evaluate $B = -A$. This leaves us with

$$Y_0(X) = A[1 - \exp(-X)] \quad (97)$$

Similarly, we could determine $Y_1(X)$, $Y_2(X)$ and so on. If we stop after the leading term, we have for our inner solution:

$$y^i = A[1 - \exp(-X)] + O(\varepsilon)$$

This remaining integration constant A must be chosen so as to match the inner and outer solutions. One possible choice is to take the outer limit of the inner solution [denoted $(y^i)^o$] and equate it with the inner limit of the outer solution [denoted $(y^o)^i$]:

$$\underbrace{\lim_{X \rightarrow \infty} [y^i(X, \varepsilon)]}_{(y^i)^o} = \underbrace{\lim_{x \rightarrow 0} [y^o(x, \varepsilon)]}_{(y^o)^i}$$

which is called the **Primitive Matching Principle** and was originally used by Prandtl. Taking the limit of (93) as $x \rightarrow 0$ and the limit of (97) as $X \rightarrow \infty$:

$$A = e$$

The inner expansion becomes: $y^i = e[1 - \exp(-X)] + O(\varepsilon)$

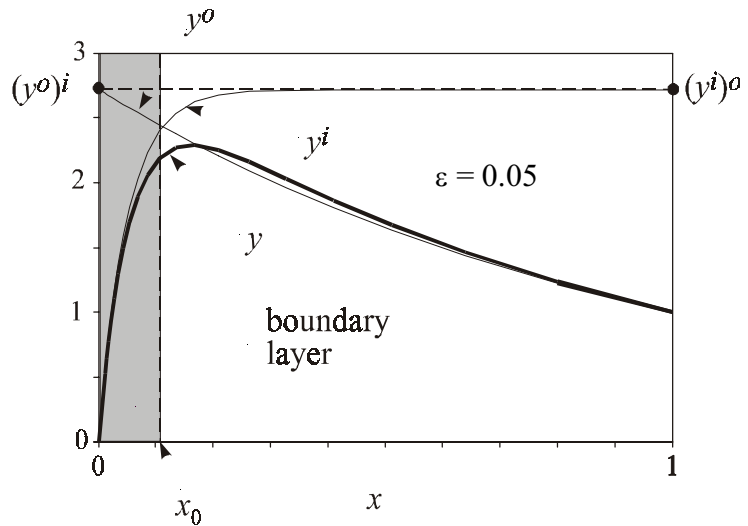
which means that for $x \leq \delta$: $y \approx e[1 - \exp(-x/\varepsilon)]$

whereas for $x \geq \delta$: $y \approx \exp(1-x)$

A convenient choice of δ is where y^i and y^o intersect. Of course, this intersection point depends on ε :

$$\delta = \delta(\varepsilon)$$

which is $O(\varepsilon)$ in this problem. From the figure at right, neither y^i nor y^o is a good approximation to y in the vicinity of δ . However, in most transport problems, the quantity of greatest interest is dy/dx at $x=0$ and dy^i/dx does seem to match dy/dx at $x=0$ quite well. If required, y^i and y^o can be blended together to obtain a single smooth function over the entire domain:

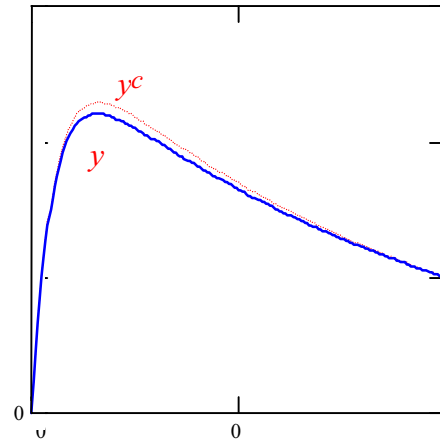


$$y^c = y^i + y^o - (y^i)^o$$

which is called the **composite solution**. For the present example, this is obtained by adding (92) and (97), expressing them in the same independent variable, and subtracting e :

$$y^c = e^{1-x} + e\left(1 - e^{-x/\varepsilon}\right) - e = e\left(e^{-x} - e^{-x/\varepsilon}\right)$$

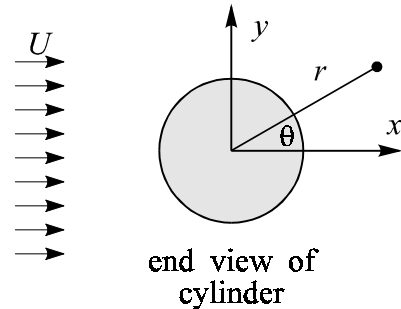
The plot at right compares y^c (the dotted curve), and the exact y (the solid curve). Note that y^c is a reasonably good approximation to y . The agreement gets much better as $\varepsilon \rightarrow 0$. Better agreement could be obtained for any ε by including the next order term in the expansions. Greenberg (p508f) obtains the second term in both the inner and outer expansions; the corresponding composite solution for $\varepsilon = 0.05$ is virtually indistinguishable from the exact solution.



MAE'S APPLIED TO 2-D FLOW AROUND CYLINDER

Let's now try to apply MAE's to solve the problem of flow around a cylinder at high Reynolds number. First, let's write the equation in dimensionless form: we will denote the dimensionless variables using an asterisk:

$$\begin{aligned} \mathbf{v}^* &\equiv \frac{\mathbf{v}}{U} & \mathbf{r}^* &\equiv \frac{\mathbf{r}}{R} & \nabla^* &\equiv R\nabla \\ p^* &\equiv \frac{p - p_\infty}{\rho U^2} & \varepsilon &\equiv \frac{\mu}{\rho UR} = \text{Re}^{-1} \end{aligned} \quad (98)$$



This nondimensionalizing differs from our previous attempt which was for the opposite limit of small Reynolds number (see p111 of Notes). First, we have used ρU^2 to nondimensionalize the pressure instead of $\mu U/R$. This is because the disturbance to pressure caused by flow is proportional to ρU^2 in the potential flow solution (see Hwk #4, Prob. 4). The second difference is that ε is defined as the reciprocal of the Reynolds numbers, rather than the Reynolds number itself. This choice makes ε a small parameter in the limit $\text{Re} \rightarrow \infty$.

The Navier-Stokes equations for steady flow become:

$$\mathbf{v}^* \cdot \nabla^* \mathbf{v}^* = \varepsilon \nabla^{*2} \mathbf{v}^* - \nabla^* p^* \quad (99)$$

and
$$\nabla^* \cdot \mathbf{v}^* = 0 \quad (100)$$

Boundary conditions are

as $r^* \rightarrow \infty$:
$$\mathbf{v}^* \rightarrow \mathbf{e}_x \quad \text{and} \quad p^* \rightarrow 0 \quad (101)$$

at $r^*=1$:
$$\mathbf{v}^* = \mathbf{0} \quad (102)$$

Expecting a boundary layer to arise at $r^*=1$ as $\text{Re} \rightarrow \infty$ (or as $\varepsilon \rightarrow 0$), we will now use the technique of MAE to solve it:

Outer Expansion

In the outer region, the regular perturbation expansion in powers of ε uses r^* as the position variable:

$$\mathbf{v}^*(r^*, \theta; \varepsilon) = \mathbf{v}_0(r^*, \theta) + \varepsilon \mathbf{v}_1(r^*, \theta) + \varepsilon^2 \mathbf{v}_2(r^*, \theta) + \dots$$

$$p^*(r^*, \theta; \varepsilon) = p_0(r^*, \theta) + \varepsilon p_1(r^*, \theta) + \varepsilon^2 p_2(r^*, \theta) + \dots$$

Substituting this perturbation expansion into (99) through (101), collecting terms of like order in ε , and setting their coefficients to zero, produces a series of well-posed mathematical problems for the coefficient functions. The first problem (the only one we will worry about in this analysis) is (dropping the $*$'s):

$$\varepsilon^0: \quad \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = -\nabla p_0 \quad (103)$$

$$\nabla \cdot \mathbf{v}_0 = 0 \quad (104)$$

The outer expansion is required to satisfy the outer boundary condition:

$$\text{as } r \rightarrow \infty: \quad \mathbf{v}_0 \rightarrow \mathbf{e}_x \quad \text{and} \quad p_0 \rightarrow 0 \quad (105)$$

Although we should not generally require the outer expansion to satisfy the inner b.c.'s, when we later match inner and outer expansions (see footnote on page 129), the outer expansion will have to satisfy:

$$v_{r0} = 0 \quad \text{at} \quad r = 1 \quad (106)$$

Note that the viscous term does not appear in this result because the lowest order viscous term is $O(\varepsilon^1)$, whereas other terms are $O(\varepsilon^0)$. It turns out that (103) through (106) is the same problem we previously solved by potential flow:

$$\mathbf{v}_0 = \nabla \phi$$

$$\text{where } \phi(r, \theta) \text{ is chosen to satisfy (104):} \quad \nabla^2 \phi = 0$$

For potential flow, $\nabla \times \mathbf{v}_0 = \mathbf{0}$ and (103) becomes:

$$\nabla \left(p_0 + \frac{1}{2} v_o^2 \right) = \mathbf{0}$$

which is just Bernoulli's equation for the pressure profile. After imposing the outer boundary conditions in (105) (and $v_r = 0$ at $r = 1$) we get the potential flow solution (see Hwk Set #4, Prob. 4). Eventually, we will need the inner limit of the outer solution for matching:

$$v_r^o(r = 1, \theta) = 0$$

$$v_{\theta}^0(r=1, \theta) = -2 \sin \theta \quad (107)$$

$$p^0(r=1, \theta) = \frac{1 - 4 \sin^2 \theta}{2} \quad (108)$$

Inner Expansion

For the inner expansion, we use a “stretched” radial coordinate:

$$Y = \frac{r^* - 1}{\varepsilon^n} \quad \text{or} \quad r^* = 1 + \varepsilon^n Y \quad (109)$$

and rename the tangential coordinate $X = \theta$

Unlike our simple example problem in the last section, the stretching parameter n in this problem is not an integer. More generally, the inner expansion should be a power series in ε^n rather than a power series in ε itself:

$$v_{\theta}^*(r^*, \theta; \varepsilon) = u_0(X, Y) + \varepsilon^n u_1(X, Y) + \varepsilon^{2n} u_2(X, Y) + \dots \quad (110)$$

$$v_r^*(r^*, \theta; \varepsilon) = v_0(X, Y) + \varepsilon^n v_1(X, Y) + \varepsilon^{2n} v_2(X, Y) + \dots \quad (111)$$

$$p^*(r^*, \theta; \varepsilon) = p_0(X, Y) + \varepsilon^n p_1(X, Y) + \varepsilon^{2n} p_2(X, Y) + \dots \quad (112)$$

In cylindrical coordinates for 2-D flow, the equation of continuity (100) becomes:

$$\nabla^* \cdot \mathbf{v}^* = \frac{1}{r^*} \frac{\partial(r^* v_r^*)}{\partial r^*} + \frac{1}{r^*} \frac{\partial v_{\theta}^*}{\partial \theta} = 0$$

We construct the first term using (109) through (111):

$$r^* v_r^* = (1 + \varepsilon^n Y)(v_0 + \varepsilon^n v_1 + \dots) = v_0 + (Y v_0 + v_1) \varepsilon^n + \dots$$

$$\frac{\partial(r^* v_r^*)}{\partial r^*} = \underbrace{\frac{\partial Y}{\partial r^*}}_{\varepsilon^{-n}} \frac{\partial}{\partial Y} [v_0 + (Y v_0 + v_1) \varepsilon^n + \dots] = \frac{\partial v_0}{\partial Y} \varepsilon^{-n} + \left(v_0 + Y \frac{\partial v_0}{\partial Y} + \frac{\partial v_1}{\partial Y} \right) \varepsilon^0 + \dots$$

$$\begin{aligned}\frac{1}{r^*} \frac{\partial(r^* v_r^*)}{\partial r^*} &= \underbrace{(1 + \varepsilon^n Y)^{-1}}_{1 - \varepsilon^n Y + \dots} \left\{ \frac{\partial v_0}{\partial Y} \varepsilon^{-n} + \left(v_0 + Y \frac{\partial v_0}{\partial Y} + \frac{\partial v_1}{\partial Y} \right) \varepsilon^0 + \dots \right\} \\ &= \frac{\partial v_0}{\partial Y} \varepsilon^{-n} + \left(v_0 + \frac{\partial v_1}{\partial Y} \right) \varepsilon^0 + \dots\end{aligned}$$

In the analysis above, we needed to expand $(1 + \varepsilon^n Y)^{-1}$ as a power series in ε^n . This was accomplished using the **Binomial Series**, which will be quite useful in later problems as well:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$$

which is just a Taylor series expansion about $x=0$. The Binomial Series is known to converge provided $|x| < 1$.

Similarly

$$\frac{\partial v_\theta^*}{\partial \theta} = \frac{\partial u_0}{\partial X} + \varepsilon^n \frac{\partial u_1}{\partial X} + \dots$$

The continuity equation becomes:

$$\frac{\partial v_0}{\partial Y} \varepsilon^{-n} + \left(v_0 + \frac{\partial v_1}{\partial Y} + \frac{\partial u_0}{\partial X} \right) \varepsilon^0 + \dots = 0$$

To satisfy “no slip” at the inner boundary, we must require:

at $Y=0$:

$$u_0 = u_1 = \dots = 0 \quad \text{and} \quad v_0 = v_1 = \dots = 0 \quad (113)$$

Setting the coefficients of each term separately to zero:

ε^{-n} :

$$\frac{\partial v_0}{\partial Y} = 0 \quad \text{for all } Y$$

$$v_0(Y) = \text{const} = 0$$

which means that v_0 must be constant inside the boundary layer. To satisfy the no-slip condition (113), the constant must be zero. Then v_0 must vanish everywhere.* The next term of the continuity equation is more useful:

$$\varepsilon^0: \quad \underbrace{v_0}_0 + \frac{\partial v_1}{\partial Y} + \frac{\partial u_0}{\partial X} = 0$$

$$\text{or} \quad \frac{\partial u_0}{\partial X} + \frac{\partial v_1}{\partial Y} = 0 \quad (114)$$

Next, we will look at the principle component of the NSE, which will be the component tangent to the surface. For steady 2-D flow, the θ -component of (99) is (BSL, p85):

$$\text{NSE}_\theta: \quad v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \varepsilon \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial^2 \theta} + \frac{2}{r} \frac{\partial v_r}{\partial \theta} \right]$$

Next, we transform each term using (109) through (112) with $v_0 = 0$:

$$\text{NSE}_\theta: \quad \underbrace{v_1 \frac{\partial u_0}{\partial Y}}_{O(\varepsilon^0)} + \underbrace{u_0 \frac{\partial u_0}{\partial X}}_{O(\varepsilon^0)} + \dots = -\underbrace{\frac{\partial p_0}{\partial X}}_{O(\varepsilon^0)} + \dots + \underbrace{\varepsilon^{1-2n} \frac{\partial^2 u_0}{\partial Y^2}}_{O(\varepsilon^{1-2n})} + \dots \quad (115)$$

As a general rule, we don't want to lose the highest order derivative inside the boundary layer, so we want this term to be lowest order in ε , but not the only term with this order. The largest inertial terms are $O(\varepsilon^0)$, so we choose n such that $1-2n = 0$:

$$1-2n = 0: \quad n = \frac{1}{2} \quad (116)$$

After collecting like-power terms, the r -component of (99) is:

$$\text{NSE}_r: \quad \underbrace{u_0^2}_{O(\varepsilon^0)} + \dots = -\underbrace{\varepsilon^{-1/2} \frac{\partial p_0}{\partial Y}}_{O(\varepsilon^{-1/2})} - \underbrace{\frac{\partial p_1}{\partial Y}}_{O(\varepsilon^0)} + \dots$$

*This result could be used to evaluate the undetermined integration constant in the outer expansion. Since v_0 represents the leading term in the inner expansion for v_r , this result means that the outer limit of the inner expansion for v_r is zero, and this must match with the inner limit of the outer expansion. Indeed, our expressions for the outer solution already make use of this result.

The lowest order term in this equation is $\partial p_0 / \partial Y$. Since there is no other term with this order, we must require that:

$$\varepsilon^{-1/2} \text{ of NSE}_r: \quad \frac{\partial p_0}{\partial Y} = 0$$

which integrates to yield: $p_0 = c(X)$

where the integration “constant” $c(X)$ can be evaluated by matching the outer limit ($Y \rightarrow \infty$) of this inner solution with the inner limit ($r \rightarrow 1$) of the outer solution (108):

$$p_0 = c(X) = \frac{1 - 4 \sin^2 X}{2} \quad (117)$$

We still have two unknowns: u_0 and v_1 . We can formulate two equations from the above:

$$\varepsilon^0 \text{ of (115):} \quad u_0 \frac{\partial u_0}{\partial X} + v_1 \frac{\partial u_0}{\partial Y} - \frac{\partial^2 u_0}{\partial Y^2} = 4 \sin X \cos X \quad (118)$$

$$\text{and (114):} \quad \frac{\partial u_0}{\partial X} + \frac{\partial v_1}{\partial Y} = 0 \quad (119)$$

No slip requires:

$$\text{at } Y=0: \quad u_0 = v_1 = 0 \quad (120)$$

Matching the outer limit of the inner solution with the inner limit of the outer solution requires:

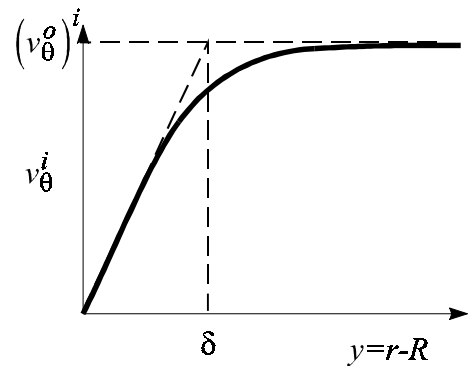
$$\text{as } Y \rightarrow \infty: \quad u_0 \rightarrow -2 \sin X \quad (121)$$

Boundary Layer Thickness

Recall the definition for “boundary layer” from page 116:

boundary layer: a very thin region near to a boundary in which the solution has a gradient which is orders of magnitude larger than its characteristic value outside the region.

So how thick is the boundary layer for 2-D flows like the one we just analyzed? A typical profile for the tangential component of velocity inside the boundary layer is sketched at right. On the scale of this sketch, the velocity appears to approach $(v_\theta^o)^i$ which is the inner limit of the outer solution. Actually, the nearly horizontal dotted line isn't quite horizontal: it has a small negative slope, but its slope is so small compared to the slope inside the boundary layer, that the outer solution appears to be flat on the scale of this drawing.



We might define the thickness of the boundary layer as the distance we have to go away from the surface to reach the apparent plateau. From the geometry of this sketch this distance, which we denote as δ , is approximated using

$$\frac{(v_\theta^o)^i}{\delta} \approx \left. \frac{\partial v_\theta^i}{\partial y} \right|_{y=0} \quad (122)$$

where all of the quantities in this equation have units. Let's try to "scale" (122) and solve for δ . Recall that the outer solution corresponds to potential flow. From HWK Set #4, Prob. 4:

$$v_\theta^o(r, \theta) = U \left[1 + \left(\frac{R}{r} \right)^2 \right] \sin \theta$$

so that

$$(v_\theta^o)^i = \lim_{r \rightarrow R} v_\theta^o(r, \theta) = 2U \sin \theta$$

or

$$(v_\theta^o)^i \approx U$$

In dimensionless quantities, the inner solution is given by (110). Substituting $n = 1/2$, we have

$$v_\theta^*(r^*, \theta) = u_0(X, Y) + \varepsilon^{1/2} u_1(X, Y) + \dots$$

$$\frac{\partial v_\theta^i}{\partial y} = \frac{U}{R} \frac{\partial v_\theta^*}{\partial r^*} = \frac{U}{R} \underbrace{\frac{\partial Y}{\partial r^*}}_{\varepsilon^{-1/2}} \frac{\partial}{\partial Y} \left[u_0(X, Y) + \varepsilon^{1/2} u_1(X, Y) + \dots \right] = \varepsilon^{-1/2} \frac{U}{R} \left(\frac{\partial u_0}{\partial Y} + \dots \right) \left(\frac{\partial Y}{\partial r^*} \right)$$

Dropping any multiplicative constants, we obtain

$$\frac{\partial v_{\theta}^i}{\partial y} \approx \varepsilon^{-1/2} \frac{U}{R}$$

Substituting this result into (122) and solving for the boundary-layer thickness δ :

$$\frac{U}{\delta} \approx \varepsilon^{-1/2} \frac{U}{R}$$

or

$$\delta \approx \sqrt{\varepsilon} R = \frac{R}{\sqrt{\text{Re}}}$$

Thus δ is proportional to R and inversely proportional to the square-root of Reynolds number. The boundary layer gets ever thinner as the Reynolds numbers increases. This is true for all boundary layers in 2-D flows.

PRANDTL'S B.L. EQUATIONS FOR 2-D FLOWS

Let's now summarize the mathematical problem which must be solved to obtain the velocity profile inside the boundary layer. The mathematical problem is represented by equations (118)-(121). For clarity, let's rewrite these equations using variables having dimensions [recall (98), (110)-(112) and (116)]. For example, u_0 is given by (110) and (98) as

$$u_0 = v_{\theta}^* = \frac{v_{\theta}}{U}$$

while Y is given by (109) and (116) as

$$Y = \frac{r^* - 1}{\varepsilon^{1/2}} = \frac{r - R}{R} \varepsilon^{-1/2} = \left(\frac{\rho U R}{\mu} \right)^{1/2} \frac{r - R}{R}$$

so

$$dY = \left(\frac{\rho U R}{\mu} \right)^{1/2} \frac{dr}{R}$$

Thus

$$\frac{\partial^2 u_0}{\partial Y^2} = \frac{\mu}{\rho U R} \frac{R^2}{U} \frac{\partial^2 v_{\theta}}{\partial r^2}$$

Making similar transformations to dimensional variables of each term, then multiplying both sides of the equation by $\rho U^2 / R$, (118) becomes

$$\rho \left(\frac{v_{\theta}}{R} \frac{\partial v_{\theta}}{\partial \theta} + v_r \frac{\partial v_{\theta}}{\partial r} \right) - \mu \frac{\partial^2 v_{\theta}}{\partial r^2} = \frac{\rho U^2}{R} 4 \sin \theta \cos \theta \quad (123)$$

(119) becomes
$$\frac{1}{R} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_r}{\partial r} = 0 \quad (124)$$

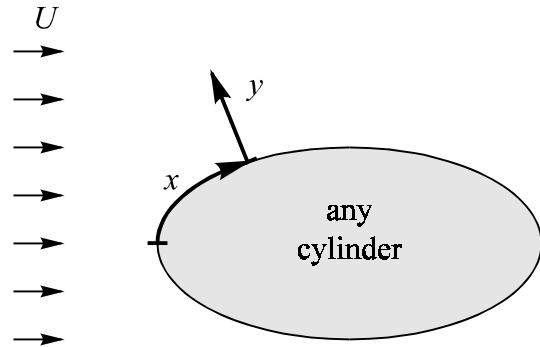
(120) becomes
$$v_r = v_\theta = 0 \quad \text{at} \quad r = R$$

(121) becomes
$$v_\theta = -2U \sin \theta \quad \text{at} \quad r \rightarrow \infty$$

Notice that (123) and (124) are approximations to NSE_θ and continuity in cylindrical coordinates.

Although the above equations for the velocity profile inside the boundary layer were derived for the specific geometry of a circular cylinder, it turns out that very similar equations are obtained for any 2-D flow, provided we express them in terms of a **local Cartesian coordinate system** (x, y) .

- x = arc length measured along the surface in the direction of flow
- y = distance from the surface measured along a normal to the surface



The more general equations for any 2-D flow are given by

$$\left. \begin{aligned} \rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) &= \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{dp_0}{dx} \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \end{aligned} \right\} \quad (125)$$

where p_0 is the inner limit of the pressure profile in potential flow, which is given by (117):

$$p_0(x) \equiv \lim_{r \rightarrow R} [p^{PF}(r, \theta)]$$

For boundary conditions, we impose “no slip” at the wall:

at $y=0$:
$$v_x(x, 0) = v_y(x, 0) = 0$$

Remaining integration constants are evaluated by matching the outer limit of the inner solution with the inner limit of the outer solution:

as $y \rightarrow \infty$:
$$v_x(x, y) \rightarrow U_0(x)$$

where $U_0(x)$ is the inner limit of the outer solution for the tangential component of velocity:

$$U_0(x) \equiv \lim_{r \rightarrow R} \left[v_0^{PF}(r, \theta) \right]$$

is the inner limit of the potential flow solution.

The the main difference from one geometry to another is a different potential flow solution applies to different geometries. From Bernoulli's equation, we have

$$\frac{p_0(x)}{\rho} + \frac{U_0^2(x)}{2} = \text{const}$$

Differentiating with respect to x and rearranging:

$$-\frac{1}{\rho} \frac{dp_0}{dx} = \frac{1}{2} \frac{dU_0^2}{dx} = U_0 \frac{dU_0}{dx}$$

Substituting into (125):

$$\boxed{\begin{aligned} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - \nu \frac{\partial^2 v_x}{\partial y^2} &= U_0 \frac{dU_0}{dx} = \text{known } f(x) \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \end{aligned}} \quad (126)$$

where $\nu \equiv \mu/\rho$. These PDE's are called ***Prandtl's boundary layer equations***. Appropriate b.c.'s include

$$\text{at } y=0: \quad v_x = v_y = 0$$

$$\text{as } y \rightarrow \infty: \quad v_x \rightarrow U_0(x)$$

ALTERNATE METHOD: PRANDTL'S SCALING THEORY*

Now we will repeat the analysis of boundary layers using Prandtl's analysis, which is more intuitive. Consider uniform flow normal to a long cylinder. Of course this is 2-D flow:

$$v_z = 0, \partial/\partial z = 0$$

At high Reynolds number, we expect the viscous terms to become negligible everywhere except inside the boundary layer. Dropping the viscous terms from the problem yields potential flow:

$$r > R + \delta: \quad \mathbf{v} = \nabla \phi$$

where δ is the thickness of the boundary layer.

Prandtl's analysis of this problem assumed that:

- outside the b.l., viscous \ll inertia, such that the velocity and pressure profiles are those obtained from potential flow. Recall the potential-flow solution from HWK 4, Prob. 3:

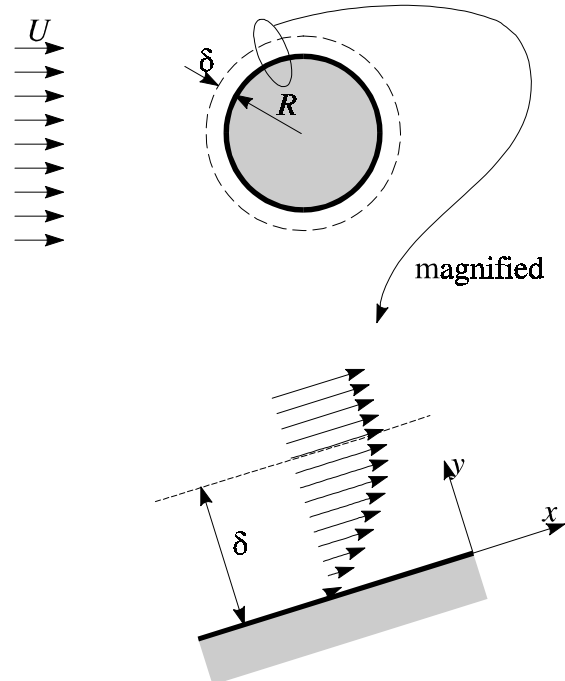
$$v_r^{PF} = U \left[1 - \left(\frac{R}{r} \right)^2 \right] \cos \theta \quad \text{for } r - R > \delta$$

$$v_\theta^{PF} = -U \left[1 + \left(\frac{R}{r} \right)^2 \right] \sin \theta$$

- $\delta \ll R$ so that fluid elements inside b.l. don't "see" the curvature of the cylinder (we call this "**Postulate #1**").
- inside the b.l., inertia \approx viscous (we call this "**Postulate #2**").

