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REPORT ON RECENT RESEARCH ON NONLINEAR OSCILLATIONS[1]

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Introduction

Since Dr. Balthazar Van der Pol, in his capacity as the President of the Physics Section of the International Congress of the ISRU, has invited us to provide a summary of recent research on nonlinear oscillations performed at the Physics Institute of Moscow University, at the Central Leningrad Radio Laboratory, at the Electrophysics Institute of Leningrad and at Gorki University, and since a large part of our work in this important field is closely related to the basic research and experiments of Dr. Van der Pol, it gives us great pleasure to submit this report. We will only be able to discuss a limited portion of this subject. So, we will be reporting on some general concepts that guided us during our research, and also on some of our more interesting experimental results as well.

Until rather recently the predominant theory of oscillations dealt with so-called linear systems (small perturbations of mechanical systems having a finite number of degrees of freedom, electronic circuits, and classical problems with boundary conditions). At the present time more interest is being focused on nonlinear systems in the various fields of pure and applied science (mechanics, acoustics, biology[2] and above all, since the invention of the electron tube, radio engineering).

The systems presently being used in radio engineering for transmitting and receiving are essentially nonlinear, and this is by no means an accidental condition. It is sufficient to merely examine – let us take the simplest example – a triode oscillator, in order to see that an autonomous linear system, i.e. a device in which the current and voltage are governed by linear differential equations in which time does not explicitly enter,[3] cannot have the properties that are possessed by, or need to be possessed by, a radio transmitter. (We shall confine ourselves to autonomous systems in order to exclude those whose oscillations are transmitted from outside the system. Naturally, when a system receives oscillations from an external source the question immediately arises as to how

[1] This report was delivered to the Radiophysics Section of the General Assembly of the International Scientific Radio Union (ISRU), London 12-18 September 1934.
[2] B. van der Pol, J. van der Mark; Philosophical Magazine, 1928: "The heartbeat considered as a relaxation oscillation and an electrical model of the heart".
[3] We define as autonomous any system whose differential equations do not contain any explicit reference to time.

oscillations from this source are produced. In order to discuss the problem of the emission of oscillations, systems functioning without external electromotive forces should be observed). Indeed the basic property of linear systems is that *the amplitude is not intrinsic to the system*, but *depends entirely on the initial conditions*. Now the distinguishing characteristic of modern oscillating devices is that, independent of the initial conditions, an oscillatory operating mode is established *whose amplitude and frequency are completely defined*. This is why modern radio engineering has had to call upon physical concepts and a mathematical approach that are able to cope with nonlinear systems. The great diversity of phenomena that are revealed in nonlinear systems makes them extremely interesting from a purely physical standpoint. It is their diversity as well as their flexibility which has enabled the broad applications that these systems have received over the course of the last few years.

Since the study of nonlinear differential equations is much more difficult and complicated than that of linear equations, the tendency from the beginning has naturally been to "linearize" the problems, i.e. to treat the essentially nonlinear problems from a linear viewpoint. It cannot be denied that, in order to clear up some aspects of the known phenomena, such a method (linearization) can sometimes have its utility. However, since it is still incomplete, artificial, and requires complementary ad-hoc hypotheses, this method of linearization often leads to errors. One of these errors, which is still encountered quite often, is referred to below[4].

After nonlinear systems had completely dominated the field of practical applications, they began to reveal phenomena absolutely foreign to linear systems, and inspired the search for a mathematical approach that could cope with these new phenomena. Very soon, publications appeared which purposely began from a nonlinear viewpoint. These are, primarily, the remarkable works of Van der Pol to which we will have occasion to refer several times. As has been stated, the results that they contain are of fundamental importance to the entire field in which we are involved. However, and this was reasonable, the first works had the production of tangible results as their goal rather than to develop a general and rigorous theory. Thus, for example, the existence of periodic solutions was accepted as an assumption. Series were often used whose convergence was questionable. Nevertheless, and let us emphasize this, the results obtained were good.

After using these methods (which originally owed their existence to the works of Van der Pol) for some time, a great quantity of valuable results were produced, and it was natural to pass on to more general points of view and look for a mathematical approach that could cope with nonlinear problems. It is in this direction that a portion of our work has been executed.