Defining a local Cartesian reference frame, the continuity and Navier-Stokes equations become:

continuity:
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$



*Schlichting, 6th ed., p117-21.

$$x: \quad v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

$$y: \quad v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

where $\nu \equiv \mu/\rho$. Next, Prandtl estimated the order of magnitude of each term, hoping that some can be dropped. This is called **scaling**. In this estimate, we are not concerned about factors of 2 or 3, but with orders of magnitude: we will try to guess the asymptotic behavior of each term in terms of the characteristic physical parameters. In particular, in scaling we try to express each term in the form of the product of the physical parameters raised to some power: In this problem the physical parameters are R , U , ν , and δ . Thus we will try to express each term as

$$R^a U^b \nu^c \delta^d$$

Let's start with the primary (x) component of velocity. Across the boundary layer, v_x must vary between zero at the surface (no-slip) and

$$\text{at } y=0: \quad v_x = 0$$

$$\text{at } y \approx \delta: \quad v_x = -v_\theta^{PF} = 2U \sin \theta$$

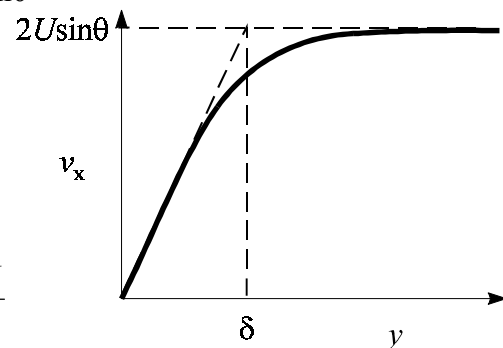
which is the potential flow solution. The assumption here is that the outer edge of the b.l. corresponds to the inner edge of the potential flow solution.

So

$$v_x \approx U$$

$$\frac{\partial v_x}{\partial y} \approx \frac{\Delta v_x}{\Delta y} \approx \frac{U}{\delta}$$

$$\frac{\partial^2 v_x}{\partial y^2} \approx \frac{\Delta \left(\frac{\partial v_x}{\partial y} \right)}{\Delta y} \approx \frac{\frac{U}{\delta} - 0}{0 - \delta} \approx \frac{U}{\delta^2}$$

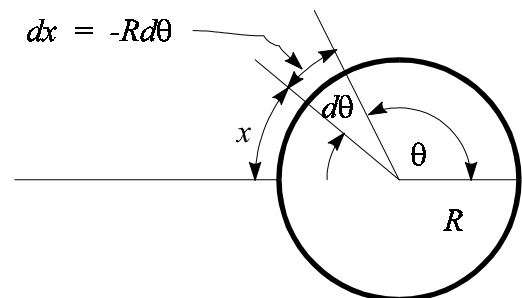


Now x is measured along the surface of the cylinder. Thinking of x as the arc length measured from the forward stagnation point, then:

$$dx = -R d\theta$$

Integrating with $x=0$ at $\theta=\pi$:

$$x = R(\pi - \theta)$$



Thus
$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_\theta^{PF}}{\partial \theta} \frac{d\theta}{dx} = (2U \cos \theta) \left(-\frac{1}{R} \right) \approx \frac{U}{R}$$

Similarly,
$$\frac{\partial^2 v_x}{\partial x^2} = -\frac{\partial^2 v_\theta^{PF}}{\partial \theta^2} \left(\frac{d\theta}{dx} \right)^2 = (2U \sin \theta) \left(-\frac{1}{R} \right)^2 \approx \frac{U}{R^2}$$

Next, we look at the equation of continuity:

$$\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} \approx \frac{U}{R}$$

Integrating across the b.l.:
$$v_y = \int \frac{\partial v_y}{\partial y} dy \approx \underbrace{-\frac{2U \cos \theta}{R}}_{\partial v_x / \partial x} y \approx \frac{U\delta}{R}$$

so
$$\frac{\partial v_y}{\partial x} \approx \frac{\partial}{\partial \theta} \left(-\frac{2Uy \cos \theta}{R} \right) \underbrace{\frac{d\theta}{dx}}_{-1/R} \approx \frac{U\delta}{R^2}$$

Estimating the inertial and viscous terms in the x -component of N-S:

inertia:
$$v_x \partial v_x / \partial x \approx (U)(U/R) = U^2/R$$

$$v_y \partial v_x / \partial y \approx (U\delta/R)(U/\delta) = U^2/R$$

viscous:
$$\nu \partial^2 v_x / \partial x^2 \approx \nu U/R^2$$

$$\nu \partial^2 v_x / \partial y^2 \approx \nu U/\delta^2$$

Now let's summarize our results as to the magnitude of each term in the x -component of the NSE:

x :
$$\underbrace{v_x \frac{\partial v_x}{\partial x}}_{\frac{U^2}{R}} + \underbrace{v_y \frac{\partial v_x}{\partial y}}_{\frac{U^2}{R}} = \underbrace{-\frac{1}{\rho} \frac{\partial p}{\partial x}}_{?} + \underbrace{\nu \frac{\partial^2 v_x}{\partial x^2}}_{\frac{\nu U}{R^2}} + \underbrace{\nu \frac{\partial^2 v_x}{\partial y^2}}_{\frac{\nu U}{\delta^2}}$$

Since $\delta \ll R$ (according to "Postulate #1"), the first viscous term must be negligible compared to the second:

$\delta \ll R$:
$$\nu U/R^2 \ll \nu U/\delta^2$$

$$\partial^2 v_x / \partial x^2 \ll \partial^2 v_x / \partial y^2$$

So we can neglect this term.

Ignoring the negligible term, we have established the following orders of magnitude for the terms in the x -component of the Navier-Stokes equation:

$$x: \quad \underbrace{v_x \frac{\partial v_x}{\partial x}}_{\frac{U^2}{R}} + \underbrace{v_y \frac{\partial v_x}{\partial y}}_{\frac{U^2}{R}} = - \underbrace{\frac{1}{\rho} \frac{\partial p}{\partial x}}_{?} + \underbrace{\nu \frac{\partial^2 v_x}{\partial y^2}}_{\frac{\nu U}{\delta^2}} \quad (127)$$

Now let's apply Prandtl's second postulate: Inside the boundary layer, viscous and inertial terms are of the same magnitude:

$$\text{inertia} \approx \text{viscous}$$

$$U^2/R \approx \nu U/\delta^2$$

This allows us to estimate the thickness of the boundary layer:

$$\delta^2 = \nu R/U = R^2/(RU/\nu)$$

$$\text{or} \quad \delta \approx \frac{R}{\sqrt{Re}} \quad (128)$$

where

$$Re \equiv RU/\nu$$

is the **Reynolds number**. Note that this correctly predicts that the boundary layer gets thinner as

$$\delta \rightarrow 0 \text{ as } Re \rightarrow \infty$$

The remaining term in (127) which has not yet been estimated is the pressure gradient. We can obtain some idea of its magnitude by looking at the potential flow solution, in which the pressure profile is given by Bernoulli's equation:

$$\text{for } y \geq \delta: \quad p/\rho + v^2/2 = \text{const}$$

Now the kinetic energy can be decomposed into contributions from the x - and y -components, whose orders of magnitude we have already estimated:

$$v^2 = v_x^2 + v_y^2 \approx v_x^2$$

Now $v_x^2 \approx U^2$ while $v_y^2 \approx (U\delta/R)^2$, so $v_y^2 \ll v_x^2$ which leaves:

$$p/\rho \approx \text{const} - v_x^2/2$$

$$\text{Differentiating:} \quad (1/\rho) \partial p / \partial x \approx v_x \partial v_x / \partial x \approx U^2/R \quad (129)$$

So this means we cannot neglect the pressure gradient in (127) since it is the same order of magnitude as the inertial terms. Scaling of the terms of the y -component of the Navier-Stokes equation yields:

$$y: \quad \underbrace{v_x \frac{\partial v_y}{\partial x}}_{\frac{U^2 \delta}{R^2}} + \underbrace{v_y \frac{\partial v_y}{\partial y}}_{\frac{U^2 \delta}{R^2}} = - \underbrace{\frac{1}{\rho} \frac{\partial p}{\partial y}}_{?} + \underbrace{v \frac{\partial^2 v_y}{\partial x^2}}_{\frac{\nu U \delta}{R^3}} + \underbrace{v \frac{\partial^2 v_y}{\partial y^2}}_{\frac{\nu U}{\delta R}} \quad (130)$$

Substituting Eq. (128) for δ shows that viscous and inertial terms again have the same magnitude:

$$\text{inertia} \approx \frac{U^2 \delta}{R^2} \approx \frac{U^2}{R^2} \sqrt{\frac{\nu R}{U}} \approx U^{3/2} R^{-3/2} \nu^{1/2}$$

$$\text{viscous} \approx \frac{\nu U}{\delta R} \approx \frac{\nu U}{R} \sqrt{\frac{U}{\nu R}} \approx U^{3/2} R^{-3/2} \nu^{1/2}$$

so $\text{inertia} \approx \text{viscous}$

What remains to be determined is the pressure gradient. At most, the pressure gradient in this equation has the same order of magnitude as the other terms:

$$(1/\rho) \partial p / \partial y \leq U^2 \delta / R^2 \quad (131)$$

Comparing this with the partial derivative in the other direction, by dividing (131) by (129):

$$(\partial p / \partial y) / (\partial p / \partial x) \leq \delta / R \rightarrow 0 \text{ as } Re \rightarrow \infty$$

which means that variations in pressure across the boundary layer are becoming insignificant at large Re . In other words, a good approximation would be to take:

$$p = p(x)$$

Since the pressure at the outer edge of the b.l. must correspond to that just outside, where potential flow occurs, we can calculate this pressure using the Bernoulli's equation and the potential flow solution:

$$\text{for } y \geq \delta: \quad p/\rho + v^2/2 = \text{const} \quad (132)$$

In potential flow at $r=R$:

$$v_r = 0$$

$$v^2(R, \theta) = v_\theta^2(R, \theta) \equiv U_o^2(x) \quad (133)$$

(133) into (132):

$$p/\rho + U_o^2/2 = \text{const}$$

$$(1/\rho)\partial p/\partial x + U_o dU_o/dx = 0 \quad (134)$$

$$(134) \text{ into } (127): \quad v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - v \frac{\partial^2 v_x}{\partial x^2} = U_0 \frac{dU_0}{dx} = \text{known} \quad (135)$$

The right-hand side of this equation is known because U_0 is the inner limit of the potential-flow solution:

$$U_0(x) \equiv \lim_{r \rightarrow R} [v_\theta^{PF}(r, \theta)]$$

Together with the continuity equation, we now have two equations in two unknowns:

2 unknowns: v_x and v_y

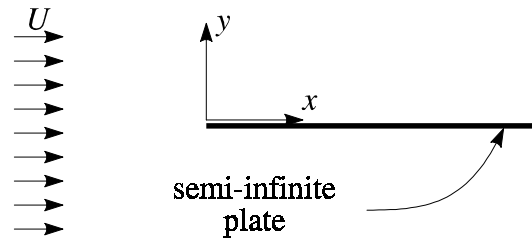
2 equations: (135), Continuity

which are known as **Prandtl's Boundary-Layer Equations** for 2-D flows.

SOLUTION FOR A FLAT PLATE

Reference: Schlichting, 6th ed., p125-33, Whitaker p430-440.

Before we continue with the analysis of flow around a circular cylinder, let's look at the simpler problem of flow tangent to a semi-infinite flat plate. The analysis begins by computing the potential-flow solution.



- Step 1: find potential flow solution

If the plate is infinitesimally thin, the uniform velocity profile is not disturbed:

P.F.: $\mathbf{v} = \mathbf{U}$ for all (x, y)

and $p = p_\infty$ for all (x, y)

Of course, this doesn't satisfy "no slip," on the plate, but then neither did potential flow around a cylinder.

Step 2: apply Prandtl's b.l. equations

From this potential-flow solution, we can calculate the U_0 appearing in Prandtl's equation:

$$U_0(x) = v_x^{PF}(x, 0) = U \text{ (a const.)}$$

$$U_o dU_o/dx = 0$$

Eq. (135) becomes:

$$v_x \partial v_x / \partial x + v_y \partial v_x / \partial y = v \partial^2 v_x / \partial y^2$$

$$\partial v_x / \partial x + \partial v_y / \partial y = 0$$

Appropriate boundary conditions are:

no slip:

$$v_x = v_y = 0 \text{ at } y=0, x>0$$

and outside the boundary layer, we obtain potential flow:

$$v_x \rightarrow U \text{ as } y \rightarrow \infty$$

Step 3: re-write the b.l. equations in terms of the streamfunction

We can contract the two equations into one by using the stream function:

$$\mathbf{v} = \nabla \times [\psi(x,y) \mathbf{k}]$$

which automatically satisfies continuity. Written in terms of the stream function, the problem becomes:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = v \psi_{yyy} \quad (136)$$

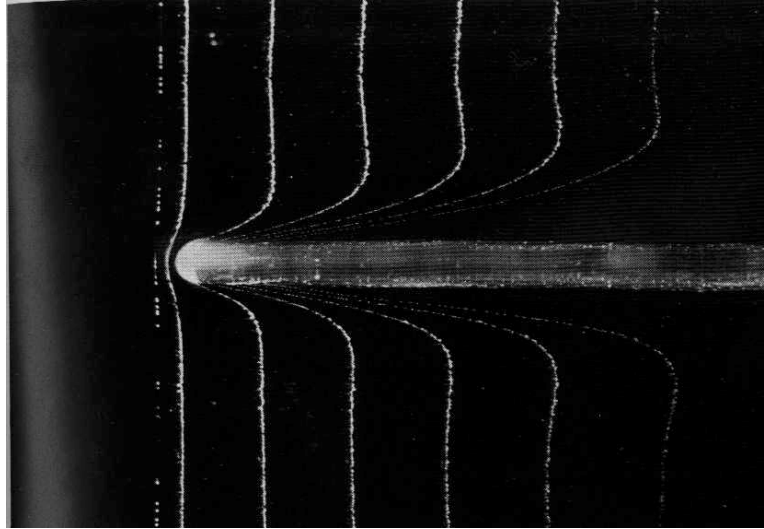
at $y=0, x>0$:

$$\psi_x = \psi_y = 0$$

as $y \rightarrow \infty$:

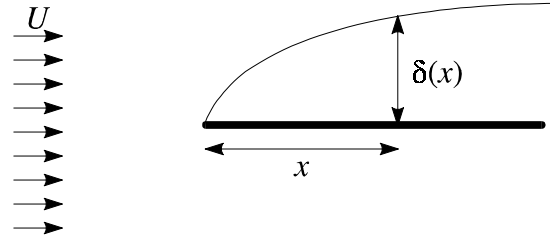
$$\psi \rightarrow Uy$$

where the subscripts denote partial differentiation.



The development of the laminar boundary layer along a flat plate is visualized by the hydrogen bubble method. A fine electrode wire is introduced upstream of the flat plate and a voltage pulse is applied repeatedly at regular intervals. The boundary layer thickness is seen to increase with the distance downstream for the leading edge. Photo taken from *Visualized Flow*, Pergamon, New York, p17 (1988).

Flow visualization studies show that the boundary-layer grows in thickness as you move downstream from the leading edge (see sketch at right). This can be rationalized by considering the inverse problem of a plate moving through a stagnant fluid.



Focus your attention on an initially stationary fluid element in the path of the moving plate.

The longer the fluid is disturbed by the moving plate, the more time there is for momentum to diffuse away from the plate.

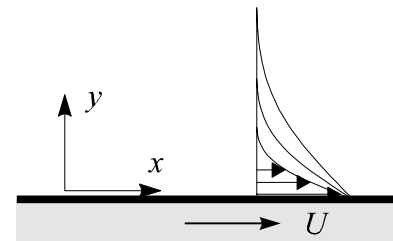
Time Out: Flow Next to Suddenly Accelerated Plate

First consider the much simpler problem of a infinite plate suddenly put in motion at $t=0$. Initially, the fluid and wall are at rest; but at time $t=0$ the wall is set in motion in the x -direction with a steady speed U . The initial and boundary conditions are:

$$\text{at } t=0 \text{ for all } y>0: \quad v_x = 0$$

$$\text{at } y=0 \text{ for all } t>0: \quad v_x = U$$

$$\text{as } y \rightarrow \infty: \quad v_x \rightarrow 0$$



The solution can be expected to have the form: $v_x = v_x(y, t)$, $v_y = v_z = 0$. The NSE becomes:

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

where $\nu \equiv \mu/\rho$. The solution to this problem is well known:*

$$v_x(y, t) = U \operatorname{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

where

$$\operatorname{erfc} \eta \equiv 1 - \underbrace{\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta'}_{\operatorname{erf} \eta} \rightarrow \begin{cases} 1 & \text{as } \eta \rightarrow 0 \\ 0 & \text{as } \eta \rightarrow \infty \end{cases}$$

is the **complementary error function**; the integral itself is the **error function**.

At any fixed t , the velocity decays monotonically from U at $y=0$ to zero as $y \rightarrow \infty$. As t gets larger, more and more fluid begins to move; we say “the motion penetrates deeper into the fluid.” Suppose we wanted to know how far from the wall we have to go before the velocity drops to 1% of the wall value.

$$v_x(y = \delta, t) = 0.01U : \quad \frac{v_x(\delta, t)}{U} = \operatorname{erfc}\left(\frac{\delta}{\sqrt{4\nu t}}\right) = 0.01$$

$$\operatorname{erf}\left(\frac{\delta}{\sqrt{4\nu t}}\right) = 1 - \operatorname{erfc}\left(\frac{\delta}{\sqrt{4\nu t}}\right) = 0.99$$

Looking up the appropriate value in a table of error functions, we obtain:

$$\frac{\delta}{\sqrt{4\nu t}} = 1.825$$

or

$$\delta = 3.65\sqrt{\nu t} \quad (137)$$

This gives us some idea how far the motion “penetrates” into the stagnant fluid. Notice that the penetration depth increases with the square-root of time.

* see BSL, p124f.

Time In: Boundary Layer on Flat Plate

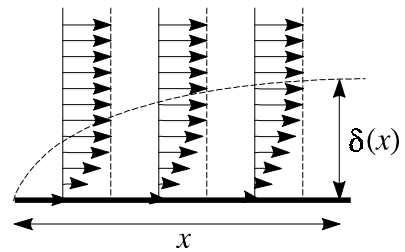
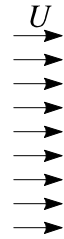
Now let's apply this result to our original problem: the boundary-layer next to a semi-infinite flat plate. Fluid farther downstream from the leading edge has been in contact with the moving plate longer; and estimate of the "contact time" is:

$$t = x/U \quad (138)$$

(138) into (137):
$$\delta \propto \sqrt{vt} \propto \sqrt{vx/U} \quad (139)$$

- Step 4: solve b.l. equation using "similarity transform"

Judging from the boundary conditions alone, we would expect v_x to vary from 0 at $y=0$ to U at $y=\delta$ with sort of a parabolic shape, with δ getting larger as we move downstream. Notice that the basic shape of this profile is not really changing



with x , only the range of y -values over which the solution departs from potential flow is increasing as we move downstream. This suggests a solution of the form:

$$v_x/U = f'(y/\delta) \quad (140)$$

Let's define a new independent variable which is scaled to the boundary-layer thickness:

Let
$$\frac{y}{\delta(x)} \approx \frac{y}{\sqrt{vx/U}} \equiv \eta(x, y)$$

To get this guess in terms of the stream function, recall:

$$v_x = \frac{\partial \psi}{\partial y}$$

thus
$$\psi = \int_{x,y=0}^{x,y} \underbrace{v_x(x,y)}_{Uf'} \underbrace{dy}_{\delta d\eta} = U\delta \int_0^\eta f'(\eta) d\eta = U\delta f(\eta)$$

substituting δ from (139):
$$\psi(x, y) = \sqrt{vUx} f(\eta) \quad (141)$$

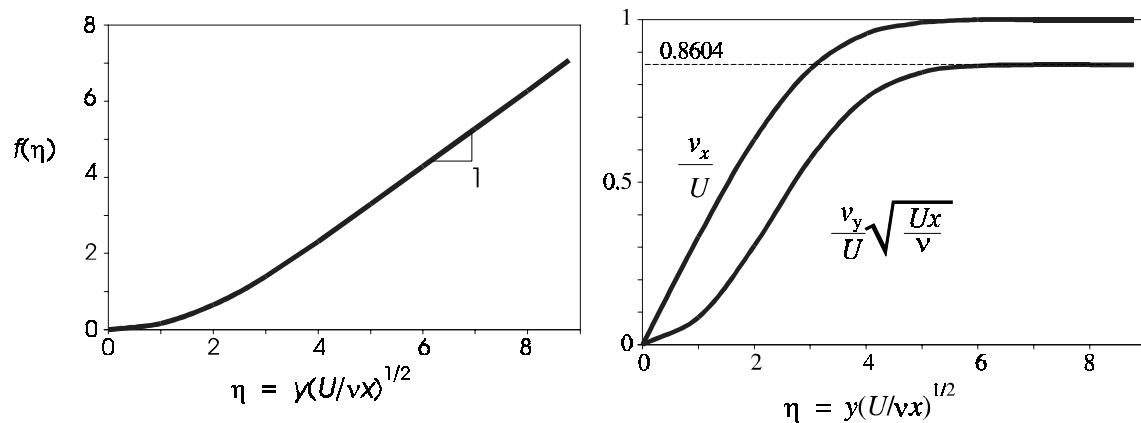
In terms of $f(\eta)$, the boundary-layer equations (136) become:

$$ff'' + 2f''' = 0$$

subject to:
$$f = f' = 0 \text{ at } \eta = 0$$

$$f' \rightarrow 1 \text{ as } \eta \rightarrow \infty$$

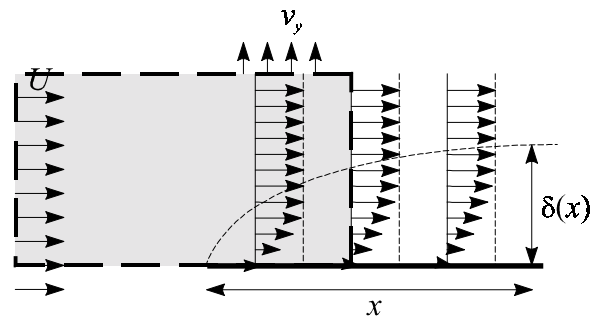
Now we have an ordinary two-point boundary-value problem, which can be easily solved. This solution was first obtained by Blasius (1908), who was a Ph.D. student of Prandtl.



Notice that the velocity component normal to the plate (i.e. v_y) does not vanish far from the plate:

$$\lim_{y \rightarrow \infty} \left\{ \frac{v_y(x, y)}{U} \right\} = 0.8604 \sqrt{\frac{\nu}{Ux}} \neq 0$$

Consider a fluid balance around the shaded rectangle in the figure at right. Owing to the need to meet the “no slip” condition on the surface of the plane, the flowrate out the right side of the system is less than the flow in the left side. The excess has to go out the top, causing $v_y > 0$ there.



Boundary-Layer Thickness

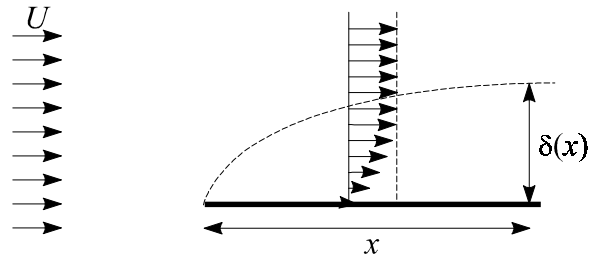
The definition of boundary layer thickness is somewhat arbitrary. Although we are tempted to say that δ is that value of y at which the boundary-layer solution for $v_x(y)$ equals U (potential flow), v_x approaches U only asymptotically as $y \rightarrow \infty$; therefore, this "definition" yields the unhelpful result that δ is ∞ . There are several ways to assign a more meaningful finite value to $\delta(x)$.

One way to define the boundary-layer thickness, δ , is that value of y at which the velocity is 99% of the asymptotic value.

- Def'n 1: $v_x(x, \delta_1) = 0.99U_0(x)$

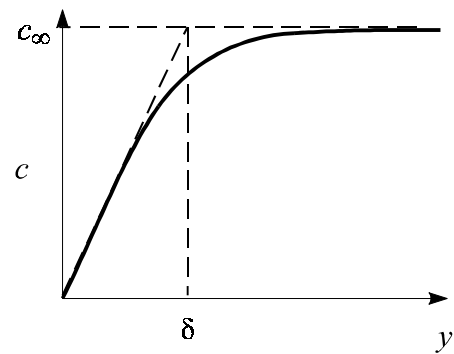
For flow over a flat plate, this convention yields:

$$\delta_1(x) = 5.0 \sqrt{\frac{\nu x}{U}}$$



Another way to define δ was suggested by Nernst who was concerned with boundary layers which arose in mass transfer problems. Nernst chose the diffusion boundary-layer thickness as the thickness of a hypothetical stagnant film which has the same diffusion resistance.

$$N_y \Big|_{y=0} = -D \frac{dc}{dy} \Big|_{y=0} = \frac{c_\infty}{\delta}$$



Graphically, this δ can be determined by drawing a tangent to the concentration profile at the surface ($y=0$). By analogy, concentration of mass is like concentration of momentum, which is just the fluid velocity.

If we define δ for momentum transfer in the same way, replacing concentration by v_x , then:

- Def'n 2: $\frac{\partial v_x}{\partial y} \Big|_{y=0} = \frac{\Delta v_x}{\Delta y} = \frac{U_0(x) - 0}{\delta_2 - 0}$

For flow tangent to a flat plate, this definition yields:

$$\delta_2(x) = 3.0 \sqrt{\frac{\nu x}{U}}$$

Recall that the normal component of the velocity profile was nonzero far from the plate:

$$\lim_{y \rightarrow \infty} \left\{ \frac{v_y(x, y)}{U} \right\} = 0.8604 \sqrt{\frac{\nu}{Ux}} \neq 0$$

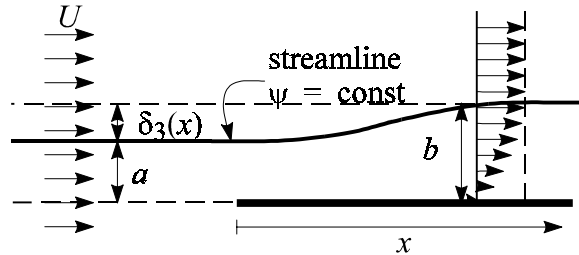
except far downstream from the leading edge:

$$\lim_{\substack{y \rightarrow \infty \\ x \rightarrow \infty}} \left\{ \frac{v_y(x, y)}{U} \right\} = \lim_{x \rightarrow \infty} \left\{ 0.8604 \sqrt{\frac{\nu}{Ux}} \right\} = 0$$

This means that the streamline moves away from the plate. A third definition for δ is the distance the streamline is displaced away from the plate.

Suppose that, far upstream, a streamline is given by

$$x \rightarrow -\infty: \quad y = a$$



Thus the streamline is initially a distance a from the x -axis. If downstream from the leading edge of the plate, the streamline is a distance b from the x -axis, then the displacement of the streamline is given by:

$$\delta_3 = b - a$$

To evaluate this displacement, recall (from the definition of streamline) that the flowrate in the x -direction between $y=0$ and the streamline is the same all along the streamline. Thus:

$$\psi = \frac{Q}{W} = aU = \int_0^{b(x)} v_x dy$$

where W is the width of the plate (assumed to be arbitrarily wide). Adding $\delta_3 U$ to both sides:

$$(a + \delta_3)U = \int_0^b v_x dy + \delta_3 U \quad (142)$$

The left-hand side of this expression can be rewritten as:

$$(a + \delta_3)U = bU = \int_0^b U dy \quad (143)$$

(142) becomes:

$$\begin{aligned} \int_0^b U dy &= \int_0^b v_x dy + \delta_3 U \\ \delta_3 U &= \int_0^b (U - v_x) dy \end{aligned}$$

Of course, the upper limit b will depend on the particular streamline we have chosen. In other words, different streamlines are displaced differently depending on how close to the plate they were initially. Thus the displacement distance δ_3 is not unique. However, distant streamlines are all displaced by the same distance (since the integral converges as $b \rightarrow \infty$). So let's define the boundary-layer thickness as the displacement of *external* streamlines:

- Def'n 3:
$$\delta_3 U = \int_0^{\infty} (U - v_x) dy$$

For flow tangent to a flat plate, this yields:

$$\delta_3(x) = 1.72 \sqrt{\frac{\nu x}{U}}$$

Notice that all three of these definitions yield a boundary-layer thickness which is proportional to $\sqrt{\nu x / U}$ although the proportionality constant varies considerably.

$$\delta(x) \propto \sqrt{\frac{\nu x}{U}} \quad (144)$$

We showed (see page 132) for any 2-D flow (which this is) that:

$$\delta \approx \frac{R}{\sqrt{\text{Re}}} = \sqrt{\frac{\nu R}{U}} \quad (145)$$

where R is the radius of the cylinder. For noncircular cylinders, R is some characteristic dimension of the cross section (e.g. the major or minor axis of an ellipse). A semi-infinite flat plate is somewhat unusual in that it has no characteristic dimension. However, if the plate were finite with length L along the direction of flow, it would seem natural to choose L as the characteristic length. If one can reasonably assume that what happens downstream with a longer plate does not significantly effect the boundary layer thickness for a plate of length L (i.e. "exit effects" don't propagate upstream). For an semi-infinite plate, the same result is obtained by choose x as the characteristic length; then

$$\delta \approx \sqrt{\frac{\nu x}{U}} \quad \text{and} \quad \text{Re} \equiv \frac{Ux}{\nu}$$

which is consistent with (144) and with Prandtl's more general result.

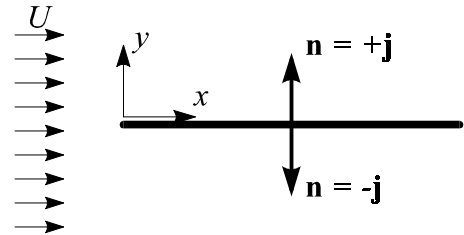
Drag on Plate

The net force exerted by the fluid on the plate is calculated from:

$$\mathbf{F} = \int_A \mathbf{n} \cdot \underline{\mathbf{T}} da = -\int_A p \mathbf{n} da + \int_A \mathbf{n} \cdot \underline{\boldsymbol{\tau}} da$$

Choosing $\mathbf{n} = +\mathbf{j}$ on the upper surface and $\mathbf{n} = -\mathbf{j}$ on the lower surface, the integral involving the pressure vanishes owing to cancellation of the contributions from the upper and lower surfaces (the pressure is the same, but \mathbf{n} has opposite direction on the two surfaces). This leaves

$$F_x = 2\mu \int_0^x \left(\frac{\partial v_x}{\partial y} \Big|_{y=0} \right) dx$$



The “2” comes from addition of the two contributions from the upper and lower surface. Evaluating the integral and expressing the result in dimensionless form:

$$C_D \equiv \frac{\frac{F_x}{2Wx}}{\frac{1}{2}\rho U^2} = \frac{1.328}{\sqrt{\text{Re}}}$$

where $\text{Re} \equiv Ux/\nu$. In defining drag coefficient this way, we have departed somewhat from the convention which uses the projected area along the direction of flow (i.e., the area of the shadow cast by the object if the light source were located very far upstream). In the case of a plate this projected area is zero, so we have used the area of the plate instead.

SOLUTION FOR A SYMMETRIC CYLINDER

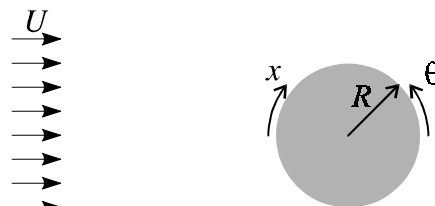
Let's now return to the problem of flow around a cylinder at large Reynolds number. We follow the same general steps as we did in solving flow tangent to a flat plate:

- Step 1: find potential flow solution and $U_0(x)$

We accomplished this in HWK #4, Prob. 4. The velocity profile in the potential flow solution is

$$v_r = U \left[1 - \left(\frac{R}{r} \right)^2 \right] \cos \theta$$

$$v_\theta = -U \left[1 + \left(\frac{R}{r} \right)^2 \right] \sin \theta$$



- Step 2: apply Prandtl's b.l. equations (see page 134)

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - v \frac{\partial^2 v_x}{\partial y^2} = U_0 \frac{dU_0}{dx}$$

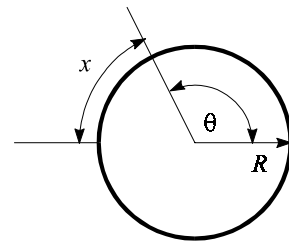
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

From this potential-flow solution, we can calculate the U_0 appearing in Prandtl's equation:

$$U_0(x) = -v_\theta(R, \theta) = 2U \underbrace{\sin \theta}_{\sin\left(\pi - \frac{x}{R}\right) = \sin \frac{x}{R}} = 2U \sin \frac{x}{R}$$

where x is the arc length, measured from the forward stagnation line, and θ is the polar coordinate, which is measured from the rear stagnation line. The two are related by

$$x = R(\pi - \theta) \quad \text{or} \quad \theta = \pi - \frac{x}{R}$$



Prandtl's boundary-layer equations (126) become

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - v \frac{\partial^2 v_x}{\partial y^2} = 4 \frac{U^2}{R} \sin \frac{x}{R} \cos \frac{x}{R}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Appropriate boundary conditions are:

no slip: $v_x = v_y = 0$ at $y=0$

and outside the boundary layer, we obtain potential flow:

$$v_x \rightarrow U_0(x) \text{ as } y \rightarrow \infty$$

The main difference between a circular cylinder and the flat plate the left-hand side of the first equation, which represents a pressure gradient along the direction of flow inside the boundary layer.

$$U_0 \frac{dU_0}{dx} = -\frac{1}{\rho} \frac{dp}{dx} \neq 0$$

- Step 3: re-write the b.l. equations in terms of the streamfunction

In terms of the streamfunction, Prandtl's boundary-layer equations are:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = \underbrace{4 \frac{U^2}{R} \sin \frac{x}{R} \cos \frac{x}{R}}_{U_0 \frac{dU_0}{dx}}$$

at $y=0$: $\psi_x = \psi_y = 0$

as $y \rightarrow \infty$: $\psi \rightarrow U_o(x)y$

Blausius obtained the solution to this problem, which in fact works for cylinders of much more general shape — not just circular cylinders.

What is required is that the potential flow solution must be an odd function of x :

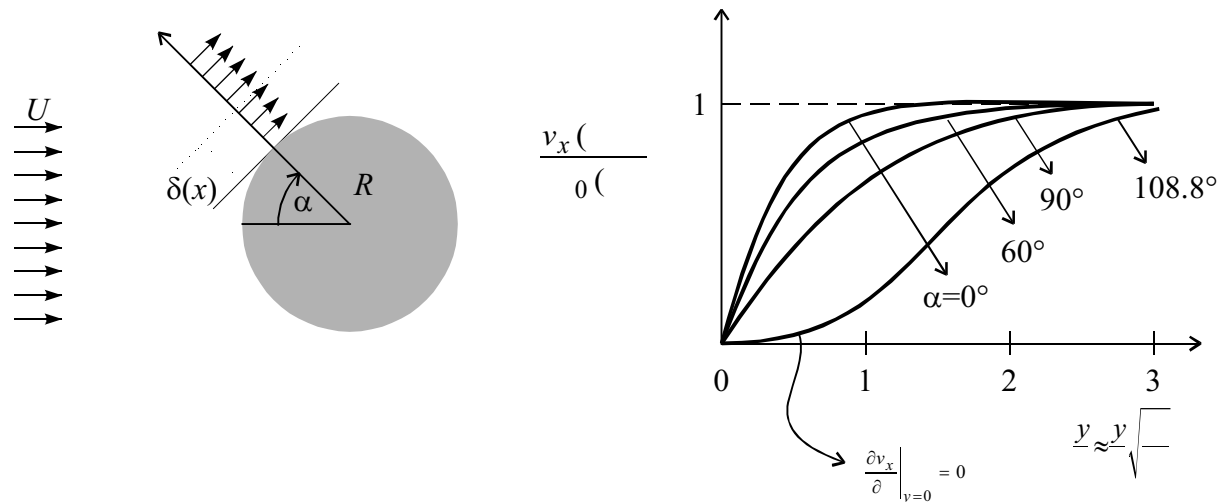
$$U_o(x) = u_1 x + u_3 x^3 + u_5 x^5 + \dots$$

Then Blasius obtained a solution with the form (see S:154-158):

$$\psi(x,y) = (\nu/u_1)^{1/2} [u_1 x f_1(\eta) + 4u_3 x^3 f_3(\eta) + 6u_5 x^5 f_5(\eta; u_1, u_3, u_5) + \dots]$$

where $\eta = y(u_1/\nu)^{1/2}$

The velocity profile obtained this way for a circular cylinder is sketched below:



This result is quite different from what was obtained with the flat plate. With the flat plate, the velocity profile v_x as a function of y had the same shape for different x 's. Indeed the shape at different x 's were similar and we were able to use a "similarity transform." Here, for flow perpendicular to a circular cylinder, the basic shape changes with x (or α or θ). In particular

notice that the initial slope $(\partial v_x / \partial y)|_{y=0}$ is positive for $\alpha=0$ and becomes zero for $\alpha=108.8^\circ$. At even larger α 's, the initial slope becomes negative.

This turning around of the velocity profile coincides with a profound event called **boundary-layer separation**. Separation does not occur on the flat plate. The main difference in Prandtl's boundary-layer equations which causes separation is the form of the pressure gradient.

Boundary-Layer Separation

To see what boundary-layer separation is and why it comes about, let's first recall the potential flow solution for the pressure on the surface of the cylinder. Using the velocity profile obtained in Hwk #4, Prob. 4, we can calculate the kinetic energy per unit volume at any point in the flow

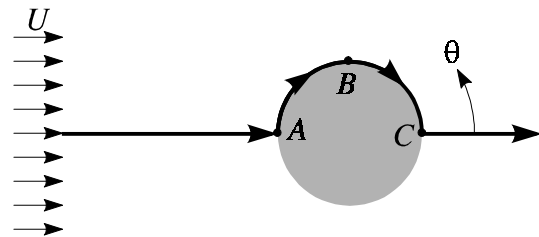
$$\frac{1}{2} v^2 = U^2 \left[\frac{1}{2} + (1 - 2 \cos^2 \theta) \left(\frac{R}{r} \right)^2 + \frac{1}{2} \left(\frac{R}{r} \right)^4 \right]$$

Then using Bernoulli's equation:

$$\frac{p}{\rho} + \frac{v^2}{2} = \frac{p_\infty}{\rho} + \frac{U^2}{2}$$

we can compute the pressure profile around the cylinder.

Consider a fluid element approaching the cylinder along the stagnation line shown in the sketch at right. As the fluid element moves toward the stagnation point A ($\theta = \pi$ and r changes from ∞ to R), the pressure rises to a maximum. Moving away from Point A along the surface of the cylinder ($r = R$ and θ decreases from π to $\pi/2$), the fluid element now accelerates until reaching its maximum speed and minimum pressure at Point B and so on.



The results for the pressure changes or kinetic energy changes during this journey are summarized on the graph at right. Note that Bernoulli's equation balances an increase in kinetic energy with a decrease in pressure and vice versa. So that the sum of the two curves equals a constant. Inside the boundary-layer, the same pressure gradient applies but we also have viscous dissipation in addition to kinetic energy. The upshot is:

just outside b.l.: potential flow (no irrev. loss of energy). The kinetic energy of the fluid at *B* is just enough to overcome the pressure hill at *C*. Fluid elements arrive at *C* with $v=0$ (no kinetic energy to spare).

inside b.l.: dp/dx same, but viscous dissipation consumes some of the kinetic energy, leaving insufficient energy to climb the pressure hill.

Consequently fluid elements in the boundary layer stop their forward advance at some point before reaching *C*, which we will label *S*. Fluid elements between *S* and *C* are driven toward *S* by falling down the pressure hill.

To conserve mass, fluid must be pushed away from the cylinder at *S*. This is known as **separation** of the boundary layer.

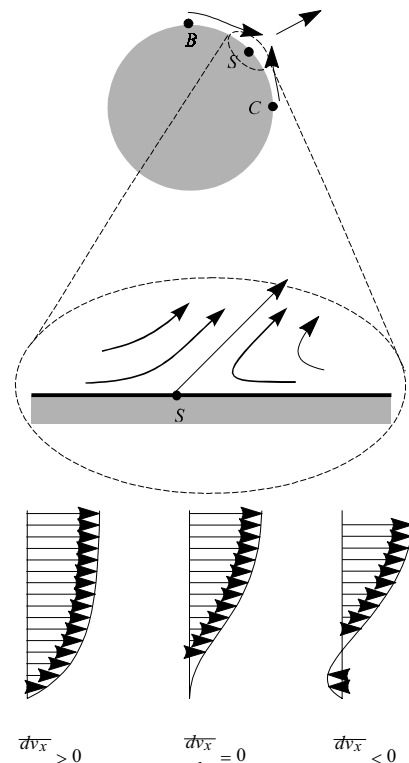
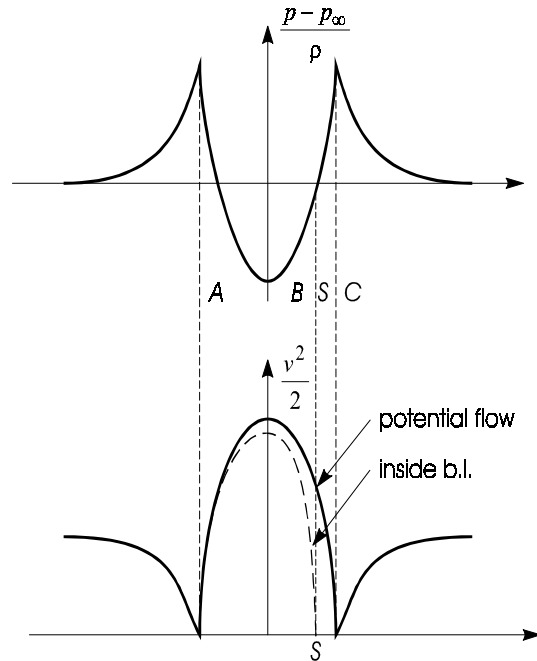
The x -component of flow is toward the separation point on either side of *S*. Thus at $y=0$ $dv_x/dy > 0$ for $x < x_S$ and $dv_x/dy < 0$ for $x > x_S$. At the separation point the derivative changes sign:

$$\left. \frac{\partial v_x}{\partial y} \right|_{x=x_S, y=0} = 0$$

which serves as a convenient way to locate the separation point in any mathematical solution to the flow problem.

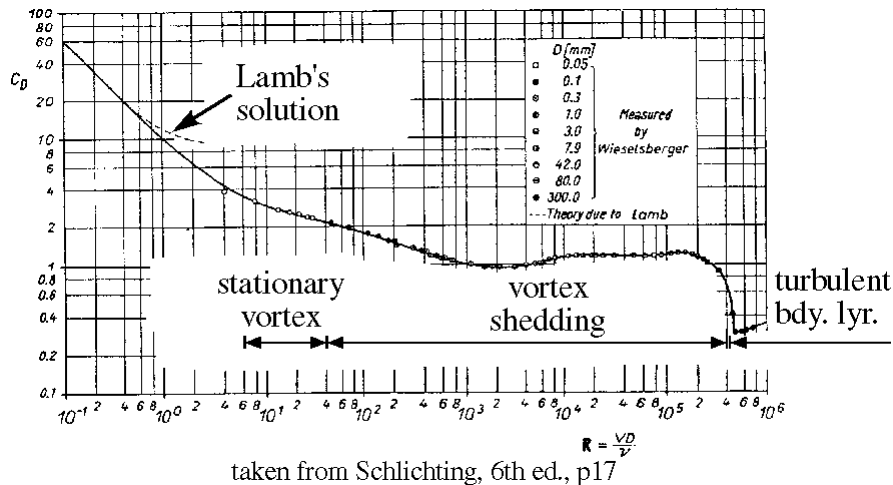
Necessary conditions for separation include:

1. decelerating flow or $Dp/Dt > 0$
2. irreversible losses of energy



Drag Coefficient and Behavior in the Wake of the Cylinder

Blausius solution does not apply downstream from the separation point (i.e. $x > x_s$). Indeed, the boundary layer is not thin enough to be described by Prandtl's equations. Since the behavior on the downstream side significantly influences the drag and since this behavior is not predicted by Blasius solution, Blasius solution does not correctly predict the drag force. Measurements of the drag coefficient for flow normal to a circular cylinder are summarized below.



Several different regions are apparent in this log-log graph. Some of these correspond to major changes in the shape of the velocity profile.

$Re \ll 1$. Upstream and downstream halves of streamline are mirror images. This is what you would expect if you looked for a streamfunction solution with the form (see Hwk #7, Prob. 2):

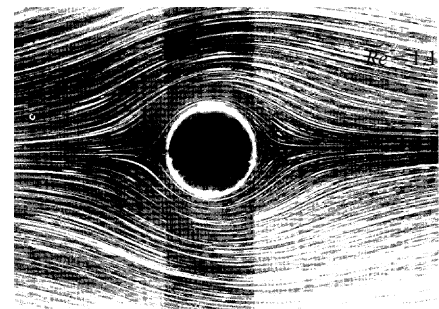
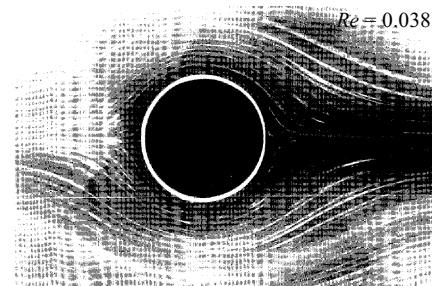
$$\psi(r, \theta) = f(r) \sin \theta$$

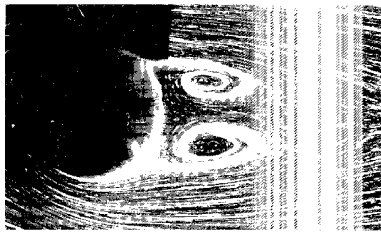
Although inertial terms are never negligible. The measured drag coefficients agree well with the prediction of Lamb.

As we increase the Reynolds number, inertia becomes more important. Generally large inertia has a tendency to make streamlines straight.

$1 < Re < 6$. Streamlines are no longer symmetric about $\theta = \pi/2$. Streamlines are somewhat farther from the cylinder on the downstream side.

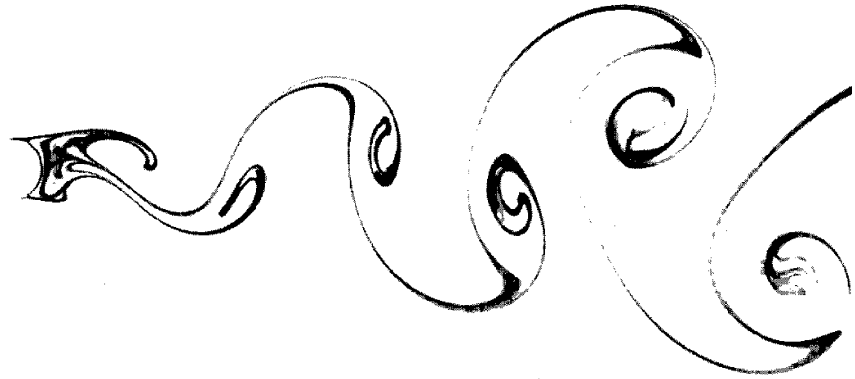
Drag coefficients are significantly smaller than predicted by Lamb.



 $Re = 19$

$6 < Re < 40$. Two stationary vortices are formed in the wake of the cylinder -- a consequence of boundary-layer separation.

$Re > 40$. Periodic vortex shedding. Large vortices, the size of the cylinder, are created on a periodic basis. These vortices detach and move downstream. Vortex shedding alternates between the top and bottom of the cylinder. Vortices shed from the top have a vorticity ($\nabla \times \mathbf{v}$) with a sign opposite from vortices shed from the bottom.

 $Re=140$

These vortices persist many cylinder diameters downstream from the cylinder. Most of the irreversible losses of energy occur in forming these vortices, whose ultimate fate is to dissipate their kinetic energy as heat.

$Re \approx 4 \times 10^5$. Onset of turbulence in the boundary layer. Point of separation moves further back toward rear stagnation point. Drag is significantly reduced -- almost a discontinuous drop.

The Lubrication Approximation

Consider a plate sliding on a lubricating film past a second stationary surface. If the distance separating the two plates is small compared to the dimensions of the plate, we can assume fully developed flow applies through out most of the oil film. Then:

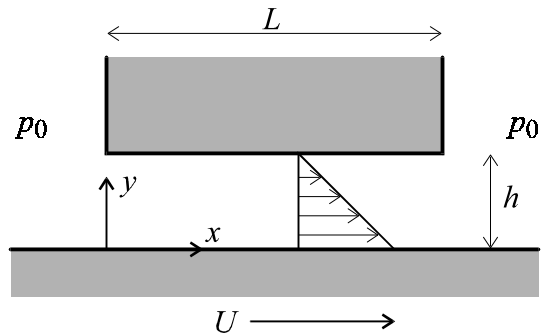
$$\text{for } h \ll L: \quad v_x(y) = U \frac{h-y}{h}$$

$$p = p_0 \text{ for all } x, y$$

$$Q = \int_0^L v_x(y) dy = \frac{1}{2} U h$$

$$\tau_{xy} = \tau_{yx} = -\frac{\mu U}{h}$$

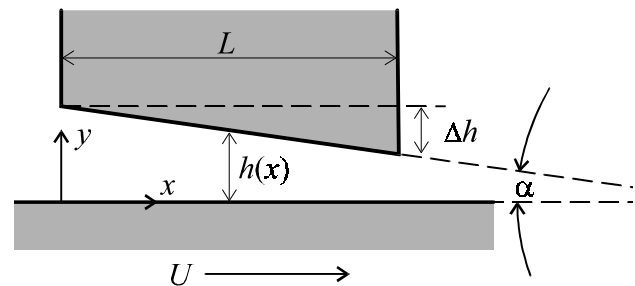
$$F_x = \frac{\mu U L W}{h}$$



Since the pressure is same inside the film as outside the slide, the sliding motion of two parallel surfaces produces no lateral component of force.

$$F_y = 0$$

Now suppose the slider is inclined ever so slightly relative to the stationary plate. We might guess, that if α is small enough, the velocity profile will not be affected. But, owing to the inclination, h is no longer independent of x , so our guess leads to:



$$\frac{Q}{W} = \frac{1}{2} U h(x) = f(x) \quad (146)$$

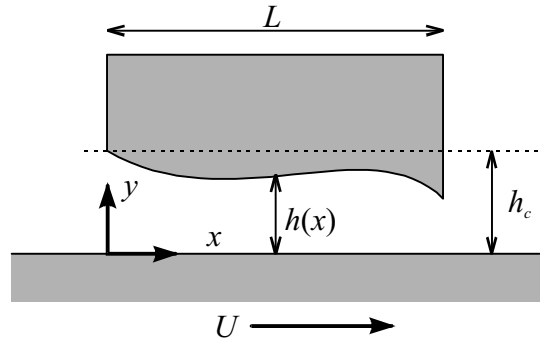
which violates continuity. It turns out that continuity is preserved by a nonzero pressure gradient, dp/dx , which causes pressure-driven flow. Thus even the primary component of the velocity profile is affected by this slight inclination. More significantly, it turns out that this inclination will produce a different pressure in the film from the fluid outside the slider block which, tends to push the two surfaces together or apart.

$$F_y \neq 0$$

Let's try to estimate this force. It turns out to be no more difficult to obtain the result for an arbitrary gap profile $h(x)$ (see figure at right), since the essential difficulty arises from the fact that h is not constant with respect to x .

Suppose the thickness of the gap is everywhere very small compared to the dimensions of the slider block.

$$h(x) \ll L \quad \text{for all } x$$



Essentially, this is a geometry with two very different length scales characterizing variations in the different directions x and y : we expect slow variations with x and rapid variations with y . We will exploit this difference using a regular perturbation in the ratio of the two length scales.

We will start by nondimensionalizing the equations of motion:

Let

$$X \equiv \frac{x}{L} \quad Y \equiv \frac{y}{h_c} \quad u \equiv \frac{v_x}{U} \quad v \equiv \frac{v_y}{v_c}$$

L is an obvious choice for the characteristic value of x (since $0 < x < L$) and U is a obvious choice for the characteristic value of v_x (since $v_x = U$ at $y=0$). Since $0 < y < h(x)$ some characteristic value h_c of the film thickness seems like a logical choice to scale y .* The choices of characteristic values for v_y and p are not obvious; so we will postpone a choice for now and just denote these values as v_c and p_c .