[4] See page 87.

It turned out that the mathematical approach that could deal with nonlinear oscillation problems had been in existence for a long time. On the one hand, it was contained in the famous works of Henri Poincaré [1], [2], and on the other hand, in the remarkable investigations of Liapunov [3].

The relationship existing between the works of Poincaré carried forward by Birkhoff [4], as well as those of Liapunov and our present physical problems was indicated out by one of us [5]. Three things should be pointed out. First of all, the qualitative theory of differential equations, developed earlier by Poincaré, has turned out to be very effective for the qualitative discussion of physical phenomena taking place in systems presently used in radio engineering. Nevertheless, neither the physicist nor, with greater reason, the engineer can be happy with a qualitative analysis. The later works of Poincaré supplied an approach that allowed the treatment of our problems on a quantitative basis. Finally, the works of Liapunov permit applying questions of stability to the mathematical discussion.

The first part of this report summarizes these mathematical theories and shows how we can apply them to our present problems. We shall virtually disregard questions of stability. The second part of this report discusses the theoretical and experimental aspects of some concrete questions. These partially involve problems for which we have perfected and supplied results, previously obtained by other authors, with a strict mathematical foundation. In addition, we shall examine resonance phenomena of the n-th degree and shall concisely report certain experiments and theoretical considerations on the so-called parametric excitation phenomena.

We will conclude by making several observations on the role of statistics in oscillatory phenomena.

Section 1: Geometric Presentation of the Motion of an Oscillatory System; The Phase Plane

There is no question but that the mathematical methods concerned here are noticeably more complicated and difficult than those used to study linear systems. This arises from the very nature of the physical problems, which are far from being simple. Also, there is no doubt that the characteristic features of this approach will prove capable of dealing with nonlinear systems on a theoretical as well as a practical basis. It is our position that a mathematical approach can only be charged with being cumbersome and overcomplicated when it leads to a result after a long succession of operations in which each operation, taken separately, has no physical interpretation. Now, this is not at all the case in the geometric approach connected with the name of Poincaré. Here, each geometrical component possesses a direct physical meaning. This is why this geometrical approach, though complicated, is far from being an obstacle. It actually simplifies the description and understanding of the physical phenomena involved.

This well-founded method, which consists of showing the operation of an oscillating system by using a geometric figure, has been used in science for quite some time. The idea is, essentially, as follows. In order to characterize the state of a system with N degrees of freedom it is necessary to provide 2N numbers (N coordinates and N velocities). These 2N numbers can be considered as specifying the position of a point in space with 2N dimensions. To each point of this space there corresponds a certain state (a certain “phase”) of the system. This is why this space is called “extension in phase”. In the case of a system with one degree of freedom, this space has two dimensions. In the simplest case it is a plane.

Let us take the simplest example of a harmonic oscillator. Its equation has the form:

$$\ddot{x} + \omega_o^2 x = 0 \tag{1}$$

or

$$\left.\begin{aligned} \dot{y} &= -\omega_o^2 x \\ \dot{x} &= y \end{aligned}\right\} \tag{2}$$

These equations, likewise, describe an electrical circuit that has capacitance and self-induction (but no resistance) if, for example, x represents the charge on the condenser. We will describe the behavior of the oscillator on a plane related to the rectangular axes $x, \dot{x}$ (the voltage-current plane). *This will be the phase plane*. Each new state of the system corresponds to a new “representative” point on the phase plane. A succession of states of the system corresponds to a movement of the representative point on the phase plane, or *phase trajectory*.

Planck has familiarized physicists with the phase trajectories of the harmonic oscillator. They form a family of ellipses, each enclosing the other and having a common center as the origin. The equation $\dot{x} = y$ indicates that the representative point is moving in a clockwise direction. The origin can be considered as an ellipse that has degenerated into a point.