We seek a solution in the form of a regular perturbation:

where

$$\alpha \equiv \frac{h_c}{L}$$

$$v_x(x, y, \alpha) = Uu(X, Y, \alpha) = U[u_0(X, Y) + \alpha u_1(X, Y) + \dots] \quad (147)$$

$$v_y(x, y, \alpha) = v_c v(X, Y, \alpha) = v_c [v_0(X, Y) + \alpha v_1(X, Y) + \dots] \quad (148)$$

$$p(x, y, \alpha) - p_\infty = p_c P(X, Y, \alpha) = p_c [P_0(X, Y) + \alpha P_1(X, Y) + \dots] \quad (149)$$

As usual an important aspect of this form is that all derivatives of the dimensionless velocity components u and v with respect to the dimensionless coordinates X and Y are $O(\alpha^0)$:

* For definiteness, we might select the largest value of $h(x)$ to be h_c .

$$\partial u/\partial X, \partial u/\partial Y, \partial v/\partial X \text{ and } \partial v/\partial Y \text{ are } O(\alpha^0) \quad (150)$$

The choice of v_c becomes apparent when we nondimensionalize the continuity equation:

$$\text{continuity:} \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (151)$$

Substituting (147) and (148) into (151):

$$\frac{U}{L} \frac{\partial u}{\partial X} = -\frac{v_c}{h_c} \frac{\partial v}{\partial Y} \quad \text{or} \quad \frac{\partial u}{\partial X} = -\frac{1}{h_c/L} \frac{v_c}{U} \frac{\partial v}{\partial Y} \quad \text{or} \quad \frac{\partial u}{\partial X} = -\frac{v_c}{\alpha U} \frac{\partial v}{\partial Y}$$

(151) requires the two terms in the continuity equation to be equal but opposite; and to be exactly the same order of α . Since $\partial u/\partial X$ and $\partial v/\partial Y$ are $O(\alpha^0)$ according to (150), we are forced to choose v_c such that $v_c/\alpha U$ is $O(\alpha^0)$. So let's choose

$$v_c = \alpha U \quad (152)$$

Substituting (147)-(149) and (152) into (151), the leading term is

$$\varepsilon^0: \quad \frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial Y} = 0 \quad (153)$$

Next we examine the principle component of the Navier-Stokes equation:

$$\text{NSE}_x: \quad v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \quad (154)$$

Substituting (147)-(149) and (152) into (154):

$$\underbrace{\frac{U^2}{L} u \frac{\partial u}{\partial X}}_{O(\alpha^0)} + \underbrace{\frac{\alpha U^2}{\alpha L} v \frac{\partial u}{\partial Y}}_{O(\alpha^0)} = \underbrace{-\frac{p_c}{\rho L} \frac{\partial P}{\partial X}}_{?} + \underbrace{\nu \frac{U}{L^2} \frac{\partial^2 u}{\partial X^2}}_{O(\alpha^0)} + \underbrace{\nu \frac{U}{(\alpha L)^2} \frac{\partial^2 u}{\partial Y^2}}_{O(\alpha^{-2})}$$

The last term in the equation is lowest-order in the small parameter α : it's $O(\alpha^{-2})$. All the other terms in the equation (except possibly for the pressure gradient) are $O(\alpha^0)$. Unless we have some other term in the equation of the same order, we will be forced to take $\partial^2 u / \partial Y^2 = 0$, which yields linear shear flow to all orders in α . We already know that this solution violates macroscopic continuity. To avoid this situation, we choose p_c so that the pressure gradient term is also of $O(\alpha^{-2})$:

$$p_c = \rho L \times \frac{\nu}{\mu/\rho} \frac{U}{(\alpha L)^2} = \frac{\mu U}{\alpha^2 L} \quad (155)$$

With this choice, the leading term in NSE_x is

$$\alpha^{-2}: \quad \frac{\partial^2 u_0}{\partial Y^2} = \frac{\partial P_0}{\partial X} \quad (156)$$

So far we have two equations [(153) and (156)] in three unknowns (u_0 , v_0 and P_0). We need another equation. So we turn to the secondary component of NSE:

$$NSE_y: \quad \nu_x \frac{\partial v_y}{\partial x} + \nu_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

Introducing our dimensionless variables, we have:

$$\underbrace{\frac{\alpha U^2}{L} u \frac{\partial v}{\partial X}}_{O(\alpha)} + \underbrace{\frac{(\alpha U)^2}{\alpha L} v \frac{\partial v}{\partial Y}}_{O(\alpha)} = -\underbrace{\frac{\mu U}{(\alpha^2 L)(\alpha L)} \frac{\partial P}{\partial Y}}_{O(\alpha^{-3})} + \underbrace{\nu \frac{\alpha U}{L^2} \frac{\partial^2 v}{\partial X^2}}_{O(\alpha)} + \underbrace{\nu \frac{\alpha U}{(\alpha L)^2} \frac{\partial^2 v}{\partial Y^2}}_{O(\alpha^{-1})}$$

After substituting the perturbation expansions (147)-(149), (152) and (155), then collecting terms, the leading term is:

$$\alpha^{-3}: \quad \frac{\partial P_0}{\partial Y} = 0 \quad \text{or} \quad P_0 = \text{const w.r.t. } Y \quad (157)$$

This conclusion suggests that the pressure gradient in (156) can be treated as a constant with respect to Y :

$$\frac{\partial^2 u_0}{\partial Y^2} = \frac{dP_0}{dX} = \text{const w.r.t. } y \quad (158)$$

We might be tempted to set this pressure gradient to zero since:

$$p(0) = p(L) = p_o$$

Of course the pressure gradient is zero when the two plates are parallel. But we already suspect that the pressure gradient is needed to avoid linear shear flow, which violates continuity. So we avoid this temptation and leave dP_0/dX nonzero.

From this point on, we will resort to the dimensional variables, using what we have learned from the leading terms of the perturbation expansion. In essence, we are truncating (147)-(149) after the first nonzero term.

Integrating (158) twice: $v_x(x, y) = \frac{1}{2\mu} \frac{dp}{dx} y^2 + c_1(x)y + c_2(x)$

Boundary conditions are given by the “no slip” requirement:

$$v_x = 0 \text{ at } y=h(x)$$

$$v_x = U \text{ at } y=0$$

Evaluating the two integration constants leads to:

$$v_x(x, y) = \underbrace{U \left(1 - \frac{y}{h}\right)}_{\text{linear shear flow}} + \underbrace{\frac{1}{2\mu} \frac{dp}{dx} (y^2 - yh)}_{\text{pressure-driven flow}}$$

The volumetric flowrate per unit width of plate is calculated as:

$$\frac{Q}{W} = \int_0^h v_x(x, y) dy = \frac{Uh}{2} - \frac{h^3}{12\mu} \frac{dp}{dx} \quad (159)$$

Q can be made to be a constant at each x if dp/dx is allowed to take on non-zero values. The necessary values can be calculated from (159):

$$Q/W = \text{const.}: \quad \frac{dp}{dx} = \frac{6\mu U}{h^2} - \frac{12\mu}{h^3} \frac{Q}{W} \quad (160)$$

Integrating with respect to x from $x=0$ where $p=p_0$ to any other x .

$$p(x) - p_0 = 6\mu U \int_0^x \frac{dx}{h^2} - 12\mu \frac{Q}{W} \int_0^x \frac{dx}{h^3} \quad (161)$$

The pressure is the same at the downstream end of the gap. Then:

$$p(L) - p_0 = 0 = 6\mu U \int_0^L \frac{dx}{h^2} - 12\mu \frac{Q}{W} \int_0^L \frac{dx}{h^3}$$

Knowing the overall pressure drop is zero allows us to compute the flowrate:

$$\frac{Q}{W} = \frac{UH}{2} \quad (162)$$

where

$$H = \frac{\int_0^L \frac{dx}{h^2}}{\int_0^L \frac{dx}{h^3}}$$

is an average gap width. If $h(x)$ is a linear function whose value changes from

$$h(0) = h_1$$

to

$$h(L) = h_2 < h_1$$

then

$$H = 2 \frac{h_1 h_2}{h_1 + h_2} \quad \text{or} \quad \frac{1}{H} = \frac{1}{2} \left(\frac{1}{h_1} + \frac{1}{h_2} \right)$$

which is called the “harmonic mean” of h_1 and h_2 . (162) into (160) gives the pressure gradient:

$$\frac{dp}{dx} = \frac{6\mu U}{h^2} \left(1 - \frac{H}{h} \right)$$

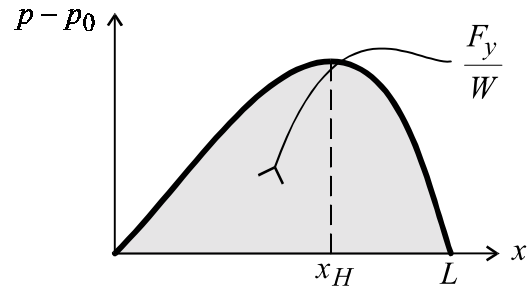
Note that

$$h_2 < H < h_1$$

$$\text{at } x=0: \quad h = h_2 > H \rightarrow dp/dx > 0$$

$$\text{at } x=x_H: \quad h = H \rightarrow dp/dx = 0$$

$$\text{at } x=L: \quad h = h_1 < H \rightarrow dp/dx < 0$$



Thus H represents not only an average gap width (with respect to flowrate) but it is also the width at the point where pressure is a maximum.

Now we are in a position to evaluate the force exerted on the plate by the fluid. The x -component of the force is

$$F_x = W \int_0^L \tau_{yx}(x, 0) dx = W \int_0^L \mu \frac{dv_x}{dy} \Big|_{y=0} dx = WL \frac{\mu U}{h} + O(\varepsilon)$$

where

$$\varepsilon \equiv \frac{h_1 - h_2}{L}$$

is the angle of tilt between the two plates ($|\varepsilon| \ll 1$). This result is the same (neglecting the $O(\varepsilon)$ correction) as would be obtained for two parallel plates. More interesting is the y -component. For a linear gap:

$$\frac{F_y}{W} = \int_0^L [p(x) - p_0] dx = \frac{6\mu UL^2}{(h_1 - h_2)^2} \left[\ln \frac{h_1}{h_2} - \frac{2(h_1 - h_2)}{h_1 + h_2} \right]$$

Does this reduce to zero when the two surfaces are parallel (i.e. $h_1 = h_2$)? Although it might appear that $F_y \rightarrow \infty$ as $h_1 \rightarrow h_2$, it turns out that the terms inside the square brackets tend to zero faster than the denominator outside the brackets. In the limiting case that

$$\varepsilon \ll 1 : \quad h_1 \approx h_2 \approx h,$$

$$\text{and} \quad h_1 - h_2 \ll \min(h_1, h_2),$$

$$\text{then we obtain:} \quad \frac{F_y}{W} = \frac{1}{2} \mu U \left(\frac{L}{h} \right)^3 \varepsilon$$

So we do recover zero force for parallel plates. The main difference between two parallel- and two nonparallel-plates is the occurrence of the nonzero y -component of lift which would not occur for parallel plates. Notice that $F_y > 0$ if $\varepsilon > 0$ ($h_1 > h_2$) and $F_y < 0$ if $\varepsilon < 0$ ($h_1 < h_2$). Thus either repulsion or attraction of the two plates is possible, depending on the direction of tilt relative to the direction of flow.

TRANSLATION OF A CYLINDER ALONG A PLATE

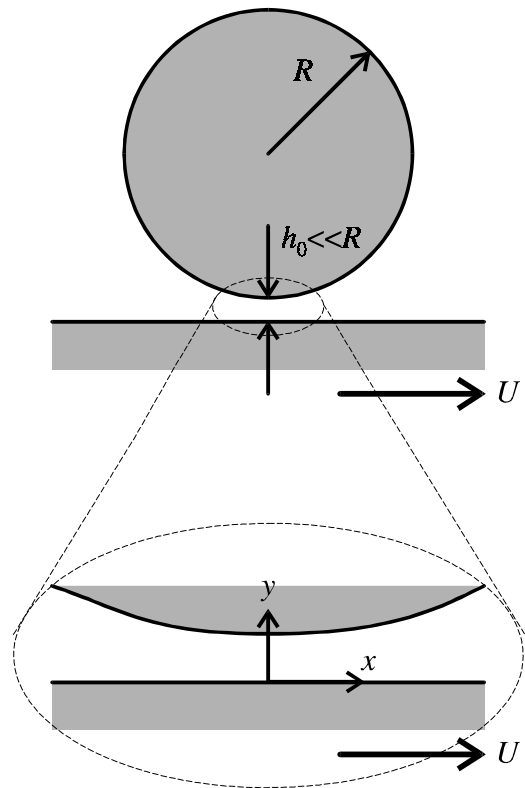
The lubrication approximations developed for the slider block can be easily extended to other geometries. For example, instead of a planar slider block, suppose I try to drag a cylinder parallel to a plate. What will be the force tending to push the two surfaces apart? The same perturbation expansion done with the slider block applies here, except we have a different profile $h(x)$ for the gap between the two surfaces.

To deduce the gap profile, recall the equation of a circle

$$(x - x_c)^2 + (y - y_c)^2 = R^2$$

where (x_c, y_c) is the location of the center of the circle and R is its radius. Substituting the coordinates of the center in our problem and $y(x) = h(x)$, we have

$$x^2 + (h - R - h_0)^2 = R^2$$



Dividing by R^2

$$\frac{x^2}{R^2} + \left(\frac{h-h_0}{R} - 1 \right)^2 = 1$$

Recognizing that $h-h_0$ is very small compared to R , we can use the Binomial Series, truncated after the second term, to obtain an approximation to the second term:

$$\left(\frac{h-h_0}{R} - 1 \right)^2 = \left(1 - \underbrace{\frac{h-h_0}{R}}_{\varepsilon} \right)^2 = (1-\varepsilon)^2 = 1 - 2\varepsilon + O(\varepsilon^2) \approx 1 - 2 \frac{h-h_0}{R}$$

The “1” cancels with the “1” on the right-hand side of the equation, leaving:

$$h(x) = h_0 + \frac{x^2}{2R} \quad (163)$$

provided that $h-h_0 \ll R$ which requires that x remain small compared to R .

$$\alpha = \frac{dh}{dx} = \frac{x}{R} \ll 1$$

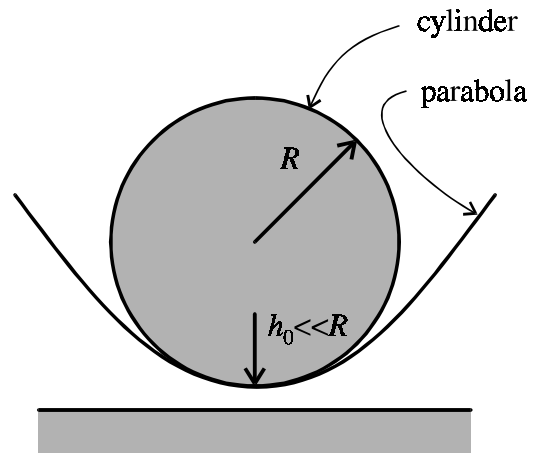
where we have moved $x=0$ to the center of the gap, which is now symmetric about $x=0$. Although (163) is only valid for x very small, it turns out virtually all of the contribution to the force comes from the region where (163) is valid — provided that h_0 is sufficiently small compared to R . In any case, let's assume that (163) is valid. If that bothers you, then replace the circular cylinder by a parabola.

As with the flat slider, the pressure profile is determined by the need to have the volumetric flowrate through any $x=\text{const}$ plane be the same for all such planes. Eq. (159) becomes:

$$\frac{Q}{W} = \frac{Uh}{2} - \frac{h^3}{12\mu} \frac{dp}{dx} = \text{const w.r.t. } x \quad (164)$$

If we view Q/W as an unknown integration constant, then we will need two boundary conditions to evaluate the two integration constants we will have after integrating this. Since the fluid held between the cylinder and the wall is in contact with the same reservoir at either end of the gap, we can require:

$$p = p_0 \text{ at } x = -\infty, +\infty$$



By “infinity”, we simply mean far from the origin. It might seem more reasonable to specify $x=R$, but if R is sufficiently large compared to h_0 , no significant error will be incurred by extending the limit to infinity. The counterpart to (161) is:

$$p(x) - p_\infty = 6\mu U \int_{-\infty}^x \frac{dx'}{h^2} - 12\mu \frac{Q}{W} \int_{-\infty}^x \frac{dx'}{h^3} \quad (165)$$

Applying the other boundary condition at $x=+\infty$ allow us to evaluate Q . The counterpart of (162) is

$$\frac{Q}{W} = \frac{UH}{2} \quad (166)$$

where

$$H = \frac{\int_{-\infty}^{\infty} \frac{dx}{h^2}}{\int_{-\infty}^{\infty} \frac{dx}{h^3}}$$

is an average gap width.

Substituting (163):

$$H = \frac{4}{3} h_0$$

As long as $h(x)$ is an even function of x , then $p(x)$ must be odd:

$$h(x)=\text{even} \rightarrow p(x)=\text{odd}$$

Thus the pressure profile given by (165) looks as shown at right. The extrema correspond to:

$$dp/dx=0$$

Substituting (166) and $dp/dx=0$ into (164) yields:

$$dp/dx=0:$$

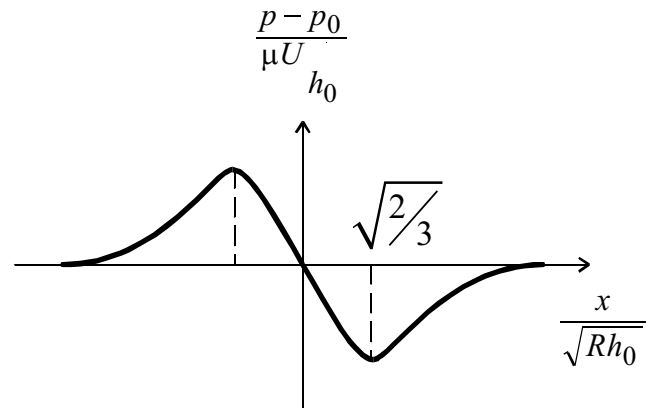
$$h = H = \frac{4}{3} h_0$$

Substituting (163) and solving for x :

$$dp/dx=0:$$

$$x = \pm \sqrt{\frac{2}{3} Rh_0}$$

Because p is an odd function, there will be no normal force tending to separate the two surfaces:



$p=\text{odd:}$

$$\frac{F_y}{W} = \int_{-\infty}^{\infty} [p(x) - p_0] dx = 0$$

$$\frac{F_x}{W} = \int_{-\infty}^{\infty} \tau_{xy}(x) dx = 2\pi\mu U \sqrt{\frac{2R}{h_0}}$$

CAVITATION

However, there is an important phenomenon which we have not discussed which can cause a lift force to push the two surfaces apart. That phenomenon is

cavitation - formation of gas bubbles caused by a lowering of pressure

If the absolute pressure of the fluid drops below the vapor pressure of the liquid, we will have boiling of the liquid and cavitation. Because pressures generated in lubrication problems can be significant compared to atmospheric, cavitation is not an uncommon event.

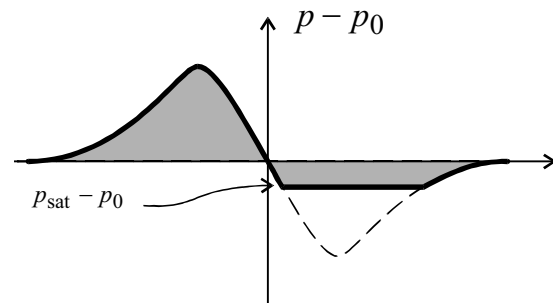
sources of gas bubbles:

- vapor of liquid (if $p < p_{\text{vapor}}$)
- air (if $p < p_{\text{saturation}}$)

Many liquids are kept in contact with air at one atmosphere and therefore become saturated with air. If the pressure on the liquid is suddenly lowered, the air will be supersaturated and air bubbles will form.

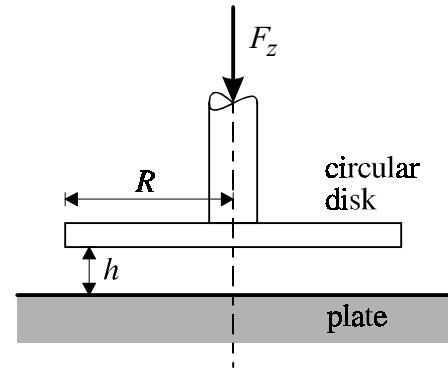
What effect will cavitation have on the pressure profile? Although an exact analysis would require consideration of two-phase flow, we can anticipate that — at the very least — the absolute pressure cannot drop below saturation.

If any of the negative portion of the pressure profile is chopped off, the profile loses its anti-symmetry. A repulsive force pushing the surfaces apart becomes likely. The resulting profile might be expected to look something like that shown at right.



SQUEEZING FLOW

The above two problems both involve the sliding of two surfaces past one another. A related lubrication problem is the squeezing motion between two bodies. Consider the “squeezing” motion in a thin film of liquid held between a circular disk and a parallel flat plate in the limit in which $h \ll R$. In the limit $\varepsilon \equiv h/R \rightarrow 0$, a solution to the Navier-Stokes equation can be found via a regular perturbation expansion of the form:



$$v_r(r, z, \varepsilon) = u_c u(\rho, \zeta, \varepsilon) = u_c [u_0(\rho, \zeta) + \varepsilon u_1(\rho, \zeta) + \dots]$$

$$v_z(r, z, \varepsilon) = U v(\rho, \zeta, \varepsilon) = U [v_0(\rho, \zeta) + \varepsilon v_1(\rho, \zeta) + \dots]$$

$$p(r, z, \varepsilon) - p_\infty = p_c P(\rho, \zeta, \varepsilon) = p_c [p_0(\rho, \zeta) + \varepsilon p_1(\rho, \zeta) + \dots]$$

where
$$U \equiv \frac{dh}{dt}, \quad \varepsilon \equiv \frac{h}{R}, \quad \rho \equiv r/R \quad \text{and} \quad \zeta \equiv z/h = z/\varepsilon R$$

Note by using the arbitrary u_c as our characteristic radial velocity, we are delaying the choice until we have a chance to inspect the continuity equation:

continuity:
$$\frac{1}{r} \frac{\partial(r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad (167)$$

Nondimensionalizing:
$$\frac{u_c}{R} \frac{1}{\rho} \frac{\partial(\rho u)}{\partial \rho} + \frac{U}{h} \frac{\partial v}{\partial \zeta} = 0$$

Dividing by U/h :
$$\underbrace{\frac{\varepsilon u_c}{U}}_1 \underbrace{\frac{1}{\rho} \frac{\partial(\rho u)}{\partial \rho}}_{O(\varepsilon^0)} = - \underbrace{\frac{\partial v}{\partial \zeta}}_{O(\varepsilon^0)} \quad (168)$$

Boundary conditions on v_z include:

at $z=0$ or $\zeta=0$:
$$v_z = 0 \quad \text{or} \quad v = 0$$

at $z=h$ or $\zeta=1$:
$$v_z = U \quad \text{or} \quad v = 1$$

This means that $\partial v_z / \partial z$ (and $\partial v / \partial \zeta$) is not zero; so that the other term in the continuity equation (167) is not zero either. The only way the two sides of (168) can have the same order in ε is for the coefficient to be $O(\varepsilon^0)$. This is accomplished by choosing:

$$u_c = \varepsilon^{-1} U$$

Next we examine the principle component of the Navier-Stokes equation. Although the flow is caused by motion in the z -direction, the r -component is much larger ($u_c \gg U$ for $\varepsilon \ll 1$).

$$r: \quad v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho_f} \frac{\partial p}{\partial r} + \frac{\mu}{\rho_f} \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right] + \frac{\mu}{\rho_f} \frac{\partial^2 v_r}{\partial z^2}$$

Here we have added the subscript “ f ” on the fluid density to avoid confusion with the dimensionless radial coordinate. Nondimensionalizing:

$$\underbrace{\frac{(\varepsilon^{-1} U)^2}{R} u \frac{\partial u}{\partial \rho}}_{O(\varepsilon^{-2})} + \underbrace{\frac{\varepsilon^{-1} U^2}{\varepsilon R} v \frac{\partial u}{\partial \zeta}}_{O(\varepsilon^{-2})} = -\frac{p_c}{\rho_f R} \frac{\partial p}{\partial \rho} + \underbrace{\frac{\mu \varepsilon^{-1} U}{\rho_f R^2} \frac{\partial}{\partial \rho} \left[\frac{1}{\rho} \frac{\partial (\rho u)}{\partial \rho} \right]}_{O(\varepsilon^{-1})} + \underbrace{\frac{\mu \varepsilon^{-1} U}{\rho_f (\varepsilon R)^2} \frac{\partial^2 u}{\partial \zeta^2}}_{O(\varepsilon^{-3})}$$

Clearly, the first viscous term and both inertial terms are negligible compared to the second viscous term. Unless we have some other term in the equation of the same order, we will be forced to take $\frac{\partial^2 u}{\partial \zeta^2} = 0$, which (after no slip is applied) yields $u = 0$ for all ζ . This would violate continuity. Thus we choose p_c so that the pressure derivative has the same order as the second viscous term:

$$\frac{p_c}{\rho_f R} = \frac{\mu \varepsilon^{-1} U}{\rho_f (\varepsilon R)^2} \quad \text{or} \quad p_c = \frac{\mu U}{\varepsilon^3 R}$$

Finally, we look at the secondary component of the Navier-Stokes equation:

$$z: \quad v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho_f} \frac{\partial p}{\partial z} + \frac{\mu}{\rho_f} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{\mu}{\rho_f} \frac{\partial^2 v_z}{\partial z^2}$$

Nondimensionalizing:

$$\underbrace{\frac{(\varepsilon^{-1} U) U}{R} u \frac{\partial v}{\partial \rho}}_{O(\varepsilon^{-1})} + \underbrace{\frac{U^2}{\varepsilon R} v \frac{\partial v}{\partial \zeta}}_{O(\varepsilon^{-1})} = -\underbrace{\frac{\mu U / \varepsilon^3 R}{\rho_f \varepsilon R} \frac{\partial P}{\partial \zeta}}_{O(\varepsilon^{-4})} + \underbrace{\frac{\mu}{\rho_f} \frac{U}{R^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial v}{\partial \rho} \right)}_{O(\varepsilon^0)} + \underbrace{\frac{\mu}{\rho_f} \frac{U}{(\varepsilon R)^2} \frac{\partial^2 v}{\partial \zeta^2}}_{O(\varepsilon^{-2})}$$

Note that $\partial P / \partial \zeta$ is the lowest order term in this equation. Moreover, it is the only term which is $O(\varepsilon^{-4})$. Therefore when the perturbation expansions are substituted, and terms of like order are collected, the leading term is

$$\varepsilon^{-4}: \quad \frac{\partial P_0}{\partial \zeta} = 0 \quad \text{or} \quad P_0 = P_0(\rho) \quad (169)$$

so that the z -dependence of pressure can be neglected compared to the r -dependence. The resulting problem to solve (going back to dimensional quantities) is:

$$\text{Continuity:} \quad \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad (170)$$

$$\text{NSE}_r: \quad \mu \frac{\partial^2 v_r}{\partial z^2} = \frac{dp}{dr} = \text{const. w.r.t. } z \quad (171)$$

Since the right-hand side is independent of z , we can integrate immediately to obtain the general solution:

$$\mu v_r = \frac{dp}{dr} \frac{z^2}{2} + c_1(r)z + c_2(r)$$

Boundary conditions are:

$$\text{at } z=0: \quad v_r = v_z = 0$$

$$\text{at } z=h: \quad v_r = 0, v_z = -U$$

Applying the b.c.'s we get:

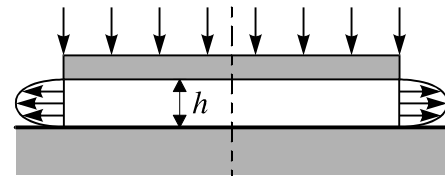
$$v_r = \frac{1}{2\mu} \frac{dp}{dr} z(z-h) \quad (172)$$

As before, dp/dr is determined such that continuity is satisfied. Now, however, macroscopic continuity requires:

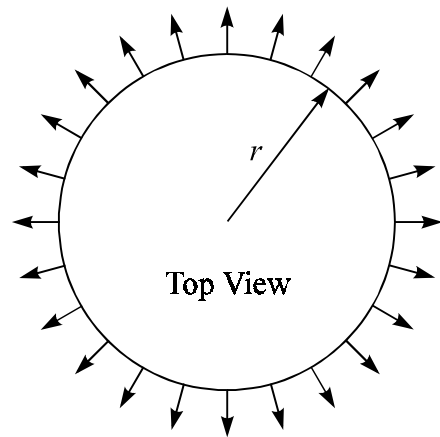
$$\underbrace{\int_0^h v_r(r, z) 2\pi r dz}_{\text{flow out through walls of cylinder}} = \underbrace{\pi r^2 U}_{\text{flow in through top}} \quad (173)$$

(172) into (173) and requiring that the result be satisfied for any r yields:

$$\frac{dp}{dr} = -6\mu U \frac{r}{h^3} \quad (174)$$



Side View



Top View

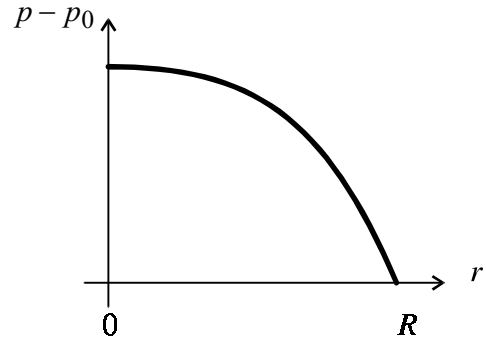
Requiring that: $p(R) = p_0$

we can integrate to obtain the pressure profile:

$$p(r) - p_0 = 3\mu U \frac{R^2 - r^2}{h^3}$$

which is sketched at right. Substituting (174) into (172):

$$v_r(r, z) = 3U \frac{r}{h} \left[\left(\frac{z}{h} \right) - \left(\frac{z}{h} \right)^2 \right] \quad (175)$$



The other component of velocity can be determined by applying microscopic continuity. Using (151) and (175):

$$\frac{\partial v_z}{\partial z} = -\frac{1}{r} \frac{\partial(rv_r)}{\partial r} = -\frac{6U}{h} \left[\left(\frac{z}{h} \right) - \left(\frac{z}{h} \right)^2 \right]$$

Integrating subject to $v_z = 0$ at $z = 0$:

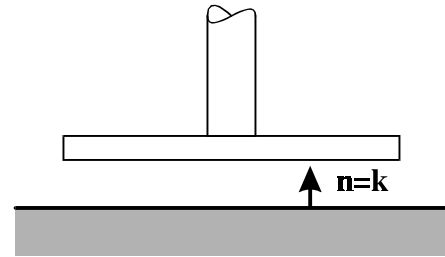
$$v_z(r, z) = -U \left[3 \left(\frac{z}{h} \right)^2 - 2 \left(\frac{z}{h} \right)^3 \right] \quad (176)$$

To calculate the force exerted by the plate on the fluid, we use the unit normal pointing out of the fluid: $\mathbf{n} = \mathbf{k}$:

$$d\mathbf{F} = \mathbf{k} \cdot \underline{\underline{\mathbf{T}}} da$$

From the axisymmetry of the problem, we anticipate that there will only be a z -component of this force, which we can calculate by post dotting the above by \mathbf{k} :

$$dF_z = \mathbf{k} \cdot \underline{\underline{\mathbf{T}}} \cdot \mathbf{k} da = (-p + \tau_{zz}) da$$



In this problem, the normal component of the deviatoric stress (τ_{zz}) vanishes. Using (176):

$$\tau_{zz} = \mu \partial v_z / \partial z|_{z=h} = \mu(-U)[6zh^{-2} - 6z^2h^{-3}]|_{z=h} = 0$$

This leaves just a contribution from the pressure. Since $p(r, h)$ is independent of θ , we choose $da = (2\pi r)dr$ to be a thin annulus of radius r and thickness dr :

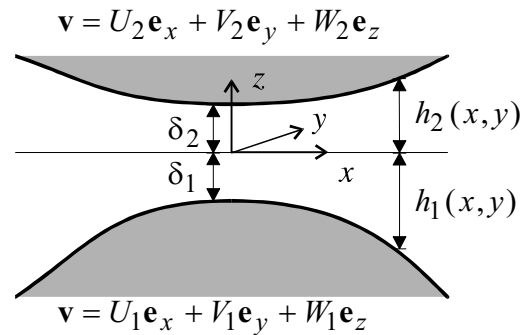
$$-F_z = 2\pi \int_0^R r p(r, h) dr = \frac{3\pi\mu UR^4}{2h^3}$$

Notice that, for a fixed U , the force required becomes unbounded as $h \rightarrow 0$.

REYNOLDS EQUATION

Sliding motion and squeezing motion are quite different. Yet the lubrication approximation for each has a lot in common. In particular, note that in either case, pressure can be taken as a constant along a normal to either surface (compare (157) and (169)). In either case, the dominant velocity component is tangent to the surfaces and the principle component of the NSE is approximated by a balance between viscous shear stress on the surface with the pressure gradient along the surface (compare (158) and (171)). It turns out that the lubrication approximation can be generalized to handle an arbitrary combination of squeezing and sliding motion in 3-D.

Consider two bodies of arbitrary (but smooth) shape moving slowly through a viscous fluid in the near vicinity of each other. A rectangular Cartesian coordinate system is chosen so that the z -axis coincides with a straight line connecting the two surfaces at the points of minimum approach. The origin is located at some arbitrary point along this line. δ_i represents the distance (along the z -axis) from the origin to the surface of body i ($i = 1, 2$), while $z = -h_1(x, y)$ and $z = h_2(x, y)$ describe a portion of their surfaces nearest the origin. Let R_{ix} and R_{iy} be the radii of curvature of body i in the x - and y -directions, respectively.



For distances $h_i(x, y)$ much less than both R_{ix} and R_{iy} , h_i can be approximated by*

$$h_i(x, y) = \delta_i + \frac{x^2}{2R_{ix}} + \frac{y^2}{2R_{iy}}$$

The total distance between the two surfaces is

$$h(x, y) = h_1(x, y) + h_2(x, y) = \delta + \frac{x^2}{2R_x} + \frac{y^2}{2R_y}$$

where $\delta = \delta_1 + \delta_2$ and R_j ($j = x, y$) can be considered to be the radii of curvature of the film:

* Actually, this assumes that the principle radii of curvature of both surfaces lie either in the x - or y -directions. Should one of the surfaces be rotated around the z -axis by an angle θ , the function acquires an addition term which is proportional to $xy \sin\theta \cos\theta$.

$$\frac{1}{R_j} = \frac{1}{R_{1j}} + \frac{1}{R_{2j}}$$

As in the two previous examples, scaling leads us to the following equations for the velocity profile in the film:

$$\mu \frac{\partial^2 v_x}{\partial z^2} = \frac{\partial p}{\partial x}$$

$$\mu \frac{\partial^2 v_y}{\partial z^2} = \frac{\partial p}{\partial y}$$

where the pressure profile is independent of z :

$$p = p(x, y)$$

Since pressure is independent of z , these equations can be easily integrated to yield the velocity profile in the film, which again turns out to be the sum of linear shear flow (from the sliding motion) and a parabolic pressure-driven flow.

$$v_x = \frac{\partial p}{\partial x} \frac{1}{2\mu} \left[z^2 - z(h_2 - h_1) - h_1 h_2 \right] + \frac{\Delta U}{h} (z + h_1) + U_1 \quad (177)$$

$$v_y = \frac{\partial p}{\partial y} \frac{1}{2\mu} \left[z^2 - z(h_2 - h_1) - h_1 h_2 \right] + \frac{\Delta V}{h} (z + h_1) + V_1$$

where

$$\Delta U \equiv U_2 - U_1 \quad \text{and} \quad \Delta V \equiv V_2 - V_1$$

Still unknown is the pressure profile, which is found by requiring the velocity profile to satisfy the continuity equation. In particular, since pressure is independent of z , we will choose p to satisfy the integral of the continuity equation with respect to z

$$\int_{-h_1(x, y, t)}^{h_2(x, y, t)} (\nabla \cdot \mathbf{v}) dz = 0$$

Expanding the divergence and separating derivatives with respect to z from those with respect to x and y :

$$\int_{-h_1}^{h_2} (\nabla \cdot \mathbf{v}) dz = \int_{-h_1}^{h_2} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dz = 0 \quad (178)$$

Applying Leibnitz' rule for differentiating an integral whose limits are functions of the differentiation variable:

$$\frac{\partial}{\partial x} \int_{-h_1(x,y)}^{h_2(x,y)} v_x(x,y,z) dz = \int_{-h_1(x,y)}^{h_2(x,y)} \frac{\partial v_x}{\partial x} dz + \underbrace{v_x(x,y,h_2)}_{U_2} \frac{\partial h_2}{\partial x} - \underbrace{v_x(x,y,-h_1)}_{U_1} \frac{\partial(-h_1)}{\partial x}$$

so

$$\int_{-h_1}^{h_2} \frac{\partial v_x}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h_1}^{h_2} v_x(x,y,z) dz - U_2 \frac{\partial h_2}{\partial x} - U_1 \frac{\partial h_1}{\partial x} \quad (179)$$

Substituting the velocity profile (177) into (179) and integrating leads to:

$$\int_{-h_1}^{h_2} v_x dz = \frac{h_1 + h_2}{2} (U_1 + U_2) - \frac{(h_1 + h_2)^3}{12\mu} \frac{\partial p}{\partial x}$$

Differentiating

$$\frac{\partial}{\partial x} \int_{-h_1}^{h_2} v_x dz = \frac{1}{2} (U_1 + U_2) \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right)$$

where

$$h = h_1 + h_2$$

(179) becomes

$$\int_{-h_1}^{h_2} \frac{\partial v_x}{\partial x} dz = -\frac{1}{12\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) - \frac{1}{2} \Delta U \frac{\partial \Delta h}{\partial x} \quad (180)$$

where

$$\Delta h = h_2 - h_1$$

There is a very similar result for the integral of $\partial v_y / \partial y$:

$$\int_{-h_1}^{h_2} \frac{\partial v_y}{\partial y} dz = -\frac{1}{12\mu} \frac{\partial}{\partial y} \left(h^3 \frac{\partial p}{\partial y} \right) - \frac{1}{2} \Delta V \frac{\partial \Delta h}{\partial y} \quad (181)$$

Finally the result for the integral of $\partial v_z / \partial z$:

$$\int_{-h_1}^{h_2} \frac{\partial v_z}{\partial z} dz = \int_{-h_1}^{h_2} dv_z = \underbrace{v_z(x,y,h_2)}_{W_2} - \underbrace{v_z(x,y,-h_1)}_{W_1} = \Delta W \quad (182)$$

Adding (180), (181) and (182), setting to zero (to satisfy (178)) and rearranging:

$$\frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(h^3 \frac{\partial p}{\partial y} \right) = -6\mu\Delta U \frac{\partial \Delta h}{\partial x} - 6\mu\Delta V \frac{\partial \Delta h}{\partial y} + 12\mu\Delta W \quad (183)$$

which is called **Reynolds lubrication equation**. The solution to this equation yields the pressure profile in the gap for any prescribed translation of the two bodies. Outside the gap, the pressure must approach the bulk pressure, which is taken to be zero

$$p \rightarrow 0 \text{ as } (x^2 + y^2) \rightarrow \infty$$

In the special case in which upper and lower surfaces are surfaces of revolution around the same axis, polar coordinates (r, θ) are more convenient than (x, y) since then $h_1 = h_1(r)$ and $h_2 = h_2(r)$. (183) can be written in invariant vector notation:

$$\nabla_s \cdot (h^3 \nabla_s p) = -6\mu(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{n}_2 - \mathbf{n}_1) \quad (184)$$

where \mathbf{v}_i is the velocity of body i ($i = 1$ or 2) and \mathbf{n}_i are local normals to body i (not necessarily of unit length). In particular \mathbf{n}_i is defined as:

$$\mathbf{n}_i = \nabla f_i$$

where

$$f_1(x, y, z) = h_1(x, y) + z$$

and

$$f_2(x, y, z) = h_2(x, y) - z$$

That \mathbf{n}_i are local normals to body i follows from the fact that f_1 is a constant on surface #1 (defined as $z = -h_1$) and f_2 is a constant on surface #2 (defined as $z = +h_2$). If we decompose the velocity of the two bodies into contributions along the z axis and in the xy plane

$$\mathbf{v}_i = \mathbf{v}_{si} + W_i \mathbf{e}_z$$

then (184) becomes

$$\boxed{\nabla_s \cdot (h^3 \nabla_s p) = \underbrace{-6\mu(\mathbf{v}_{s2} - \mathbf{v}_{s1}) \cdot \nabla_s (h_2 - h_1)}_{\text{sliding motion}} + \underbrace{12\mu\Delta W}_{\text{squeezing motion}}} \quad (185)$$

Example #1. Let's reformat the sliding-flow problem for a plate sliding past a cylinder (see page 162). In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(x) = h_0 + \frac{x^2}{2R}$$

The total gap between the two surfaces is

$$h(x) = h_1 + h_2 = h_0 + \frac{x^2}{2R}$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h(x)$$

The velocity of the lower surface is

$$U_1 = U \quad V_1 = 0 \quad W_1 = 0$$

while the upper surface is stationary:

$$U_2 = 0 \quad V_2 = 0 \quad W_2 = 0$$

The following quantities appear in Reynolds equation

$$\Delta U = -U \quad \Delta V = 0 \quad \Delta W = 0$$

Reynolds equation becomes

$$\frac{d}{dx} \left(h^3 \frac{dp}{dx} \right) = 6\mu U \frac{dh}{dx}$$

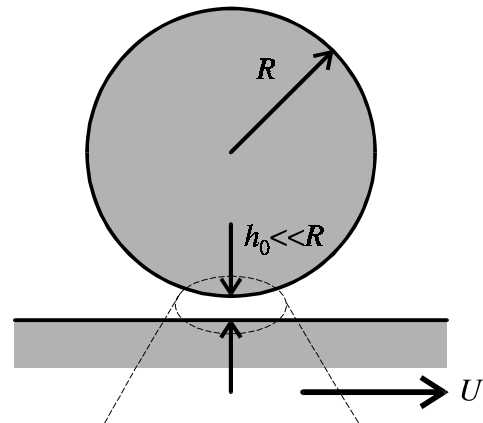
which is identical to the derivative of (164). Solving this 2nd order ODE leads to the same pressure profile we determined earlier.

Example #2.

Now let's reformulate the squeezing flow problem on page 166. In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(x,y) = h$$


The total gap between the two surfaces is

$$h = h_1 + h_2 = h$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h$$

The velocity of the upper surface is

$$U_2 = 0 \quad V_2 = 0 \quad W_2 = -U$$

while the lower surface is stationary:

$$U_1 = 0 \quad V_1 = 0 \quad W_1 = 0$$

The following quantities appear in Reynolds equation becomes

$$\Delta U = 0 \quad \Delta V = 0 \quad \Delta W = -U$$

(185) becomes
$$\nabla_s \cdot (h^3 \nabla_s p) = -12\mu U \quad (186)$$

Because the upper surface is a circular disk and the gap is uniform, we expect squeezing flow to be axisymmetric in cylindrical coordinates. In other words, we expect that $p = p(r)$ (i.e. no θ -dependence). In cylindrical (r, θ, z) or polar coordinates (r, θ) , the gradient is[♣]

$$\nabla_s p = \frac{\partial p}{\partial r} \mathbf{e}_r$$

while the divergence is
$$\nabla_s \cdot (h^3 \nabla_s p) = \frac{1}{r} \frac{d}{dr} \left(r \frac{dp}{dr} \right)$$

(186) becomes
$$\frac{1}{r} \frac{d}{dr} \left(r h^3 \frac{dp}{dr} \right) = -12\mu U$$

Multiplying through by r and integrating:

$$r h^3 \frac{dp}{dr} = -6\mu U r^2 + c$$

[♣] see http://www.andrew.cmu.edu/course/06-703/Vops_cyl.pdf

Dividing by rh^3 :

$$\frac{dp}{dr} = -6\mu U \frac{r}{h^3} + \frac{c}{rh^3}$$

When this is integrated a second time, the second term will lead to a logarithmic singularity at $r=0$. To keep pressure finite, we choose $c=0$ and then the above equation is identical to (174). Solving this 2nd order ODE leads to the same pressure profile we determined earlier.

Example #3. Sliding of a plate past a sphere

In this problem the equation of the lower surface (the plate) is just

$$h_1(x,y) = 0$$

The equation of the upper surface is

$$h_2(r) = h_0 + \frac{r^2}{2R}$$

The total gap between the two surfaces is

$$h(r) = h_1 + h_2 = h_0 + \frac{r^2}{2R} \quad (187)$$

For Reynolds equation, we also need

$$\Delta h = h_2 - h_1 = h(r)$$

The velocity of the sphere is purely along the x -axis

$$\mathbf{v}_2 = \mathbf{v}_{s2} = U\mathbf{e}_x = \mathbf{e}_r U \cos\theta - \mathbf{e}_\theta U \sin\theta$$

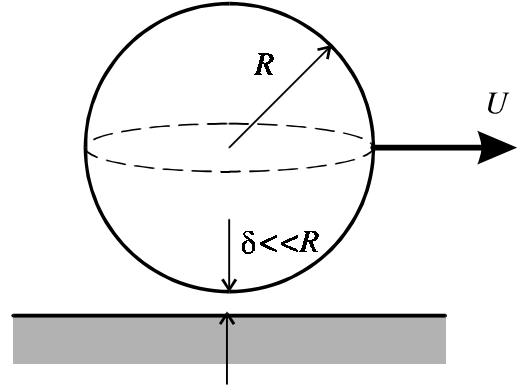
$$U_2 = U \quad V_2 = 0 \quad W_2 = 0$$

while the lower surface is stationary: $\mathbf{v}_1 = \mathbf{v}_{s1} = \mathbf{0}$

The following quantities appear in (185):

$$\nabla_s (h_2 - h_1) = \nabla_s h = \frac{dh}{dr} \mathbf{e}_r$$

$$(\mathbf{v}_{s2} - \mathbf{v}_{s1}) \cdot \nabla_s (h_2 - h_1) = (U \cos\theta \mathbf{e}_r - U \sin\theta \mathbf{e}_\theta) \cdot \left(\frac{dh}{dr} \mathbf{e}_r \right) = U \frac{dh}{dr} \cos\theta$$



(185) becomes
$$\frac{1}{r} \frac{\partial}{\partial r} \left(r h^3 \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{h^3}{r} \frac{\partial p}{\partial \theta} \right) = -6\mu U \frac{dh}{dr} \cos \theta$$

For the particular case in which h is given by (187), the solution is

$$p(r, \theta) = p_\infty + \frac{6\mu U r \cos \theta}{5h^2}$$

This produces no force in the z -direction (pressure profile is antisymmetric) but of course a force must be applied to the sphere to get it to move:

$$F_x = \frac{16}{5} \pi \mu U R \ln \frac{R}{\delta} + O(\delta^0) \quad \text{as } \delta \rightarrow 0$$

$$F_y = F_z = 0$$

Turbulence

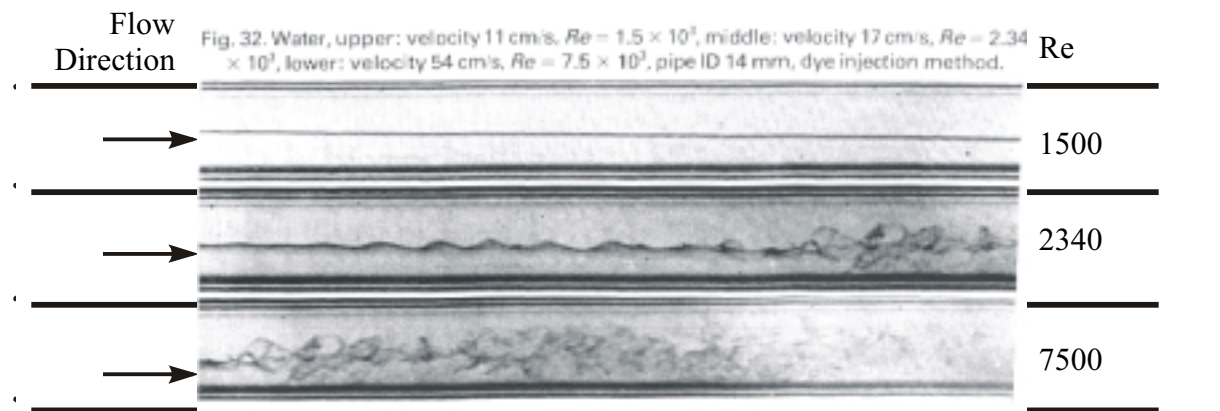
GENERAL NATURE OF TURBULENCE

In all the problems we have analyzed to date, the fluid elements travel along smooth predictable trajectories. This state of affairs is called:

laminar flow - fluid elements travel along smooth deterministic trajectories

These trajectories are straight parallel lines for simple pipe flows. However, this is not the only solution to the equations of motion. Consider the following experiment

Reynolds Experiment (1882) - inject a thin stream of dye into a fully developed flow in a pipe; observe the dye downstream. (see S:37)



- for laminar flow: dye stream appears as a straight colored thread

As the total flow rate of fluid in the pipe is increased, a sudden change in the appearance of the dye stream occurs. The thread of dye becomes more radially mixed with the fluid and, far enough downstream, its outline becomes blurred.

- for turbulent flow: irregular radial fluctuations of dye thread

Using a pipe with a sharp-edge entrance, Reynolds determined the critical flow rate for a large number of fluids and pipe sizes. He found in all cases, the transition occurred at a critical value of a dimensionless group:

$$\underbrace{\frac{\rho \langle \bar{v}_z \rangle D}{\mu}}_{Re} = 2300 \pm 200$$

where $\langle \bar{v}_z \rangle$ is the cross-sectional average velocity (= volumetric flowrate / pipe area). Today, we know this dimensionless group as the **Reynolds number**.

origin of turbulence - instability of laminar-flow solution to N-S eqns

instability - small perturbations (caused by vibration, etc.) grow rather than decay with time.

That the laminar-flow solution is metastable for $Re > 2100$ can be seen from Reynolds experiment performed with a pipe in which disturbances are minimized:

- reduce vibration
- fluid enters pipe smoothly
- smooth pipe wall

Under such conditions, laminar flow can be seen to persist up to $Re = 10^4$. However, just adding some vibrations (disturbance) can reduce the critical Re to 2100. The onset of turbulence causes a number of profound changes in the nature of the flow:

- dye thread breaks up -- streamlines appear contorted and random
- sudden increase in $\Delta p/L$
- local v_z fluctuates wildly with time
- similar fluctuations occur in v_r and v_θ

As a consequence of these changes, no simplification of the N-S equation is possible: v_r , v_θ , v_z and p all depend on r , θ , z and t .

TURBULENT FLOW IN PIPES

Velocity profiles are often measured with a **pitot tube**, which is a device with a very slow response time. As a consequence of this slow response time, the rapid fluctuations with time tend to average out. In the descriptions which follow, we will partition the instantaneous velocity \mathbf{v} into a time-averaged value $\bar{\mathbf{v}}$ (denoted by the overbar) and a fluctuation \mathbf{v}' (denoted by the prime):

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$$

Cross-sectional area averages will be denoted by enclosing the symbol inside carets:

$$\langle \bar{v}_z \rangle = \frac{\int_0^R \bar{v}_z(r) 2\pi r dr}{\int_0^R 2\pi r dr} = \frac{Q}{\pi R^2}$$

In laminar flow, the velocity profile for fully developed flow is parabolic in shape with a maximum velocity occurring at the pipe center that is twice the cross-sectional mean velocity:

In turbulent flow, the *time-averaged* velocity profile has a flatter shape. Indeed as the Reynolds number increases the shape changes such that the profile becomes even flatter. The profile can be fit to the following *empirical* equation:

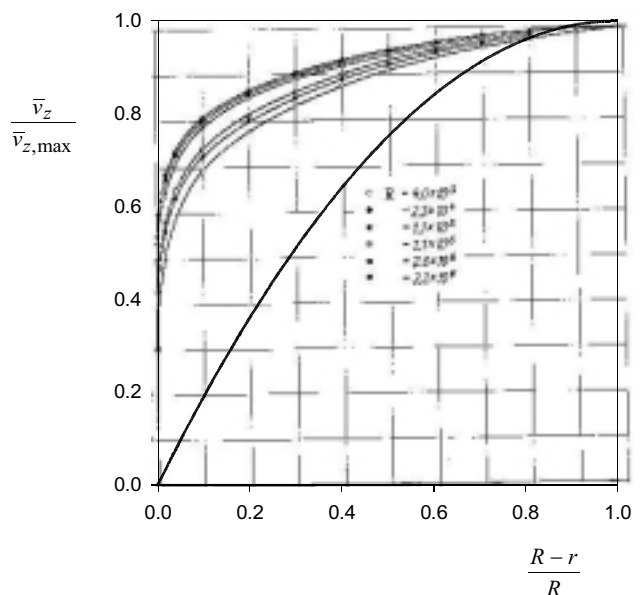
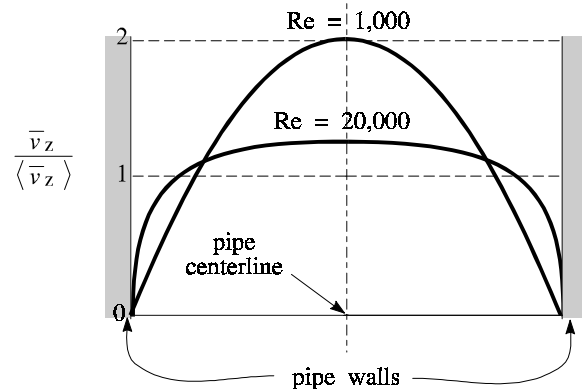
$$\bar{v}_z(r) = \bar{v}_{z,\max} \left(\frac{R-r}{R} \right)^{1/n}$$

where the value of the parameter n depends on Re :

$Re =$	4.0×10^3	2.3×10^4	1.1×10^5	1.1×10^6	2.0×10^6	3.2×10^6
$n =$	6.0	6.6	7.0	8.8	10	10
$v_{\max}/\langle v_z \rangle$	1.26	1.24	1.22	1.18	1.16	1.16

The reduction in the ratio of maximum to average velocity reflects the flattening of the profile as n becomes larger. Of course, this equation gives a “kink” in the profile at $r=0$ and predicts infinite slope at $r=R$, so it shouldn’t be applied too close to either boundary although it gives a reasonable fit otherwise.