In the case of equations (2), the point x = 0, y = 0 is a “singular point”, since at this point dy/dx = 0/0 and the direction of the phase trajectories is indeterminate. According to the description acceptable in mathematics, a singular point surrounded by a family of ellipses may be termed *the center*. It is clear that in the instance of our subject, singular points are of special interest. In order for dy/dx = 0/0, i.e. a singular point, it is sufficient that dy/dt = 0 and dx/dt = 0. However, if the current and voltage in a simple circuit are simultaneously equal to zero, the system is in equilibrium. Therefore, all the states of equilibrium of the system under study are represented by the singular points of the differential equations (2).

Each ellipse and each closed trajectory represents a periodic phenomenon corresponding to certain suitably selected initial conditions. The origin represents a stable state of equilibrium in the sense that any small disturbance remains small.

In the case under consideration, it is clear that, regardless of the initial conditions, the system describes a periodic motion, except in the situation where the initial conditions correspond to the coordinate origin.

In the same manner, we present the phase plane for a damped oscillator that is governed by the equation:

$$\ddot{x} + 2h\dot{x} + \omega_o^2 x = 0 \qquad (3)$$

or

$$\left.\begin{aligned} \dot{y} &= 2hy - \omega_o^2 x \\ \dot{x} &= y \end{aligned}\right\} \qquad (4)$$

We shall assume, first of all, that $h^2 < (\omega_0)^2$. Since x is the voltage at the condenser terminals, these equations describe an oscillating circuit that has resistance. All the integral curves are spirals that are

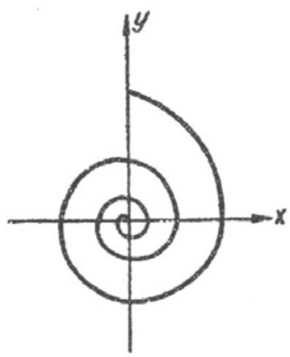

Fig. 1.

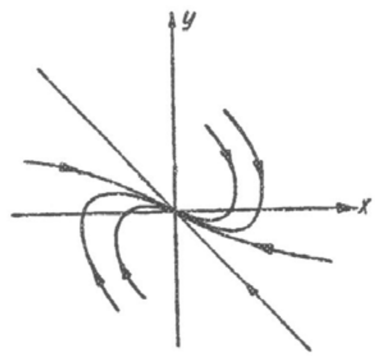

Fig. 2.

wound asymptotically with the origin of the coordinates (Figure 1). Each one of the spirals represents a damped oscillation corresponding to suitable initial conditions. Like the preceding example, the origin represents a state of equilibrium and is found to be a singular point, although of a new type. The singular points that are used as asymptotic points with a family of spirals are called *the focus*. In our case, the singular point is stable and is therefore called a *stable focus*.

Giving consideration to the phase plane of a system governed by equations (3) or (4), but whose damping is substantial enough for $h^2 > (\omega_0)^2$ to be true, we shall see parabolic curves passing through the origin substituted for the spirals (Figure 2). As in the proceeding example, the origin represents a state of equilibrium. It is a singular point of the type called *a node*. Integral curves reflect the aperiodic movement of the system towards the state of equilibrium. Therefore, we have a *stable node*.

In addition to the center, the focus, and the node, there also exists an important type of singular point for us: this is *the saddle* (Figure 3). The saddle represents, for example, the upper equilibrium position of a pendulum.

The behavior of integral curves in the vicinity of the saddle shows that the system always ends by diverging. A saddle is, therefore, always unstable.

In the case of a triode oscillator whose oscillating circuit is inserted between the filament and the grid, the voltage v at the condenser terminals satisfies the equation:

$$L\ddot{v} + R\dot{v} + \frac{1}{C}v = \frac{M}{C}f\left(v, \dot{v}\right) \tag{4}$$

or

$$\begin{aligned} \frac{dv}{dt} &= \frac{i}{C} \\ \frac{di}{dt} &= -\omega_o^2 v + f(v, i) \end{aligned}$$

The form of the function f (v, i) is provided by the plate characteristic of the tube (for simplicity's sake, we disregard the grid current). The mathematical discussion of this

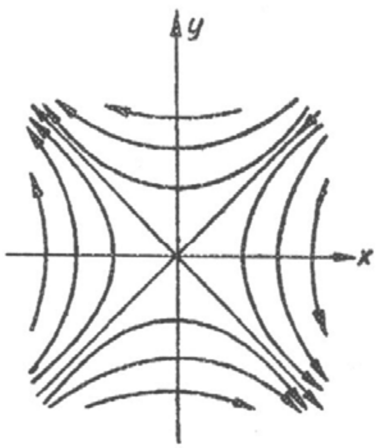

Fig. 3.