How big are the fluctuations relative to the maximum velocity? Instantaneous speeds can be obtained for air flows using a **hot-wire anemometer**. This is simply a very thin wire which is electrically heated above ambient by passing a current through it. As a result of electrical heating (I^2R) the temperature of the wire will depend on the heat transfer coefficient, which in turn depends on the velocity of

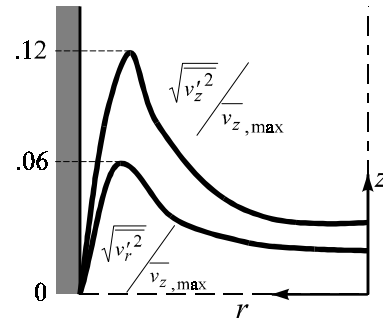


flow over the wire:

$$v_z \uparrow \rightarrow h \uparrow \rightarrow T_{\text{wire}} - T_{\text{air}} \downarrow$$

It's easy to determine the temperature of the wire from its electrical resistance, which generally increases with temperature. The reason for making the wire very thin is to decrease its thermal inertia. Very thin wires can respond rapidly to the rapid turbulent fluctuations in v_z .

Anyway, the instantaneous speed can be measured. Then the time-averaged speed and the fluctuations can be calculated. The root-mean-square fluctuations depend on radial position, as shown at right. Typically the axial fluctuations are less than 10% of the maximum velocity whereas the radial fluctuations are perhaps half of the axial.



Note that the fluctuations tend to vanish at the wall. This is a result of no-slip (applies even in turbulent flow) which requires that the *instantaneous* velocity must vanish at the wall for all time, which implies that the time average and the instantaneous fluctuations must vanish.

TIME-SMOOTHING

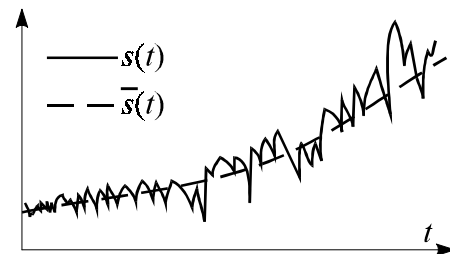
As we will see shortly, these fluctuations profoundly increase transport rates for heat, mass, and momentum. However, in some applications, we would be content to predict the time-averaged velocity profile. So let's try to time-average the Navier-Stokes equations with the hope that the fluctuations will average to zero.

First, we need to define what we mean by a **time-averaged** quantity. Suppose we have some property like velocity or pressure which fluctuates with time:

$$s = s(t)$$

We can average over some time interval of half width Δt :

$$\bar{s}(t) \equiv \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} s(t') dt'$$



We allow that the time-averaged quantity might still depend on time, but we have averaged out the rapid fluctuations due to turbulence.

Now let's define another quantity called the **fluctuation** about the mean:

$$s'(t) \equiv s(t) - \bar{s}(t)$$

TIME-SMOOTHING OF CONTINUITY EQUATION

The simple functional form of our experimentally measured velocity profile -- $\bar{v}_z(r)$ -- is exactly the same as for laminar flow. This suggests, that if we are willing to settle for the time-averaged velocity profile, then I might be able to get the result from the NSE. Let's try to time-smooth the equation of motion and see what happens. We will start with the equation of continuity for an incompressible flow:

$$\nabla \cdot \mathbf{v} = 0$$

Integrating the continuity equation for an incompressible fluid and dividing by $2\Delta t$:

$$\frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} \nabla \cdot \mathbf{v} dt' = \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} 0 dt' = 0$$

Thus the right-hand-side of the equation remains zero. Let's take a closer look at the left-hand side. Interchanging the order of differentiation and integration:

$$\frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} \nabla \cdot \mathbf{v} dt' = \nabla \cdot \left\{ \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} \mathbf{v} dt' \right\} = \nabla \cdot \bar{\mathbf{v}}$$

Substituting this result for the left-hand side of the continuity equation, leaves:

$$\nabla \cdot \bar{\mathbf{v}} = 0$$

Thus the form of the continuity equation has not changed as a result of time-smoothing.

TIME-SMOOTHING OF THE NAVIER-STOKES EQUATION

Encouraged by this simplification, we try to time-smooth the Navier-Stokes equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

After integrating both sides with respect to time and dividing by $2\Delta t$, we can break the integral of the sum into the sum of the integrals. Most of the terms transform in much the same way as the left-hand side of the continuity equation. The result is

$$\rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho \overline{\mathbf{v} \cdot \nabla \mathbf{v}} = -\nabla \bar{p} + \mu \nabla^2 \bar{\mathbf{v}} + \rho \mathbf{g}$$

With a little additional massaging (see Whitaker), the remaining term can be expressed as

$$\overline{\rho \mathbf{v} \cdot \nabla \mathbf{v}} = \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} + \nabla \cdot (\overline{\rho \mathbf{v}' \mathbf{v}'})$$

If the second term on the right-hand side were zero, then NSE after time-smoothing would have exactly the same form as before time-smoothing. Unfortunately, this term is not zero. Although the average of the fluctuations is zero, the average of the square of the fluctuations is not zero. So this second term cannot be dropped. Thus the time-smoothed Navier-Stokes equation becomes:

$$\rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = -\nabla \bar{p} + \mu \nabla^2 \bar{\mathbf{v}} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\bar{\tau}}}^{(t)}$$

where

$$\underline{\underline{\bar{\tau}}}^{(t)} = -\overline{\rho \mathbf{v}' \mathbf{v}'}$$

has units of stress or pressure and is called the **Reynold's stress**. Sometimes it is also called the **turbulent stress** to emphasize that arises from the turbulent nature of the flow. The existence of this new term is why even the time-averaged velocity profile inside the pipe is different from that during laminar flow. Of course, our empirical equation for the $\bar{v}_z(r)$ is also different from that for laminar flow.

Although we don't yet know how to evaluate this Reynolds stress, we can add it to the viscous stress and obtain a differential equation for their sum which we can solve for the simple case of pipe flow. Here's how we do it. First, recall that for incompressible Newtonian fluid, the stress is related to the rate of strain by Newton's law of viscosity. Time smoothing this constitutive equation yields:

$$\underline{\underline{\bar{\tau}}} = \mu \left[\nabla \bar{\mathbf{v}} + (\nabla \bar{\mathbf{v}})^t \right]$$

Taking the divergence:

$$\nabla \cdot \underline{\underline{\bar{\tau}}} = \mu \nabla^2 \bar{\mathbf{v}}$$

If we now make this substitution, NSE becomes

$$\begin{aligned} \rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} &= -\nabla \bar{p} + \nabla \cdot \underline{\underline{\bar{\tau}}} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\bar{\tau}}}^{(t)} \\ &= -\nabla \bar{p} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\bar{\tau}}}^{(T)} \end{aligned} \quad (188)$$

where

$$\underline{\underline{\bar{\tau}}}^{(T)} = \underline{\underline{\bar{\tau}}} + \underline{\underline{\bar{\tau}}}^{(t)}$$

is the total stress, i.e., the sum of the time-averaged viscous stress and the Reynolds stress. Thus we see that the Reynolds stress appears in the equations of motion in the same manner as the viscous stress. Indeed the sum of the two contributions plays the same role in turbulent flows that the viscous friction played in laminar flow.

ANALYSIS OF TURBULENT FLOW IN PIPES

We can make the same assumptions (i.e. the same guess) about the functional form of the *time-averaged* velocity and pressure profile in turbulent flow that we made for laminar flow: we will assume that the time-averaged velocity profile is axisymmetric ($v_\theta=0$, $\partial/\partial\theta=0$) and fully developed ($\partial/\partial z=0$).

$$\begin{aligned}\bar{v}_z &= \bar{v}_z(r) & \bar{v}_r &= \bar{v}_\theta = 0 \\ \bar{p} &= \bar{p}(z)\end{aligned}$$

Then the z -component of (188) yields:

$$0 = -\underbrace{\frac{\partial \bar{P}}{\partial z}}_{f(z)} + \underbrace{\frac{1}{r} \frac{\partial}{\partial r} (r \bar{\tau}_{rz}^{(T)})}_{g(r)} - \underbrace{\frac{1}{r} \frac{\partial \bar{\tau}_{\theta z}^{(T)}}{\partial \theta}}_0 - \underbrace{\frac{\partial \bar{\tau}_{zz}^{(T)}}{\partial z}}_0$$

where \bar{P} is the time-averaged dynamic pressure. This form of this equation was obtained using the tables in BSL (top half of p85, eqn C), after replacing the instantaneous quantities by their time averages, except that the instantaneous viscous stresses τ has been replaced by (minus) the *total* stress $\tau^{(T)}$. Expecting the time-averaged flow to be axisymmetric ($\partial/\partial\theta = 0$) and fully developed ($\partial/\partial z = 0$, except for pressure), the last two terms in this equation can be dropped and the second term is a function of r only. This leaves us with the same equation we had for laminar flow: a function of r only equal to a function of z only. The only way these two terms can sum to zero for all r and z is if both equal a spatial constant:

$$\frac{d\bar{P}}{dz} = \frac{1}{r} \frac{d}{dr} (r \bar{\tau}_{rz}^{(T)}) = -\frac{\Delta P}{L} < 0$$

This implies that pressure \bar{P} varies linearly with z . Solving for the *total* stress $\bar{\tau}_{rz}^{(T)}$ by integrating:

$$\bar{\tau}_{rz}^{(T)} = -\frac{1}{2} \frac{\Delta P}{L} r + \frac{c}{r} = \bar{\tau}_{rz} + \bar{\tau}_{rz}^{(t)} \quad (189)$$

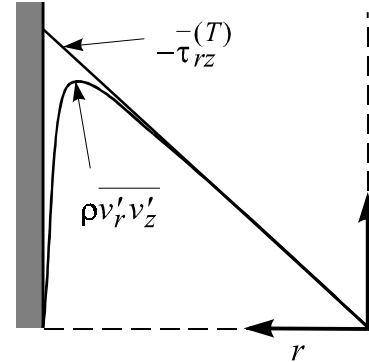
The integration constant c was chosen to be zero to avoid having the stress unbounded at $r=0$. Now this is the *total* stress: the sum of the Reynolds stress

$$\tau_{rz}^{(t)} = -\rho \overline{v'_r v'_z}$$

and a viscous contribution from time-smoothing Newton's law of viscosity:

$$\bar{\tau}_{rz} = \mu \frac{d\bar{v}_z}{dr}$$

The latter can be determined by differentiating the time-averaged velocity profile. If we subtract this from the total we can determine $\bar{\tau}_{rz}^{(t)}$ -- one of the components of the Reynolds stress tensor. The result is shown in the figure at right. Notice that the turbulent stress tends to vanish near the wall. This can be explained by noting that at the wall, "no slip" between the fluid and the stationary wall requires that the instantaneous velocity, as well as its time average, must be zero:



$$v_z = \bar{v}_z = 0 \Rightarrow v'_z = 0 \Rightarrow \bar{\tau}_{rz}^{(t)} = -\rho \overline{v'_r v'_z} = 0$$

In terms of the relative importance of these two contributions to the total, one can define three regions:

1. **turbulent core:** $\tau^{(t)} \gg \tau$. This covers most of the cross section of the pipe.
2. **laminar sublayer:** $\tau^{(t)} \ll \tau$. Very near the wall, the fluctuations must vanish (along with the Reynolds stress) but the viscous stress are largest.
3. **transition zone:** $\tau^{(t)} \approx \tau$. Neither completely dominates the other.

When applied to the situation of fully developed pipe flows, continuity is automatically satisfied and the time-smoothed Navier-Stokes equations yields only one equation in 2 unknowns:

2 unknowns: $\bar{v}_z(r)$ and $\rho \overline{v'_r v'_z}$

Clearly another relationship is needed to complete the model. This missing relationship is the **constitutive equation** relating the Reynolds stress to the time-smoothed velocity profile. One might be tempted to define a quantity like the viscosity to relate stress to the time-averaged velocity.

$$\bar{\tau}_{rz}^{(t)} \stackrel{?}{=} \mu^{(t)} \frac{d\bar{v}_z}{dr}$$

But if you define the "turbulent viscosity" this way, its value turns out to depend strongly on position.

$$\frac{\mu^{(t)}}{\mu} = \begin{cases} \approx 100 & \text{near pipe centerline} \\ 0 & \text{at pipe wall} \end{cases}$$

So unlike the usual viscosity, $\mu^{(t)}$ is not a material property (since it depends on position rather than just the material).

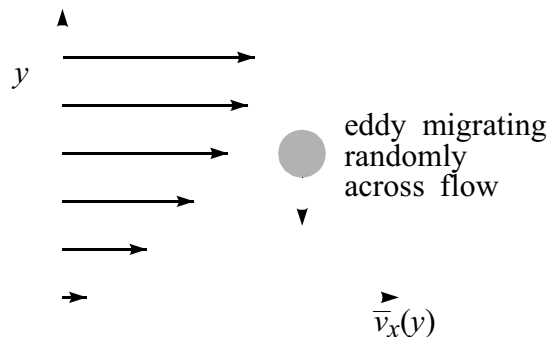
PRANDTL'S MIXING LENGTH THEORY

The first successful constitutive equation for turbulence was posed by Prandtl in 1925. Prandtl imagined that the fluctuations in instantaneous fluid velocity at some fixed point were caused by **eddies** of fluid which migrate across the flow from regions having higher or lower time-averaged velocity.

eddy - a packet of fluid (much larger than a fluid element) which can undergo random migration across streamlines of the time-smoothed velocity field.

These eddies have a longitudinal velocity which corresponds to the time-average velocity at their previous location.

As this eddy moves across the streamlines, it gradually exchanges momentum with the surrounding fluid which is moving at a different longitudinal velocity. But this exchange does not occur instantaneously. The eddy retains its original velocity for a brief period of time. We might call this the **mixing time**. During this time, the eddy migrates laterally a distance l called the mixing length:



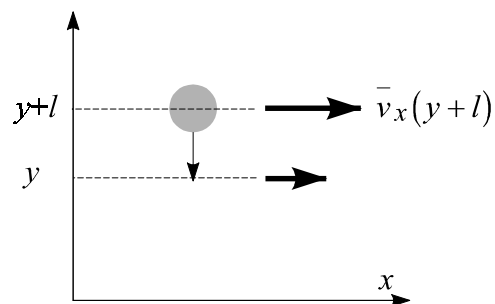
mixing length (l) - characteristic distance an eddy migrates normal to the main flow before mixing

Although momentum exchange between eddies occurs continuously in actual turbulent flow, Prandtl imagined that a migrating eddy keeps all of its original velocity until it migrated a distance l and then suddenly it exchanges it. This is like molecules of a gas retaining its momentum until it collides with another gas molecule, which causes a sudden exchange of momentum. Indeed, you might find it helpful to think of the mixing length as being the analogue of **mean-free-path** in the kinetic theory of gases.

Recall that:

mean-free path - average distance a gas molecule travels before colliding with another gas molecule.

Now suppose we are monitoring the instantaneous velocity at a distance y from the wall when an eddy drifts into our location from $y+l$. Because this migrating eddy has a higher velocity than the average fluid at y , we will observe an positive fluctuation



when the eddy arrives. To estimate the magnitude of the fluctuation, we can expand the time-smoothed velocity profile in Taylor series about $y=y$:

$$\bar{v}_x(y+l) = \bar{v}_x(y) + \left. \frac{d\bar{v}_x}{dy} \right|_y l + \frac{1}{2} \left. \frac{d^2\bar{v}_x}{dy^2} \right|_y l^2 + \dots$$

Assuming that l is sufficiently small that we can truncate this series without introducing significant error:

$$(v'_x)_{above} = \bar{v}_x(y+l) - \bar{v}_x(y) \approx l \frac{d\bar{v}_x}{dy}$$

where the subscript “above” is appended to remind us that is the fluctuation resulting from an eddy migrating from above. At some later time, another eddy might migrate to our location from below, producing a negative fluctuation in velocity:

$$(v'_x)_{below} = \bar{v}_x(y-l) - \bar{v}_x(y) \approx -l \frac{d\bar{v}_x}{dy}$$

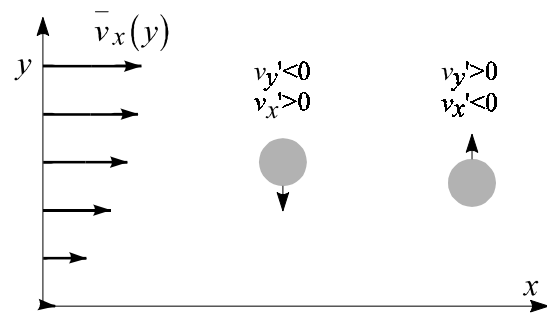
Of course the average fluctuation is zero: $\overline{v'_x} = 0$, but the average of the squares is not:

$$\overline{(v'_x)^2} \approx \frac{1}{2} \left\{ \overline{(v'_x)_{above}^2} + \overline{(v'_x)_{below}^2} \right\} \approx l^2 \left(\frac{d\bar{v}_x}{dy} \right)^2 \quad (190)$$

Now let's turn our attention to v'_y . This is related to how fast the eddies migrate, and the sign depends on whether they are migrating upward or downward.

If the eddy migrates from above, it represents a negative y -fluctuation (it is moving in the $-y$ direction). Such an eddy will have a greater x -velocity than the fluid receiving it, consequently generating a positive x -fluctuation:

$$v'_y < 0 \rightarrow v'_x > 0 \rightarrow v'_x v'_y < 0$$



On the other hand, if the eddy migrates from below, it represents a positive y -fluctuation but has less x -velocity than the fluid receiving it, generating a negative x -fluctuation:

$$v'_y > 0 \rightarrow v'_x < 0 \rightarrow v'_x v'_y < 0$$

Finally, if there is no vertical migration of eddies, there is no reason for the x -velocity to fluctuate:

$$v_y' = 0 \rightarrow v_x' = 0$$

These three statements suggest that the y -fluctuations are proportional to the x -fluctuations, with a negative proportionality constant:

$$v_y' \approx -\alpha v_x'$$

where $\alpha > 0$. Alternatively, we can write:

$$v_x' v_y' = -\alpha (v_x')^2$$

Time averaging and then substituting (190):

$$\overline{v_x' v_y'} = -\alpha \overline{(v_x')^2} = -\alpha l^2 \left(\frac{d\bar{v}_x}{dy} \right)^2$$

Absorbing the unknown α into the (still unknown) mixing length parameter:

$$\bar{\tau}_{xy}^{(t)} = -\rho \overline{v_x' v_y'} = \rho l^2 \left(\frac{d\bar{v}_x}{dy} \right)^2 \quad (191)$$

Comparing this result with Newton's law of viscosity:

$$\bar{\tau}_{xy} = \mu \frac{d\bar{v}_x}{dy}$$

we could conclude that an apparent **turbulent viscosity** is given by:

$$\mu^{(t)} = \rho l^2 \left| \frac{d\bar{v}_x}{dy} \right|$$

Of course, this viscosity is not a true fluid property, because it depends strongly on the velocity profile.

For this theory to be useful, we need a value for the “mixing length” l . There are two properties of l which we can easily deduce. First of all, l was defined as the distance normal to the wall which the eddy travels before becoming mixed with local fluid. Clearly, this mixing must occur before the eddy “bumps” into the wall, so:

Property #1:

$$l < y$$

where y is the distance from the wall. Secondly, we know from no-slip that the fluctuations all vanish at the wall. Consequently, the Reynolds stress must vanish at the wall. Since the velocity gradient does not vanish, we must require that the mixing length vanish at the wall:

Property #2:

$$l=0 \text{ at } y=0$$

If it's not a constant, the next simplest functional relationship between l and y which satisfies both these properties is:

$$l = ay \quad (192)$$

where a is some constant and $0 < a < 1$.

PRANDTL'S "UNIVERSAL" VELOCITY PROFILE

The velocity profile in turbulent flow is essentially flat, except near the wall where the velocity gradients are steep. Focussing attention on this region near the flow, Prandtl tried to deduce the form for the velocity profile in turbulent flow. Recall from (189) that in pipe flow, the total stress varies linearly from 0 at the center line to a maximum value at the wall:

$$\bar{\tau}_{rz}^{(T)} = -\frac{1}{2} \frac{\Delta P}{L} r = \tau_0 \frac{r}{R} < 0 \quad (193)$$

where we have defined

$$\tau_0 \equiv -(1/2)(R/L)\Delta P > 0$$

which represents the stress on the wall. In the "turbulent core", the Reynolds stress dominates the "laminar" stress; then substituting (191) through (193):

$$\begin{aligned} \bar{\tau}_{xy}^{(t)} &\approx \bar{\tau}_{rz}^{(T)} \\ \rho \frac{l^2}{a^2 y^2} \left(\frac{d\bar{v}_x}{dy} \right)^2 &\approx \tau_0 \left(1 - \frac{y}{R} \right) \end{aligned} \quad (194)$$

or

$$ay \frac{d\bar{v}_x}{dy} = \underbrace{\sqrt{\frac{\tau_0}{\rho}}}_{v^*} \sqrt{\left(1 - \frac{y}{R} \right)} = v^* \sqrt{\left(1 - \frac{y}{R} \right)}$$

The general solution to this 1st order ODE is

$$v^+(y^+) = C + \frac{2}{a} \sqrt{1 - \frac{y^+}{R^+}} - \frac{2}{a} \tanh^{-1} \sqrt{1 - \frac{y^+}{R^+}} \quad (195)$$

where C is the integration constant, v^* is called the **friction velocity** and where we have introduced dimensionless variables:

$$l = ay \quad (192)$$

where a is some constant and $0 < a < 1$.

PRANDTL'S "UNIVERSAL" VELOCITY PROFILE

The velocity profile in turbulent flow is essentially flat, except near the wall where the velocity gradients are steep. Focussing attention on this region near the flow, Prandtl tried to deduce the form for the velocity profile in turbulent flow. Recall from (189) that in pipe flow, the total stress varies linearly from 0 at the center line to a maximum value at the wall:

$$\bar{\tau}_{rz}^{(T)} = -\frac{1}{2} \frac{\Delta P}{L} r = \tau_0 \frac{r}{R} < 0 \quad (193)$$

where we have defined $\tau_0 \equiv -(1/2)(R/L)\Delta P > 0$

which represents the stress on the wall. In the "turbulent core", the Reynolds stress dominates the "laminar" stress; then substituting (191) through (193):

$$\bar{\tau}_{xy}^{(t)} \approx \bar{\tau}_{rz}^{(T)}$$

$$\rho \frac{l^2}{a^2 y^2} \left(\frac{d\bar{v}_x}{dy} \right)^2 \approx \tau_0 \left(1 - \frac{y}{R} \right) \quad (194)$$

or

$$ay \frac{d\bar{v}_x}{dy} = \underbrace{\sqrt{\frac{\tau_0}{\rho}}}_{v^*} \sqrt{\left(1 - \frac{y}{R} \right)} = v^* \sqrt{\left(1 - \frac{y}{R} \right)}$$

The general solution to this 1st order ODE is

$$v^+(y^+) = C + \frac{2}{a} \sqrt{1 - \frac{y^+}{R^+}} - \frac{2}{a} \tanh^{-1} \sqrt{1 - \frac{y^+}{R^+}} \quad (195)$$

where C is the integration constant, v^* is called the **friction velocity** and where we have introduced dimensionless variables:

$$v^+ = \frac{\bar{v}_x}{v^*}$$

$$y^+ = \frac{v^*}{\nu} y \quad \text{and} \quad R^+ = \frac{v^*}{\nu} R$$

Near the wall (i.e. for $y \ll R$ or $y^+ \ll R^+$), we can simplify (195):

$$\text{for } y^+ \ll R^+: \quad \sqrt{1 - \frac{y^+}{R^+}} = 1 - \frac{y^+}{2R^+} + O(y^{+2}) \quad (196)$$

$$\tanh^{-1} \sqrt{1 - \frac{y^+}{R^+}} = \frac{1}{2} \ln \frac{4R^+}{y^+} + O(y^+)$$

Dropping the higher-order terms:

$$v^+(y^+) = \underbrace{\frac{2 - \ln(4R^+)}{a}}_c + C + \frac{1}{a} \ln y^+ \quad (197)$$

where c is a collection of constants.

This result can be derived more easily by starting over with a simplified (194):

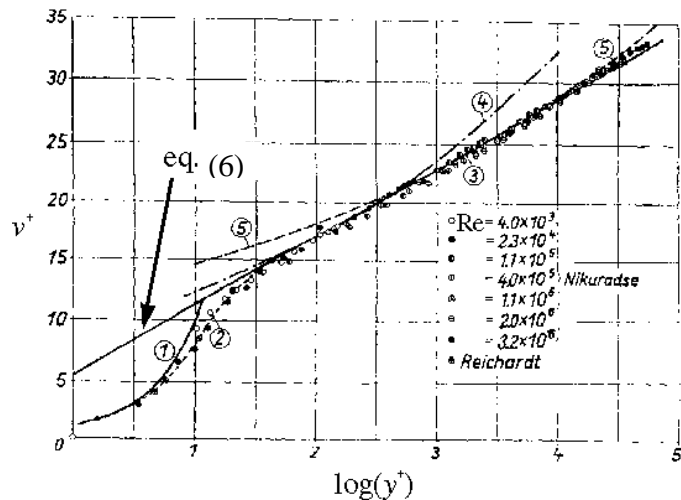
$$\text{for } y \ll R: \quad ay \frac{d\bar{v}_x}{dy} = v^* \quad \text{or} \quad \underbrace{\frac{d\bar{v}_x}{v^*}}_{dv^+} = \frac{1}{a} \frac{dy}{y} = \frac{1}{a} \frac{dy^+}{y^+} \quad (198)$$

which integrates to

$$v^+ = \frac{1}{a} \ln y^+ + c$$

where c is some integration constant. When plotted on semi-log coordinates, experimental velocity profiles do indeed show a linear region which extends over a couple of decades of y^+ values:

Moreover, the slope and intercept of this straight line don't seem to depend on the Reynolds number. Indeed, the slope and intercept also don't seem to depend on the shape of the conduit. Rectangular conduits yields the same velocity profile on these coordinates. This is called **Prandtl's Universal Velocity Profile**:



$$y^+ > 26: \quad v^+ = 2.5 \ln y^+ + 5.5 \quad (199)$$

which applies for $y^+ > 26$ (the **turbulent core**). This coefficient of $\ln y^+$ corresponds to $a=0.4$. so (192) becomes:

$$l = 0.4 y$$

Recall that we reasoned that a had to be in the range of 0 to 1 to be physically realistic.