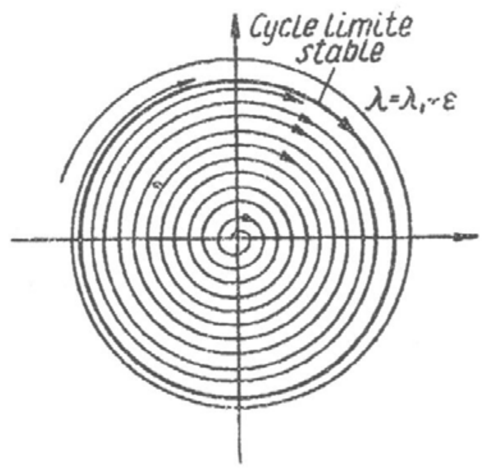

Fig. 4.

equation on the phase plane gives us the following picture. In the case of a soft mode and a weak excitation (M small), the integral curves are spirals which uncoil in a closed curve. Some approach one another from the outside, coming from infinity. Others approach one another from the inside, unwinding beginning from the point of origin (Figure 4, 16). It is easy to establish the relationship between the essential lines of the geometrical figure and those of the physical system. The origin of the coordinates still represents an equilibrium state; it is an *unstable focus*. In the circuit we will observe the appearance of oscillations whose amplitude will gradually increase, even if the values of voltage and current differ very little from zero (as will be the case, for example, if the initial perturbation i_0,v_0 is produced by random fluctuations). After some time, the increase will slow down and then stop, and we shall see the setting up of a stationary oscillatory mode of operation, which may be depicted on the phase plane by a closed curve. If the initial conditions correspond to a point located outside of the closed curve, the circuit will oscillate with decreasing amplitude until a stationary mode of operation is established.

The closed curves, on which the integral curves are wound, or from where they are unwound, are *the limiting cycles* of Poincaré. This mathematical concept has a very simple physical interpretation: the limiting cycles depict periodic stationary modes of operation. In the same way as with singular points, the limiting cycles can be stable (if all of the neighboring integral curves come closer) or unstable (if all of the neighboring integral curves diverge). It is clear that only the stable limiting cycles represent the actual periodic motion of a physical system.

If a strong excitation M is given in the same case as the previous “soft” mode of operation, there will still exist a limiting cycle with which the other integral curves will be wound from the inside and from the outside since the property was suitably selected. Nevertheless, in the vicinity of an equilibrium state the behavior of the system will be essentially different. The integral curves will diverge from the singular point in an aperiodic rather than an oscillatory manner, and the singular point will be an unstable node. The presence of the cycle shows that, no matter what the initial conditions may be, there will definitely be a well-defined periodic “amplitude” phenomenon. This periodic phenomenon will, again, be independent of the initial conditions. However, the transitory phenomenon assumes a different characteristic than in the case of a small excitation. It should be sufficient to study the beginning of the transitory phenomenon of a linear idealization in the case where the initial perturbation is assumed to be small. Nevertheless, in the case of an unstable node, the absence of oscillatory phenomena in a linear system does not permit one to conclude anything concerning the absence of periodic oscillatory phenomena out at a distance from the state of equilibrium, i.e. - out at some point where the system cannot be considered as linear.

The case of the node enclosed in a limiting stable cycle is the most striking example of the lack of capability of linear methods to decide on the existence of periodic phenomena in a self-exciting system. Furthermore, if this circumstance is disregarded, it is possible to commit a serious error, as has been done by several authors……

The singular points and the limiting cycles constitute the geometric components characterizing, to a certain degree, stationary movements in the systems. According to Poincaré, the knowledge of these components is enough to judge the properties of all other movements. The coexistence of these components is, likewise, controlled by general topological laws. Also, if the properties of one of these components is known, it is often possible to deduce the existence of the others. If, for example, far from the point of origin, all the integral curves converge towards the origin, with the latter being an unstable focus or node, and, provided that there are no other singular points, there exists at least one stable limiting cycle. If there exist several limiting cycles enclosing one another and between which there are no singular points, then there is an alternation of stable and unstable cycles. From the point of view of “Analysis Situs” (topology), these statements are almost syllogisms. Nevertheless, the physical phenomena corresponding to these geometric properties are far from being trivial. The qualitative theory of Poincaré is so valuable because it permits the formation of an overview of the physical phenomenon by a relatively simple analysis.

The following is a very simple example. Although without practical significance, it clearly illustrates what has been said. Let us assume that the characteristic of the tube has the shape that can be seen in Figure 5. Under what conditions will the oscillator have stationary (periodic) oscillations? In order to know the behavior of the system at infinity, it is clearly possible to assume that the operational point is at the apex of the angle formed by the two rectilinear parts of the characteristic. Depending on the slope of the inclined part, two cases can be seen: either the cycle at infinity will be stable, i.e., all the integral curves will go towards infinity (if the slope exceeds a certain critical value), or else the cycle at infinity will be unstable (if the slope is less than this critical value). It can easily be seen that it is only possible to have one finite limited cycle for finite values of the slope. If the operational point is located in the horizontal part of the characteristic, then the origin is a stable point of equilibrium and, consequently, owing to general topological laws, there can be no limiting cycle. Therefore, it is not possible to have oscillations. If the operational point is located on the inclined portion of the characteristic, three cases can occur. If the singular point is stable, there are no oscillations. If the slope increases, the singular point becomes unstable. If the cycle at infinity is likewise unstable, the oscillations certainly continue to exist. However, when the slope increases beyond a certain critical value, the cycle at infinity becomes stable and oscillations again become impossible.

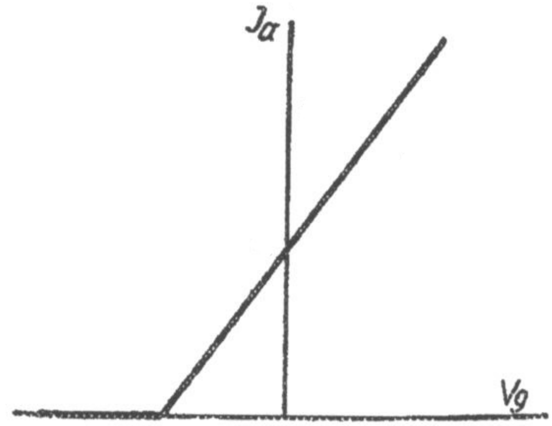

Fig. 5.

Section 2: Analytical Methods for the Study of Nonlinear Systems

The general qualitative theory of differential equations, partially discussed above, is still in the developmental stage. It allows analysis, however incomplete, in the case of two and possibly three autonomous equations (incomplete analysis in this last case), provided that the second terms are either polynomials of not too high a degree (third, fifth), or functions that can be geometrically characterized with sufficient simplicity. However, the radio engineer cannot be content with a qualitative study of the problem. He requires a quantitative study that, alone, can be used as a basis for practical calculations. On the other hand, the radio engineer can accept a quantitative theory with only an approximate basis provided that it satisfactorily takes into account the cases that are important for practical purposes.

From this point, there is a clear necessity to prepare approximate methods for the study of nonlinear systems. These methods should, of course, take into account what these systems have in the way of specifics. One approximate quantitative method which can deal with the analysis of nonlinear systems is the one involving coefficients with slow variation, or, as we shall term it, the Van der Pol method. Although this method has actually been used for quite some time in celestial mechanics, it was Van der Pol who was the first to systematically apply it to problems of radio engineering. He produced a series of basic results concerning forced synchronization, "resistance", etc. [6], [7].