In the **laminar sublayer**, Reynolds stress can be totally neglected, leaving just viscous stress. This close the wall, the total stress is practically a constant equal to the wall shear stress τ_0 :

$$y \ll R: \quad \begin{cases} \bar{\tau}_{xy} \approx \bar{\tau}_{rz}^{(T)} \\ \mu \frac{d\bar{v}_x}{dy} = \tau_0 \end{cases}$$

Then we can integrate the above ODE for \bar{v}_x , subject to $\bar{v}_x = 0$ at $y=0$ (i.e. no slip):

$$\bar{v}_x = \frac{\tau_0}{\mu} y$$

Dividing both sides by v^* we can make the result dimensionless:

$$y^+ < 5: \quad v^+ = y^+ \quad (200)$$

which applies for $0 < y^+ < 5$ (the **laminar sublayer**). Of course (199) also does not apply near the center of the pipe, since the $y^+ \approx R^+$ there, whereas (199) was derived by assuming that $y^+ \ll R^+$ (see (196)).

PRANDTL'S UNIVERSAL LAW OF FRICTION

Let's try to figure deduce the analog of Poiseuille's Formula (see page 87) for turbulent flow. Poiseuille's Formula is the relationship between volumetric flowrate through the pipe and pressure drop. Volumetric flowrate Q is calculated by integrating the axial component of fluid velocity of the cross section of the pipe:

$$\langle \bar{v}_z \rangle = \frac{Q}{\pi R^2} = \frac{2}{R^2} \int_0^R r \bar{v}_z(r) dr$$

Now we are going to use (199) for the velocity profile, although we assumed in (198) that $y \ll R$ (where $y = R-r$).

Although the derivation of (199) assumed that we are very close to the wall ($y^+ \ll R^+$, see (196)), (199) works remarkably well right out to the centerline $y^+ = R^+$. The plot at right shows the velocity profiles (with different wall roughness on the walls) compared with predictions based on (199). The ordinate is

$$\begin{aligned} \frac{\bar{v}_z(R) - \bar{v}_z(r)}{v^*} &= v^+ (R^+) - v^+ (y^+) = \left[2.5 \ln \frac{R}{y} \right] \\ &= 2.5 (\ln R^+ - \ln y^+) = 2.5 \ln \frac{R}{y} \\ &= 2.5 \ln \frac{R}{y} = 2.5 \ln \frac{R}{R-r} \end{aligned}$$

Note that (199) predicts an infinite velocity-difference at $y=0$, whereas the actual velocity must be finite. Of course, (199) does not apply right up to the wall because very near the wall the Reynolds stresses are not dominant.

Substituting (199) and integrating:

$$\langle \bar{v}_z \rangle = v^* \left[2.5 \ln \left(\frac{v^* R}{v} \right) + 1.75 \right] \quad (201)$$

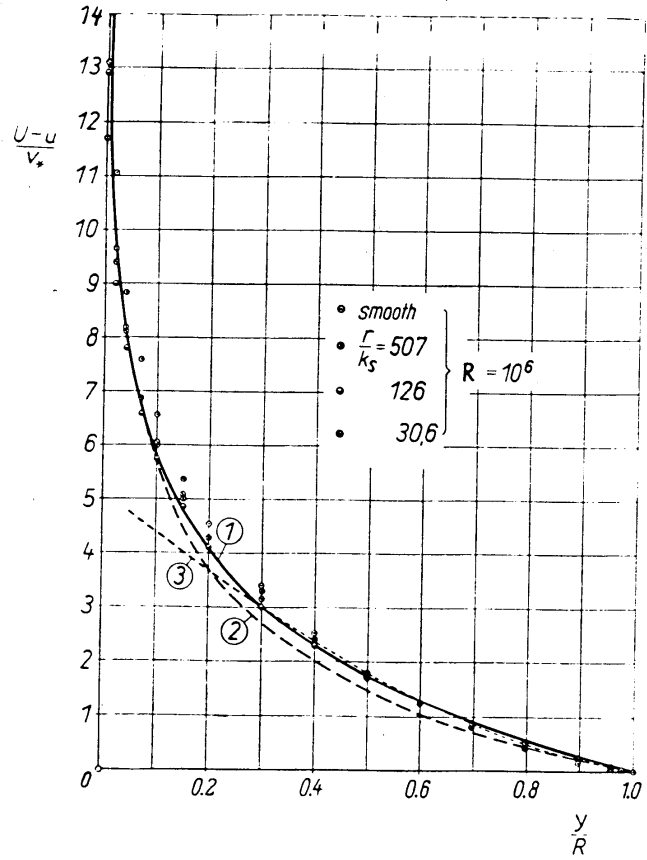
Now the friction velocity can be related to the **friction factor**, whose usual definition can be expressed in terms of the variables in this analysis:

$$f \equiv \frac{\tau_0}{\frac{1}{2} \rho \langle \bar{v}_z \rangle^2} = 2 \left(\frac{v^*}{\langle \bar{v}_z \rangle} \right)^2$$

Thus

$$\frac{\langle \bar{v}_z \rangle}{v^*} = \sqrt{\frac{2}{f}}$$

Likewise, the usual definition of Reynolds number yields



$$\text{Re} \equiv \frac{2\langle \bar{v}_z \rangle R}{\nu}$$

Thus

$$\frac{\nu^* R}{\nu} = \frac{\langle \bar{v}_z \rangle R}{\underbrace{\nu}_{\text{Re}/2}} \times \frac{\nu^*}{\underbrace{\langle \bar{v}_z \rangle}_{\sqrt{f}/2}} = \frac{\text{Re} \sqrt{f}}{2\sqrt{2}}$$

Relating ν^* to f and $\langle \bar{v}_z \rangle$ to Re , (201) can be written as:

$$\frac{1}{\sqrt{f}} = 1.77 \ln(\text{Re} \sqrt{f}) - 0.60$$

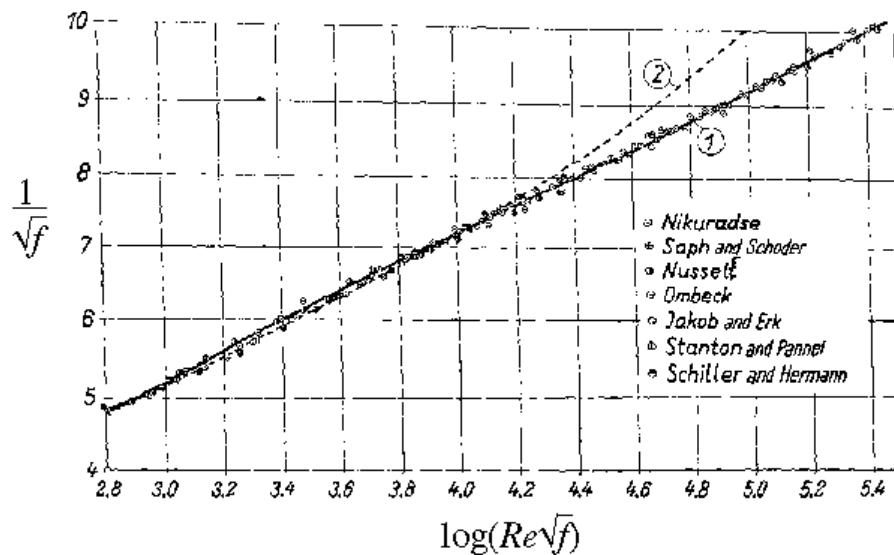
or

$$\frac{1}{\sqrt{f}} = 4.07 \log_{10}(\text{Re} \sqrt{f}) - 0.60$$

which fits experimental data remarkable well. A slightly better fit can be obtained by adjusting the coefficients:

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10}(\text{Re} \sqrt{f}) - 0.40 \quad (202)$$

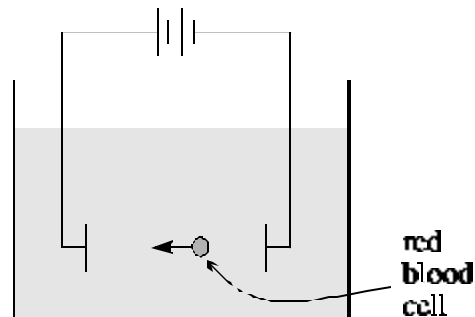
which is called **Prandtl's (universal) law of friction**. It applies virtually over the entire range of Reynolds numbers normally encountered for turbulent pipe flow: $2100 < \text{Re} < 5 \times 10^6$.



Electrohydrodynamics

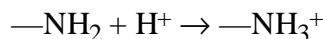
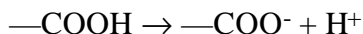
ORIGIN OF CHARGE

Nearly all particles, bubbles or drops in water are charged. We know this because when you apply an electric field across water containing a particle, bubble or drop, you can observe it to move. An example is the red blood cell experiment at right.



Interface between the two immiscible phases of the particle and water acquires charge by a number of mechanisms:

- adsorption of ions (e.g. SDS and FeOH^{+2})
- dissolution of ions from ionic solids (e.g. AgI)
- dissociation of acidic or basic sites on interface

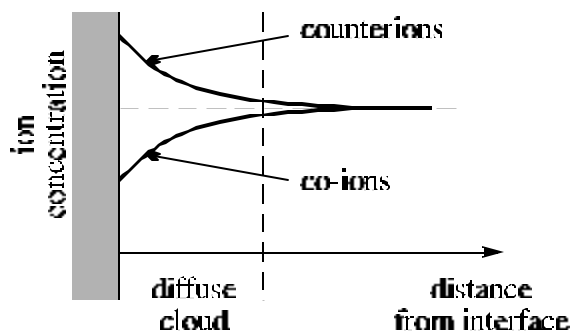


Regardless of how the surface acquires its charge, overall electroneutrality must be obeyed. So the solution must contain an excess of counterions. For example, if the surface acquires its charge by the dissociation of COOH groups at the interface, the COO^- remaining at the interface gives the interface a negative charge, but the H^+ joins the rest of the ions in the aqueous solution, thereby lending it a net positive charge.

Counterions would experience an electrostatic attraction for the surface which tends to pile up the ions next to the interface. On the other hand, the ions undergo diffusion (Brownian motion) which tends to disperse them uniformly throughout the phase. At equilibrium, a balance is achieved between these two opposing tendencies. In this equilibrium state, a diffuse cloud of counter-ions is formed next to the charged interface.

double layer - layer of charged fixed on interface + diffuse cloud of counter-ions.

Even though there are equal concentrations of positive and negative ions far from the surface, this is not true inside the counterion cloud. Fluid elements inside the



cloud bear a net charge. Just like we define the mass per volume to be the density of the fluid, we can define:

$$\rho_e(\mathbf{x}, t) = \text{local charge density (charge/volume)}$$

Global electroneutrality requires:

$$\sigma + \int_0^{\infty} \rho_e(y) dy = 0$$

$$\sigma = \text{surface charge density (charge/area)}$$

GOUY-CHAPMAN MODEL OF DOUBLE LAYER

Let's try to quantify this description. In particular, we would like to know how thick this cloud is. Because of the charge on the interface, ions feel a electrostatic force. The force per unit charge is called the *electric field*:

$$\mathbf{E}(\mathbf{x}) = \text{electric field (force/charge)}$$

Like gravity, this vector field is *conservative*, thus there must exist an scalar potential, $\psi(\mathbf{x})$ such that:

$$\mathbf{E} = -\nabla\psi \quad (203)$$

Like the gravitational potential, the minus sign is included by convention. ψ is called the *electrostatic potential*.

$$\psi [=] \text{volts (energy/charge)}$$

In thermodynamics, we learn that the criteria for phase equilibrium is equality of *chemical potential* μ_i for each chemical species i in each phase. For an ideal solution, recall that changes in chemical potential at constant temperature and pressure is given by

$$\left(\frac{\partial \mu_i}{\partial c_i} \right)_{T,P} = \frac{kT}{c_i} \quad \text{or} \quad (d\mu_i)_{T,P} = kT d \ln c_i \quad (204)$$

where kT is the thermal energy possessed by every ion or molecule and μ_i is the chemical energy per ion. When the species is charged, we must add an electrical contribution to the chemical energy to obtain the *electrochemical potential* of each species.

electrical potential energy:

$$z_i e \psi$$

where z_i is the number of elemental charges per unit ion (including sign) and e (>0) is the magnitude of the elemental charge (on proton); thus $z_i e$ is the charge on one ion. At equilibrium, ionic species distribute themselves in an electric field such that their electrochemical potential is constant everywhere:

$$\mu_i^e(\mathbf{x}) = \mu_i(\mathbf{x}) + z_i e \psi(\mathbf{x}) = \text{const} \quad (205)$$

For example, diffusion of nonelectrolytes (i.e. $z_i = 0$) takes place so that μ_i is constant throughout the solution, which means that the concentration c_i is uniform at equilibrium. If the electrostatic potential varies from one location to the next, the concentration of charged species will also not be uniform. (205) can be re-written in differential form as

$$d\mu_i^e = 0 = d\mu_i + z_i e d\psi$$

After substituting (204), we have $kT d \ln c_i + z_i e d\psi = 0$

which relates changes in ion concentration to changes in potential. This can be integrated to obtain **Boltzmann's Equation** relating the local ion concentration to the local electrostatic potential

$$c_i(\mathbf{x}) = A \exp \left[-\frac{z_i e \psi(\mathbf{x})}{kT} \right]$$

where A is some integration constant. In our problem, we expect the ion concentration to depend only on y and we expect that the ionic concentration will approach the bulk value $c_{i\infty}$ at some distance from the charged interface. For this to happen, the electrostatic potential must tend to some constant (say $\psi \rightarrow 0$ as $y \rightarrow \infty$) or $\mathbf{E} \rightarrow \mathbf{0}$:

$$\text{as } y \rightarrow \infty: \quad c_i \rightarrow c_{i\infty}, \psi \rightarrow 0$$

To satisfy these boundary conditions, Boltzmann's equation can be rewritten as

$$c_i(y) = c_{i\infty} \exp \left[-\frac{z_i e \psi(y)}{kT} \right] \quad (206)$$

So if I knew the $\psi(y)$, I could calculate the concentration profile. But alas $\psi(y)$ is still unknown. I need another equation. The extra equation is provided by **Coulomb's law** of electrostatics. If I knew the distribution of charges within a system, I could calculate the force on them using Coulomb's law. For a continuum, Coulomb's law can be written as:

$$\nabla \cdot \mathbf{E} = \frac{4\pi}{\epsilon} \rho_e \quad \text{or} \quad \nabla^2 \psi = -\frac{4\pi}{\epsilon} \rho_e \quad (207)$$

where ϵ is another material property called the **electric permittivity** of the medium between the ions (e.g. the water):

$$\epsilon_{\text{vacuum}} = 1.113 \times 10^{-10} \frac{\text{Coul}^2}{\text{N} \cdot \text{m}^2}$$

$$\epsilon_{\text{water}} = 78 \epsilon_{\text{vacuum}}$$

This equation is called **Poisson's Equation**. The charge density of the fluid arises from the charge born by each ion. Adding up the charges from each species i , then substituting Boltzman's equation, gives

$$\rho_e(y) = \sum_i z_i e c_i(y) = \sum_i z_i e c_{i\infty} \exp\left[-\frac{z_i e \psi(y)}{kT}\right]$$

Special Case: symmetric binary electrolyte, which means there are only two different species of ions and they have the same magnitude of charge:

$$z_+ = -z_- = z$$

$$c_{+\infty} = c_{-\infty} = c_\infty$$

Then Boltzmann's equation can be used to calculate the charge density:

$$\begin{aligned} \rho_e(y) &= z_+ e c_{+\infty} \exp\left[-\frac{z_+ e \psi}{kT}\right] + z_- e c_{-\infty} \exp\left[-\frac{z_- e \psi}{kT}\right] \\ &= z e c_\infty \left[\exp\left(-\frac{ze\psi}{kT}\right) - \exp\left(+\frac{ze\psi}{kT}\right) \right] = -2ze c_\infty \sinh\left(\frac{ze\psi}{kT}\right) \end{aligned} \quad (208)$$

Substituting this into Poisson's equation, we get:

$$\frac{d^2 \psi}{dy^2} = \frac{8\pi e c_\infty}{\epsilon} \sinh\left(\frac{ze\psi}{kT}\right)$$

The argument of the sinh must be dimensionless, so let's use this combination to define a new dimensionless potential

$$\Psi \equiv \frac{ze\psi}{kT}$$

We can make the ψ on the left-hand side dimensionless by multiplying both sides by ze/kT . Then we have

$$\frac{d^2 \Psi}{dy^2} = \kappa^2 \sinh \Psi \quad (209)$$

where

$$\kappa^2 = \frac{8\pi z^2 e^2 c_\infty}{\epsilon kT} [=] \frac{1}{\text{cm}^2}$$

Eq. (209) is a nonlinear ODE. Despite its innocuous appearance, it is much more difficult to solve than linear equations. However, a particular solution is possible:

subject to: $\Psi = \Psi_0$ at $y=0$

$\Psi = 0$ at $y=\infty$

$$\tanh\left[\frac{\Psi(y)}{4}\right] = \tanh\left(\frac{\Psi_0}{4}\right) e^{-\kappa y} \quad (210)$$

which is called the **Gouy-Chapman model** for the double layer. In the special case in which the surface potential is small, these expressions simplify to:

$$|\Psi_0| \ll kT/ze : \Psi(y) = \Psi_0 \exp(-\kappa y) \quad \text{and}$$

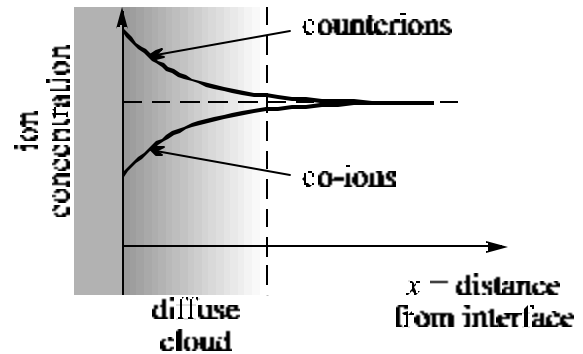
$$c_{\pm}(y) = c_\infty \left(1 \mp \frac{ze\Psi_0}{kT} e^{-\kappa y} \right)$$

Clearly κ is associated with the thickness of the diffuse cloud. κ^{-1} is called the **Debye length**:

$$\kappa^{-1} = 10^{-9} \text{m} = 1 \text{nm} \quad \text{for} \quad c_\infty = 10^{-1} \text{M}$$

$$\kappa^{-1} = 10^{-6} \text{m} = 1 \mu\text{m} \quad \text{for} \quad c_\infty = 10^{-7} \text{M}$$

This provides some idea of the thickness of the charge cloud.



ELECTROSTATIC BODY FORCES

Since the fluid inside the diffuse cloud is charged,

from (208):

$$\rho_e(y) = -2ze c_\infty \sinh\left(\frac{ze\Psi}{kT}\right) \approx -2ze c_\infty \underbrace{\frac{ze\Psi}{kT}}_{\approx \frac{ze\Psi_0}{kT}} e^{-\kappa y}$$

any electric field in the fluid will exert a force on the fluid elements. This body force needs to be included in the Navier-Stokes equations:

$$\text{electrostatic body force/volume} = \rho_e \mathbf{E}$$

which is analogous to the body force exerted by gravity:

$$\text{gravitational body force/volume} = \rho \mathbf{g}$$

Including electrostatics, but neglecting gravity, the Navier-Stokes equation becomes:

$$\rho D\mathbf{v}/Dt = \mu \nabla^2 \mathbf{v} - \nabla p + \rho_e \mathbf{E}$$

In the Gouy-Chapman model described in the previous section, the fluid is stagnant. At hydrostatic equilibrium, any electric field applied to a charged fluid element will lead to a pressure gradient:

$$\mathbf{v} = \mathbf{0}: \quad \nabla p = \rho_e \mathbf{E}$$

To see how large the pressure gradient is, let's return to the Gouy-Chapman model where the electric field is oriented normal to the charged surface: i.e. $\psi = \psi(y)$. Then the pressure gradient must also be normal to the wall so that $p = p(y)$. Hydrostatic equilibrium requires

$$\frac{dp}{dy} = \rho_e E_y = \left[-2ze c_\infty \sinh\left(\frac{ze\psi}{kT}\right) \right] \left(-\frac{d\psi}{dy} \right) \quad (211)$$

where the second equality results from substituting (122) and (203). The right-hand side turns out to be an exact differential, so that the equation can be re-written as

$$\frac{dp}{dy} = c_\infty kT \frac{d}{dy} \left(\cosh \frac{ze\psi}{kT} \right)$$

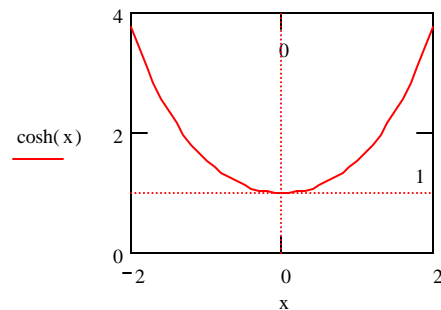
Multiplying both sides by dy and integrating from $y=y$ to $y=\infty$:

$$\begin{aligned} \int_{p_\infty}^{p(y)} dp &= c_\infty kT \int_{y=\infty}^{y=y} d \left(\cosh \frac{ze\psi}{kT} \right) = c_\infty kT \left[\cosh \frac{ze\psi(y)}{kT} - \cosh \frac{ze\psi(\infty)}{kT} \right] \\ p(y) - p_\infty &= c_\infty kT \left\{ \cosh \left[\frac{ze\psi(y)}{kT} \right] - 1 \right\} \end{aligned}$$

The last step involves substituting $\psi(\infty)=0$ and remembering that $\cosh(0)=1$. You might also recall that

$$\cosh x \equiv \frac{e^x + e^{-x}}{2} = \begin{cases} 1 & \text{for } x = 0 \\ > 1 & \text{for } x \neq 0 \end{cases}$$

Thus the pressure inside the cloud is always greater than the pressure in the bulk solution outside the cloud. This is because the diffuse cloud is always oppositely charged compared to the surface, so the cloud is also being pulled by electrostatic forces toward the surface — just like gravity always pulls us downward.



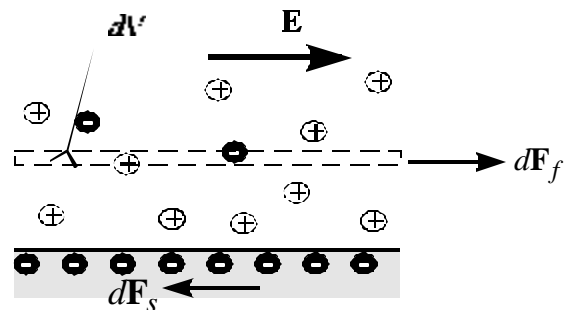
ELECTROKINETIC PHENOMENA

So far in our discussion, there has been no fluid motion. But since the fluid inside the diffuse cloud is charged, we can exert a force on it by applying an electric field tangent to the surface.

Consider again the distribution of ions near a charged interface. The solution is not electrically neutral. So if an electric field is applied tangent to the surface, the fluid will experience an electrostatic body force:

$$d\mathbf{F}_f = (dq)\mathbf{E} = (\rho_e dV)\mathbf{E}$$

where dq is the net charge on a fluid element having volume dV . The solid is oppositely charged, so it feels a force in the opposite direction. Consequently, at steady state, we can expect the fluid to move relative to the solid, or vice versa.



At least two phenomena are associated with the relative motion generated by this externally applied electrostatic force:

electrophoresis - migration of charged particles through an otherwise quiescent fluid

electro-osmosis - flow of fluid through a porous solid bearing a surface charge

In both cases, relative motion between the fluid and the solid arises when an electric field is applied externally. Conversely, forcing a fluid to move tangent to a charged interface generates an electric field:

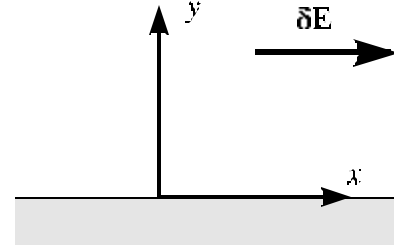
sedimentation potential - the electrical potential (gradient) which arises during sedimentation of charged particles in a gravitational field

streaming potential - the electrical potential (gradient) which arises during flow of fluid through a porous solid which bears a surface charge

These four phenomena depend on the magnitude of the surface charge. Measurements of these induced velocities or induced potentials are the most commonly used techniques for determination of surface charge.

SMOLUCHOWSKI'S ANALYSIS (CA. 1918)

Consider an infinite plate in contact with an electrolyte solution. Suppose the plate bears a uniform surface charge density and we somehow externally apply an electric field E_x tangent to the surface in the $+x$ direction. Let's try to find the velocity profile \mathbf{v} induced by this electric field.



continuity:

$$\nabla \cdot \mathbf{v} = 0$$

NSE:

$$\rho D\mathbf{v}/Dt = \mu \nabla^2 \mathbf{v} - \nabla p + \rho_e \mathbf{E} \quad (212)$$

From the arguments above, we expect the fluid to move in the direction of the applied electrostatic force. The simplest solution to this problem which has this property is:

$$v_x = v_x(y), \quad v_y = v_z = 0$$

Since v_x is independent of x , this solution automatically satisfies the mass balance. Of course, we must allow v_x to vary with y so we can meet the no-slip condition at the surface. Note that at steady-state, the inertial terms identically vanish (like gravity-driven flow down inclined plane).

In the absence of flow, the ion concentration profiles are independent of x [see (206)]. Any flow tangent to the plate is not expected to perturb the ion concentration profiles because tangential flow just replaces fluid having a particular concentration with fresh fluid having the same composition (also $\mathbf{v} \cdot \nabla c_i = 0$). This suggests that the charge density profile is not perturbed by flow caused either by an externally applied electric field (tangent to plate) or by an externally applied pressure gradient:

$$\rho = \rho^{eq}$$

where we will use the superscript "eq" to denote the equilibrium state of the variable (in the absence of flow); we will also use the prefix "δ" to denote perturbations from equilibrium (caused by the applied electric field). Thus the above statement is equivalent to

$$\rho \equiv \rho^{eq} + \delta\rho \quad \text{where} \quad \delta\rho = 0$$

If we used two large vertical plate electrodes to apply the electric field, as shown at right, we would expect the electric field lines to be straight and tangent to the surface* and the electrostatic potential to vary linearly with distance from one electrode to the other. Thus the *applied* electrostatic forces should act perpendicular to the electrodes and parallel to our surface. In other words, the *applied* electric field is expected to have only an x -component which independent of x and y .

This expectation is fulfilled if the electrostatic potential has the following form:

$$\psi(x, y) = \psi^{eq}(y) + \delta\psi(x)$$

$$\underbrace{\nabla\psi}_{-\mathbf{E}} = \underbrace{\frac{d\psi^{eq}}{dy}}_{-\mathbf{E}^{eq}} \mathbf{j} + \underbrace{\frac{d\delta\psi}{dx}}_{-\delta\mathbf{E}} \mathbf{i}$$

where ψ^{eq} is the equilibrium potential found in the absence of an externally applied electric field or an applied pressure gradient (see previous section on “Gouy-Chapman model”), and where $\delta\psi$ is the perturbation from equilibrium caused by the externally applied electric field or pressure gradient. The pressure field suffers a similar perturbation:

$$p(x, y) = p^{eq}(y) + \delta p$$

The y -component of (212) yields:

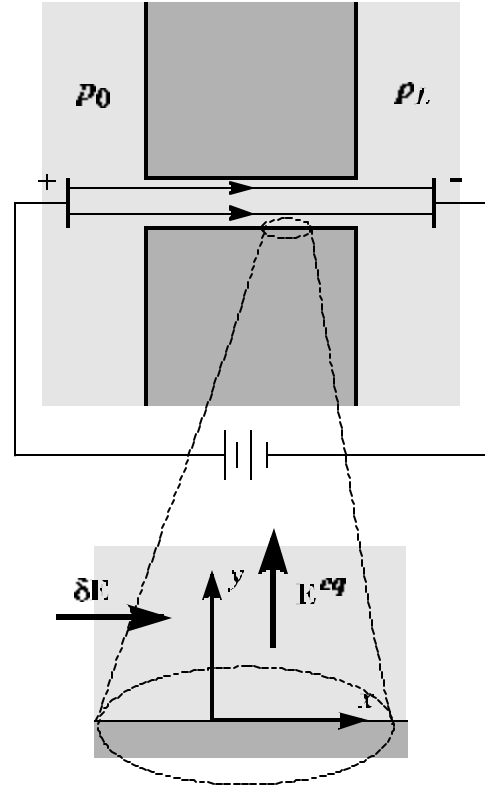
$$\frac{dp^{eq}}{dy} + \frac{\partial\delta p}{\partial y} = \rho^{eq} E_y^{eq}$$

but adding the superscript eq to terms in (211) yields:

$$\frac{dp^{eq}}{dy} = \rho^{eq} E_y^{eq}$$

subtracting:

$$\frac{\partial\delta p}{\partial y} = 0 \quad \text{or} \quad \delta p = \delta p(x)$$



* The field lines might not be straight near the entrance or exit of the capillary. Thus what follows strictly only applies in the central region away from the ends.

Poisson's equation (207) can also be written as:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{4\pi}{\epsilon} \rho \\ \nabla \cdot \mathbf{E}^{eq} + \nabla \cdot \delta \mathbf{E} &= \frac{4\pi}{\epsilon} \rho^{eq}\end{aligned}$$

but in the absence of flow, Poisson's equation requires

$$\nabla \cdot \mathbf{E}^{eq} = \frac{4\pi}{\epsilon} \rho^{eq}$$

subtracting leaves

$$\nabla \cdot \delta \mathbf{E} = 0$$

or

$$\frac{d\delta E_x}{dx} = 0 \quad \text{or} \quad \delta E_x = \text{const} \quad (213)$$

The x -component of (212) becomes

$$\underbrace{\mu \frac{d^2 v_x}{dy^2} + \rho^{eq} \delta E_x}_{g(y)} = \underbrace{\frac{d\delta p}{dx}}_{f(x)} = \text{const} = \frac{p_L - p_0}{L - 0} = 0 \quad (214)$$

By applying a pressure gradient along the surface, we could “pump” the fluid past the solid, but this is not the problem we want to address. Thus we set the pressure gradient to zero ($p_0 = p_L$) so there is no pressure-driven flow. Boundary conditions include

at $x=0$:

$$v_x = v_{x,s}$$

as $x \rightarrow \infty$:

$$dv_x/dy \rightarrow 0 \quad \text{or} \quad v_x \rightarrow v_{x,f}$$

where $v_{x,s}$ is the velocity of the solid and $v_{x,f}$ is the velocity of the bulk fluid outside the cloud. The first condition is just “no-slip” which says that at the interface between the fluid and the solid the two take on the same velocity. The second condition says there is no shear force being applied to the fluid outside the double layer.

To go further, we need to know how ρ^{eq} changes with y . This is provided by Poisson's equation (207):

$$\rho^{eq}(y) = -\frac{\epsilon}{4\pi} \frac{d^2 \psi^{eq}}{dy^2} \quad (215)$$

(215) into (214):

$$\mu \frac{d^2 v_x}{dy^2} = \frac{\epsilon}{4\pi} \frac{d^2 \psi^{eq}}{dy^2} \delta E_x$$

Since $\delta E_x = \text{const}$, this can easily be integrated to

$$\mu[v_x(y) - v_{x,s}] = \frac{\varepsilon}{4\pi} \underbrace{[\psi^{eq}(y) - \psi^{eq}(0)]}_{-\zeta} \delta E_x$$

Substituting $\psi^{eq}(y)$ from (210), we obtain a velocity profile like that shown at right.

Let $v_x(\infty) \equiv v_{x,f}$

Then the above equation can be rewritten as

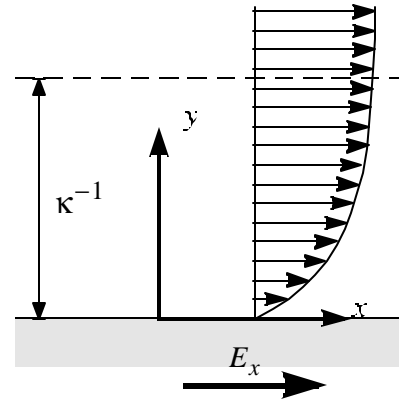
$$v_{x,s} - v_{x,f} = \frac{\varepsilon \zeta}{4\pi\mu} \delta E_x \quad (216)$$

where

$$\zeta \equiv \psi^{eq}(0) - \psi^{eq}(\infty)$$

is called the

zeta potential - the electrostatic potential drop across the diffuse part of the double layer.



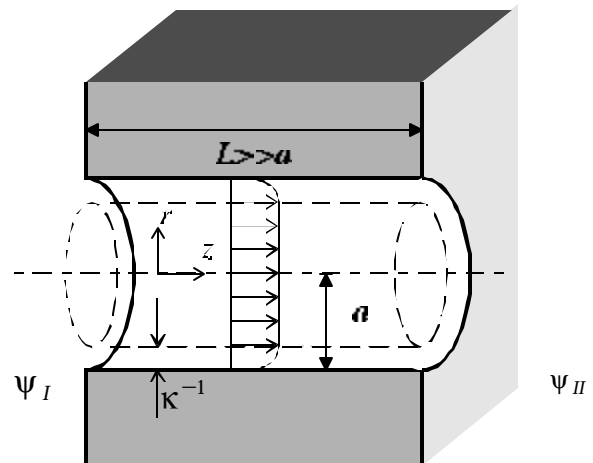
ELECTRO-OSMOSIS IN CYLINDRICAL PORES

Consider a circular hole drilled through a plate. If this is submerged in an electrolyte solution and becomes charged, then applying a voltage across the length of the pore will give rise to electro-osmosis through the hole.

If the pore radius a is much larger than the Debye length κ^{-1} , then the velocity profile is essentially plug flow except inside the double layer where the velocity suddenly drops to zero at the stationary solid surface

$$v_s = 0$$

$$v_{fluid}(r) = -\frac{\varepsilon \zeta}{4\pi\mu} \delta E_x$$



for all r outside the double layer. Here we have applied (216), but dropped the prefix 'δ' since we have no further need to distinguish between the equilibrium and the applied fields.

Example: A typical value of the ζ potential for aqueous solutions is about 50mV. Plugging in the values of the other parameters

$$\epsilon\zeta/4\pi\mu = 3.5 \text{ } \mu\text{m/s per V/cm}$$

as a typical value of v_f/E_x .

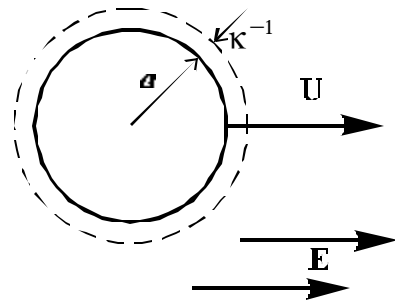
Consequently, the volumetric flowrate through the pore is just this velocity times the cross-sectional area:

$$\kappa a \gg 1 : \quad Q = \pi a^2 (\epsilon\zeta/4\pi\mu) E_x$$

So a measurement of the flowrate for a known applied electric field allows us to determine the zeta potential.

ELECTROPHORESIS

Consider a rigid sphere immersed in a quiescent electrolyte solution. Somehow we apply an electric field of strength \mathbf{E} . In this case the solid moves while the fluid far from the sphere remains stationary:



$$v_{x,f} = 0$$

(216) predicts that $v_{x,s}$ is

$$\kappa a \gg 1 : \quad \mathbf{U} = (\epsilon\zeta/4\pi\mu)\mathbf{E}$$

where $\epsilon\zeta/4\pi\mu$ is now called the *electrophoretic mobility*. This result is called **Smoluchowski's Equation**.

This applies for particles which are large compared to the Debye length. In the opposite limit of very small small particles, another simple expression can be obtained:

$$\kappa a \ll 1 : \quad \mathbf{U} = (3/2)(\epsilon\zeta/4\pi\mu)\mathbf{E}$$

which is called **Huckel's equation**. For the most general case of intermediate size, the problem is much more complicated. A numerical solution is required (see Fig. 9.11.3, p560 of Hunter).

STREAMING POTENTIAL

So far in our discussion of electrokinetic phenomena, we have considered the motion of either fluid or solid that results from the application of an electric field:

Electrophoresis, Electro-osmosis: $\mathbf{E} \neq \mathbf{0} \rightarrow \mathbf{U} \neq \mathbf{0}$

Now we would like to consider the inverse situation in which relative motion across a charged interface generates an electric field:

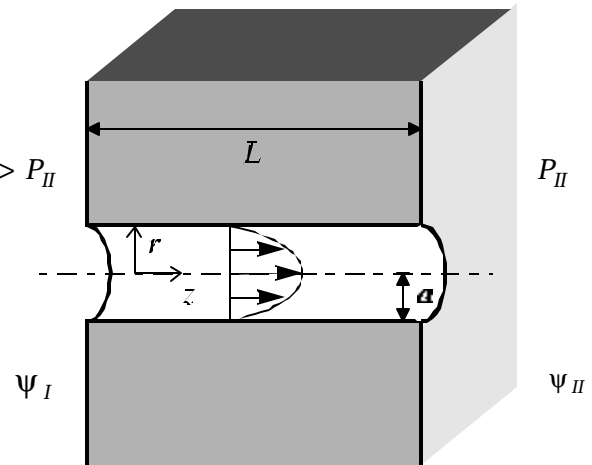
Streaming Potential,
Sedimentation Potential: $\mathbf{U} \neq \mathbf{0} \rightarrow \mathbf{E} \neq \mathbf{0}$

Consider the pressure-driven flow through a large circular pore.

$$\kappa^{-1} \ll a \ll L$$

Owing to the applied pressure drop, we generate a fully developed parabolic velocity profile so familiar for laminar flow.

$$v_z(r) = 2 \underbrace{\frac{Q}{\pi a^2}}_{\langle v_z \rangle} \left[1 - \left(\frac{r}{a} \right)^2 \right]$$



The main idea is that this flow causes convection of charge in the counterion cloud which tends to generate a current. However in the absence of electrodes in the two reservoirs, there is no way to form a complete electric circuit. So charge tends to pile up in the reservoir — positive charges on one side of the membrane and negative charges on the other. Coulomb's law exerts a force on these charges which tends to restore electroneutrality to the system and to prevent any steady-state *rate of accumulation* of charge.

Clearly any steady-state electrical current would eventually lead to infinite charge separation and infinite attractive forces. The only way to avoid this is to have **zero current** at steady state. This is achieved by an electric field which spontaneously arises inside the bulk of the fluid which creates an electrical current equal but opposite to the convective current:

$$I = I_{conv} + I_{elec} = 0 \quad (217)$$

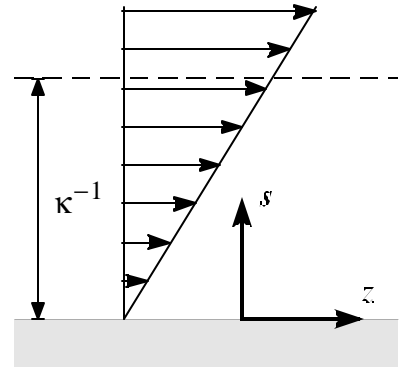
The electrostatic potential drop associated with this induced electric field is called the **streaming potential**. To calculate it, let's first calculate the current generated by convection of charges inside the cloud.

If the charge cloud is very thin compared to the pore radius, then we need to concentrate on the velocity profile right next to the wall. Here the velocity profile becomes linear:

$$\text{for } s \ll a: \quad v_z(s) = \Gamma s$$

$$\text{where} \quad s \equiv a - r$$

$$\text{and} \quad \Gamma = -dv_z/dr \big|_{r=a}$$



The net current through any surface S (open or closed) is just the integral of the normal component of the current density over the surface:

$$I = \int_S \mathbf{n} \cdot \mathbf{i} da \quad [=] \text{ amps}$$

In our case, we choose a disk of radius a whose normal is parallel to the tube axis.

$$I_{conv} = \int_0^a \rho v_z 2\pi r dr \approx 2\pi a \int_0^\infty \rho \Gamma s ds \quad (218)$$

Since $\rho=0$ for most of the fluid (except for $r \approx a$), we can approximate this integral as shown by the second equality above. Next, we substitute Poisson's equation (215) for the space charge density:

$$\rho = -(\epsilon/4\pi) d^2\psi^{eq}/ds^2 \quad (219)$$

(219) into (218) and integrating by parts:

$$I_{conv} = -\frac{\epsilon a \Gamma}{2} \int_0^\infty \frac{d^2\psi^{eq}}{ds^2} s ds = -\frac{\epsilon a \Gamma}{2} [\psi^{eq}(z,0) - \psi^{eq}(z,\infty)]$$

The potential drop across the counterion cloud is the ***zeta potential***. Thus

$$I_{conv} = -(1/8\pi)\epsilon a \Gamma \zeta \quad (220)$$

According to (217), this convective contribution to the current must be balanced an electrical contribution from an induced electric field. The relationship between current and electric field is given by ***Ohm's law*** for electrolyte solutions, which takes the form of

$$\mathbf{i}_{elec} = K\mathbf{E}$$

where K is called the ***specific conductance*** of the solution. Integrating over the cross-section of the tube:

$$I_{elec} = \int_0^a KE_z 2\pi r dr \approx \pi a^2 KE_z \quad (221)$$

where

$$E_z = -\partial\psi/\partial z = \text{const} = -\Delta\psi/L$$

is required by (213), where L is the length of the cylindrical pore. (220) and (221) into (217):

$$\Delta\psi = \varepsilon\zeta L\Gamma/2\pi aK \quad (222)$$

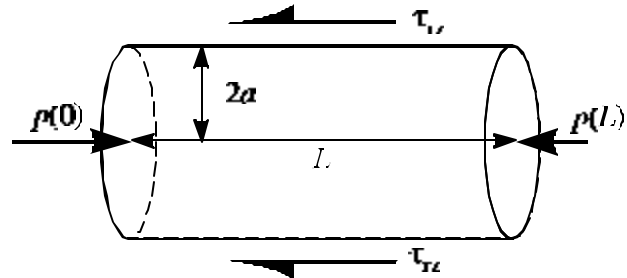
Finally, we can relate Γ to the applied pressure drop by performing a macroscopic force balance on the tube. Recognizing that the shear stress on the tube wall τ_{rz} is $\mu dv_z/ds|_{r=a} = \mu\Gamma$,

$$\mu\Gamma(2\pi aL) = \Delta p(\pi a^2)$$

or
$$L\Gamma = \frac{a}{2\mu} \Delta p \quad (223)$$

(223) into (222):
$$\Delta\psi = \frac{\varepsilon\zeta}{4\pi\mu K} \Delta p$$

which is *Smoluchowski's equation* for streaming potential.



Surface Tension

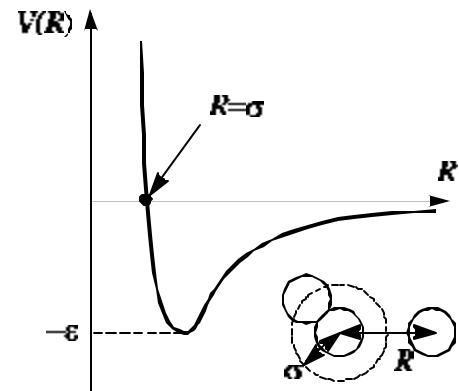
MOLECULAR ORIGIN

Many of the fascinating effects we saw in Trefethen's film are a consequence of surface tension. What is surface tension and where does it come from?

Surface tension is a consequence of intermolecular forces which are always present. The interaction between two molecules is often represented by the *Lennard-Jones 6-12* formula:

$$V(R) = 4E \left[\underbrace{\left(\frac{\sigma}{R} \right)^{12}}_{\text{Born}} - \underbrace{\left(\frac{\sigma}{R} \right)^6}_{\text{vdW}} \right]$$

This expression is partly based on theory and partly empirical. The inverse 6th power term represents *van der Waals*



attraction which is based on a rigorous theory. The inverse 12th power term is empirical. It represents **Born repulsion** which results from the overlap of electron clouds. σ is called the **collision diameter** — this is the center-to-center distance at which the electron clouds begin to overlap and a significant repulsion is experienced. As a first approximation, we can think of condensation of a liquid from a vapor in terms of the molecules falling into this potential energy well:

vapor: $R \rightarrow \infty \quad V_{\text{vapor}} = 0$

liquid: $R \approx R_{\text{min}} = 1.12\sigma \quad V_{\text{liq}} = -E$

The **heat of vaporization** is approximately the energy gained by falling in this well:

$$U_{\text{vap}} \approx EN_A$$

This is all only approximate because we have only considered two-body interactions. In a condensed state, multi-body interactions must also be considered. The existence of surface tension can be expected from the difference in energy between molecules located in the bulk of a liquid and molecules located at surface between the liquid and its vapor.

Let's pick a molecule from the bulk of the liquid phase. On the average, it will be some average distance R_{min} from its nearest neighbor. If there are z_{bulk} nearest neighbors, the energy of this typical molecule from the bulk liquid is

$$e_{\text{bulk}} = (z_{\text{bulk}}/2)E$$

where the factor of 2 comes from assigning half of the energy of interaction between a pair of atoms to each atom. For a hexagonal close-packed array:

$$z_{\text{bulk}} = 12$$

Now consider a molecule which resides at the interface between the liquid and the vapor. The same basic formula can be written for the energy of this surface molecule:

$$e_{\text{surf}} = (z_{\text{surf}}/2)E$$

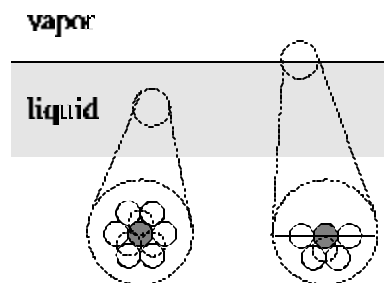
Except that the number of nearest neighbors will be significantly less. Again for a hexagonal close-packed array:

$$z_{\text{surf}} = 6$$

then

$$e_{\text{surf}} \approx (1/2)e_{\text{bulk}}$$

Now $e < 0$ for attractive interactions, so:



$$e_{bulk} < e_{surf} < 0$$

Thus molecules located at the surface do not have as low an energy as molecules in the bulk. So when we calculate the free energy of the liquid, we should treat the surface molecules differently from the bulk molecules. Indeed, it turns out that this contribution to the free energy of the system is proportional to the number of surface molecules, which in turn is proportional to the surface area. The free energy per unit area is called the *surface tension*:

$$\gamma = \left(\frac{\partial G}{\partial A} \right)_{T,P}$$

BOUNDARY CONDITIONS FOR FLUID FLOW

Surface tension causes a number of effects including:

- makes homogeneous nucleation of bubbles virtually impossible: boiling is usually from heterogeneous nucleation
- increases the vapor pressure of small drops for same T: Kelvin effect
- capillary rise
- determines the shape of drops and bubbles

These are all hydrostatic effects; there is no fluid flow occurring. Since surface tension depends on temperature, temperature gradients can gradients in surface tension, which in turn can generate flow. Since the origin of surface tension is intermolecular forces, which are very short range (range is a few Angstroms), surface tension does not change the form of the basic equations of motion. Instead, surface tension is manifested in the boundary conditions applied to those equations.

Recall that when surface tension effects are negligible, the general boundary condition on stress was given by

$$\gamma \approx 0: \quad \mathbf{n} \cdot \underline{\underline{T}}_I = \mathbf{n} \cdot \underline{\underline{T}}_{II}$$

Surface tension makes both the normal and tangential components of stress continuous across an interface. The more general boundary condition is

$$\mathbf{n} \cdot (\underline{\mathbf{T}}^I - \underline{\mathbf{T}}^II) = \mathbf{n}(\nabla_s \cdot \mathbf{n})\gamma - \nabla_s \gamma \quad (224)$$

where ∇_s is called the *surface divergence*; it represents two of the three terms in the usual divergence. By convention, \mathbf{n} is pointing from phase I into phase II. For example, suppose we define a local rectangular Cartesian coordinate system such that the z -axis is normal to the interface:

$$\mathbf{n} = \mathbf{e}_z$$

The usual divergence is given by

$$\nabla \cdot \mathbf{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z$$

The surface divergence does not include the term which represents the derivative along the normal

$$\nabla_s \cdot \mathbf{n} = \partial_x n_x + \partial_y n_y \equiv -2H [=] \text{ m}^{-1}$$

This particular surface divergence represents the *curvature* of the interface, often denoted by H or J .

Example: calculate the curvature of a sphere of radius r .

Solution: the unit normal to a sphere in spherical coordinates is

$$\mathbf{n} = \mathbf{e}_r; \quad n_r = 1, n_\theta = n_\phi = 0$$

In this particular case, the normal is independent of the normal coordinate r so that the surface divergence is the same as the regular divergence:

$$\nabla_s \cdot \mathbf{n} = r^{-2} \partial_r (r^2 \cdot 1) = 2/r$$

Thus the curvature is inversely proportional to the radius of the sphere.

Similarly the *surface gradient* represents that part of the usual gradient which lies in the surface

$$\nabla_s \gamma = \partial_x \gamma \mathbf{e}_x + \partial_y \gamma \mathbf{e}_y \quad (225)$$

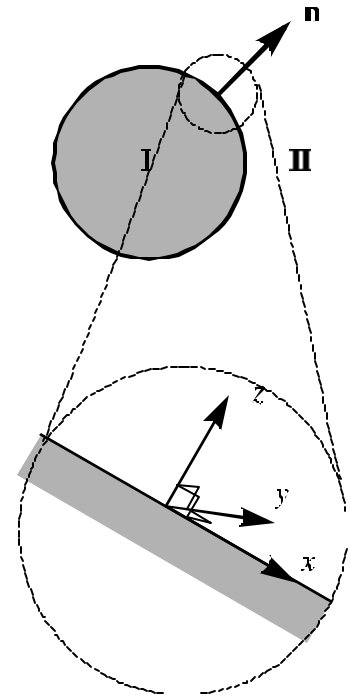
Suppose I dot both sides of (224) by the normal

$$T_{zz}^I - T_{zz}^{II} = -2H\gamma$$

Thus the normal component of the stress is discontinuous at a curved surface. In the particular case of hydrostatic equilibrium, there is no deviatoric contribution to the stress — just the hydrostatic pressure

hydrostatic equil:

$$p^I - p^{II} = -2H\gamma = 2\gamma/r$$



Index

2		double-dot product	2
2-D flow	58	drag coefficient	110
B		Dyadic Product	2
basis	3	Dyadic Product	1
Big "Oh"	121	dynamic pressure	86
body forces	38	E	
Born repulsion	211	eddies	189
boundary layer	118	eddy	189
boundary-layer separation	154	electric permittivity	199
C		electrochemical potential	198
Cartesian (implied) summation		electro-osmosis	203
convention	4	electrophoresis	203
cavitation	168	electrophoretic mobility	208
closed	28	electrostatic potential	198
collision diameter	212	equation of continuity	27
conservation of circulation	44	equation of state:	71
conservative	32, 198	Euler's Equation	41, 70
constitutive equation	75, 188	F	
Constitutive Equation	37	fluctuation	184
Continuity of stress	89	flux	73
continuously differentiable	4	form drag	108
Continuum Hypothesis	37	friction factor	195
Coulomb's law	199	friction velocity	193
creeping flow	102	G	
Cross Product	1	Gouy-Chapman model	201
Curl	16, 17	gradient	5
curvilinear	10	Gradient	16, 18
D		gravitational potential	47
d'Alembert's paradox	55	H	
Debye length	201	heat of vaporization	212
derivable from a potential	31	hot-wire anemometer	183
deviatoric stress	71	Huckel's equation	208
Divergence	16	hydrostatic pressure	39
Divergence	17	I	
Dot Product	1	ideal fluid	40, 117
double layer	197		

Idem Factor	33	path independence	32
Identity Tensor	33	Path Line	61
inner-outer product	2	persistence of irrotationality	46
instability	182	Poisson's Equation	200
Invariant	25	Poisson's ratio	76
irrotational	32	Poiseuille's Formula	88
isotropic	66	position vector	4
K		potential flow	117
Kelvin's Theorem	44	Prandtl's (universal) law of friction	196
kinematic viscosity	94	Prandtl's Boundary-Layer Equations	142
Kronecker delta	3	pressure	66
L		Primitive Matching Principle	126
Lamb	116	Proudman & Pearson	114
laminar flow	181	purely elastic solids	76
laminar sublayer	188, 194	R	
Laplace's Equation	47	rate of deformation	79
Lennard-Jones	211	rate of strain	80, 81
linear shear flow	98	regular perturbation expansion	122
linearly independent	3	Reynolds Experiment	181
M		Reynolds lubrication equation	175
matched-asymptotic expansions	114	Reynolds number	103, 140, 182
material derivative	15	Reynold's stress	186
Material point	13	Reynolds Transport Theorem	28
mean-free path	189	S	
mixing length	189	Scalar	1
mixing time	189	scalar components	3
modulus of elasticity for pure strain	77	scalar field	4
momentum flux	72	scalar potential	31
N		scaling	138
Navier-Stokes Equation	82	second coefficient of viscosity	81
Newton's law of viscosity	81	secondary flow	101
Newton's Law of Viscosity	80	second-rank tensor	3
no slip condition	83	sedimentation potential	203
O		separation	156
Ohm's law	210	shear strain	77
origin of turbulence	182	singular perturbation expansion	122
orthonormal	3	singularly different	119
Oseen	114	skin friction	109
P		Smoluchowski's equation	211
partial derivative	14	Smoluchowski's Equation	208
		solenoidal	57
		specific conductance	210
		state of stress	70
		Stokes	111

Stokes Equation	103	turbulent viscosity	191
Stokes Law	109	U	
Stokes Paradox	116	uniform flow	49
Stokes Theorem	29	unit dyads	3
strain	76	unit n-ads	3
Streak Line	61	Universal Velocity Profile	194
stream function	58	unsteady simple shear flow	72
Stream Function	57	V	
streaming potential	204, 209	van der Waals attraction	211
Streamline	61	Vector	1
stress	66	vector potential	57
stress tensor	67	velocity circulation	44
stress vector.	67	velocity potential	46
stretch transformation	123	viscoelastic	38
surface forces	39	viscous heating	66
surface tension	84, 213	viscous stress	71
symmetric	77	W	
Symmetric Tensor	33	Whitehead's Paradox	114
T		Y	
Tensor	1	Young's Modulus	76
thermodynamic pressure	71	Z	
time-averaged	184	zero current	209
total differential	4	zeta potential	207, 210
trace	71		
transition zone	188		
transpose	33		
turbulent core	188, 194		
turbulent stress	186		