But it was only recently that this method was supported on a mathematical basis. Additionally, there still remains a certain amount of uncertainty in the very method of its application. The chief difficulty, in this respect, was obviated [8] in a way which we shall explain using the equation:

$$\ddot{x} + x = \mu f\left(x, \dot{x}, t\right) \tag{5}$$

in which the second term is a periodic function of t of period 2π, and μ is a "small parameter" on which the degree of approximation will depend, as we shall see. It is possible to reduce the equation of a regenerative receiver, etc to this form. According to Van der Pol, one should pose that:

$$x = u \cos t + v \sin t \tag{6}$$

u and v being functions of time t with "slow variation," i.e. whose derivatives are small with respect to u and v, and whose second derivatives are small with respect to the primary derivatives. Introducing the hypotheses involved in expressions (6) and (5), and disregarding all the terms of higher degrees as well as the harmonics, we now obtain the approximate equations of Van der Pol:

$$\frac{du}{d\tau} = a_o(u, v), \qquad \frac{dv}{d\tau} = b_o(u, v)$$

in which $\tau = \mu t$ and $a_o(u,v)$, $b_o(u,v)$ are functions of u and v.

Let us view the problem from another point of view. Let us substitute two new variables, u and v for the variable x, u and v being defined as follows:

$$x = u \cos t + v \sin t$$

Substituting two variables u and v for a single variable x enables us to impose upon them the supplementary condition:

$$\dot{u} \cos t + \dot{v} \sin t = 0$$

and replace equation (5), so that the variable x may be checked from the equations:

$$\left.\begin{aligned} \dot{u} &= \mu f\left(u \sin t - v \cos t, u \cos t + v \sin t\right) \cos t \\ \dot{v} &= \mu f\left(u \sin t - v \cos t, u \cos t + v \sin t\right) \sin t. \end{aligned}\right\} \tag{7}$$

Replacing the second terms by their averages, we again arrive at the “truncated” or Van der Pol equations. By producing them in the same way, it is possible to clearly state an approximation problem: it is a matter of establishing when and how much (as a function of the value of μ) the solutions of the truncated equations are themselves close to those of the exact equations (7). This is purely a mathematical question that has been studied by P. Fatou [9] in a memorandum, which only came to our attention after our investigations concerning the Van der Pol method.

Our works, as well as the mathematical results of Fatou that apply to our present problems, while reporting the conditions and ranges that the truncated equations of Van der Pol can take into account, can also, with sufficient approximation, deal with transitory phenomena. In addition, the results of Fatou confirm that when μ is sufficiently small, each equilibrium position of the truncated system corresponds to a periodic solution of the exact system, and that if this position of equilibrium is stable the periodic solution is likewise stable. Therefore, the question relating to the mathematical basis of the Van der Pol method has been clarified. It is possible to hope that this method will likewise be justified for more complicated cases.

Let us explain again the advantages of the Van der Pol method. In the case of an autonomous system with one degree of freedom, the Van der Pol equations can be reduced to a single equation, which may be easily solved by quadrature. In the case of non-autonomous systems with one degree of freedom – the case which has just been reported above – the Van der Pol equations are autonomous, and consequently, soluble by the methods of Poincaré [29]. In the case of more complicated systems, for example those with two degrees of freedom (autonomous or subject to external effects), the Van der Pol equations are systems of autonomous equations of the first degree – two equations in the most simple cases – which can be processed by the methods of Poincaré [10][5]. The Van der Pol method therefore allows replacing a system of nonlinear equations by another, more simple one. It is possible to use the Van der Pol equations successfully, which has been done in a study of the concepts of extension of phase, singular points, limiting cycles, and the theory of bifurcations (Section 3). We shall see below that by applying the Poincaré methods to the approximate equations of Van der Pol, it is possible to produce some new results that are of physical interest.

We have seen that the Van der Pol method provides a solid basis for dealing with the simplest cases with which we are concerned. But it only provides a “zero order approximation”. For some questions arising in radio technology this would be sufficient. But there are other questions that require further approximations: these mainly concern questions about the correctness of frequency, the latter only appearing in many problems as a second approximation.

[5] There are 3 van der Pol equations for a self-oscillating system with two weakly coupled circuits. This case was discussed by Mayer at Gorki.

A theory that allows improvement of precision and calculation is therefore necessary. Unfortunately, such a theory only exists in the case of purely periodic phenomena. This is the "small parameter method" to which we likewise are indebted to Poincaré. This method allowed the latter to scientifically demonstrate the existence of periodic solutions of a very general character for the three-body problem in celestial mechanics. In substance, this method consists of the following. Let us assume that when the parameter $\mu = 0$, our system exhibits certain periodic motions. A search is then made for the motion existing when $\mu \neq 0$ in the form of an ordinate series according to the powers of μ, in which the zero approximation is one of the solutions corresponding to $\mu = 0$. If, with μ being zero, the system produces a family of periodic solutions, it includes a discontinuous system of periodic solutions close to those existing when $\mu \neq 0$ and which should be determined. This method is especially convenient when used with the zero approximation; this system is linear and conservative[6].

We have applied this method to a whole series of self-exciting problems [12], [13], [14], [15].

Section 3: Variation in One Parameter; Stability on a Large Scale

In order to study the important subject of the transformations undergone by the phase plane in the case of variation of one parameter, we should consult Poincaré once again. Poincaré was led, in his famous theory of the equilibrium of rotating fluid masses, to state and brilliantly solve questions relative to the development of equilibrium states of a conservative system in the case of the variation of one parameter. The concept created by Poincaré concerning the *bifurcation value* of the parameter can be generalized and applied to the problems in which we are involved. One value of the parameter $\lambda = \lambda_0$ is called ordinary if there exists a finite quantity ε ($\varepsilon > 0$) such as for $|\lambda - \lambda_0| < \varepsilon$. The integral curves on the phase plane have the same qualitative appearance and show *bifurcation* in the contrary case.

In the general case, the theory of the development undergone by the qualitative appearance of the phase plane, in the case of variation of the parameter, is quite complicated and insufficiently perfected. However, in the case of approximately sinusoidal oscillations, the theory is simplified in the extreme and returns to the theory of Poincaré for to the equilibrium states of a conservative system. It is sufficient to replace the coordinates of the states of equilibrium by the squares of the amplitudes of stationary motion (limiting cycles and singular points) [16]. Without explaining the Poincaré theory, we will provide an example of its application.

Let us take the two principal types of excitation, the "soft" excitation and the "abrupt" excitation. We shall concern ourselves with the oscillator of Figure 6 and shall select the coefficient of mutual induction as a parameter.

[6]Pontryagin has provided a general method for the case in which, with zero approximation, the system is Hamiltonian.

Soft Excitation: Let $\lambda = \lambda_1$ be the value of this parameter corresponding to the excitation (Figure 7). In the case of $\lambda < \lambda_1$, the only stable stationary state is the state of equilibrium depicted by one focus (Figure 8). Regardless of the initial position of the representative point, at the end of a certain length of time it will be found in the vicinity of this focus.

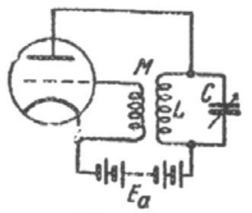

Fig. 6.

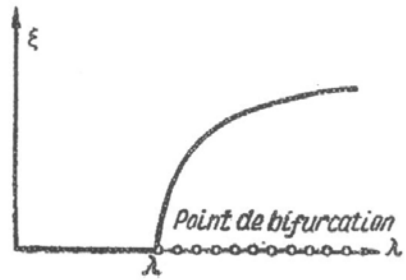

Fig. 7.

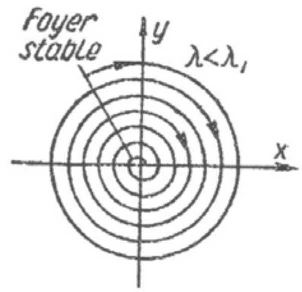

Fig. 8.

When $\lambda = \lambda_1$ is a bifurcation value of the parameter: the focus loses its stability at the same time that it generates a small stable limiting cycle (Figure 9) on which the representative point begins to rotate. In the parlance of physics, we say that the oscillator has been excited.

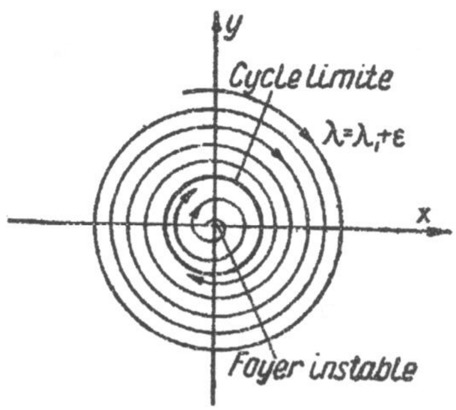

Fig. 9.

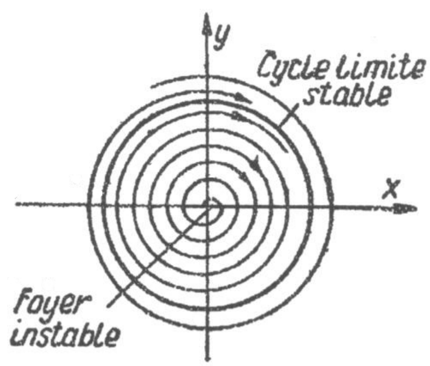

Fig. 10.

With λ increasing the radius of the limiting cycle becomes larger (Figure 10), and with λ decreasing all the phenomena are reproduced in the reverse direction: the limiting cycle is reduced to one point, and the oscillations cease. On the physical diagram I^2, (λ being the amplitude of the current), we obtain a "soft" transition from the state of equilibrium to the periodic motion and vice versa: the amplitude of the oscillations varies in a continuous manner (Figure 11).

Abrupt Excitation: If, in the case of small values for λ, the system is found to be in the proximity of the state of equilibrium, it remains there until λ assumes the value $\lambda = \lambda_1$ (Figure 12, 13, 14, 15). The creation of two twin limiting cycles -- one stable and the other unstable -- at the instant in which $\lambda = \lambda_0$ does not disturb our representative point since it leaves the stability of the equilibrium state intact.

In the range λ_0 to λ_1, the unstable cycle becomes smaller. Then when $\lambda = \lambda_1$, it disappears and dissipates, so to say, the singular point by its instability. At this instant, the representative point, following the integral curves, rejoins the stable limiting cycle whose amplitude has gradually increased from the instant at which $\lambda = \lambda_0$ (Figure 16).

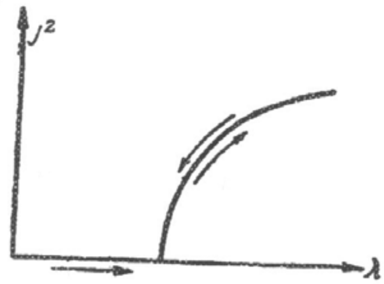

Fig. 11.

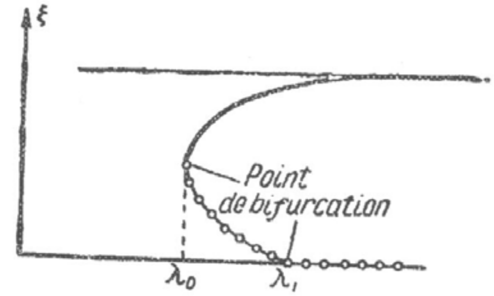

Fig. 12.

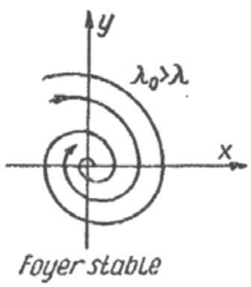

Fig. 13.

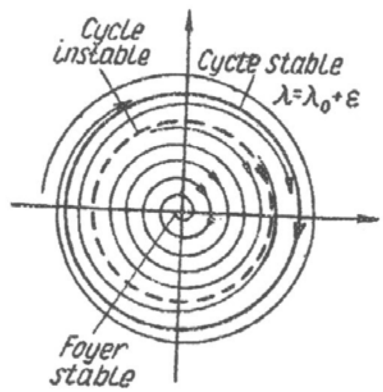

Fig. 14.

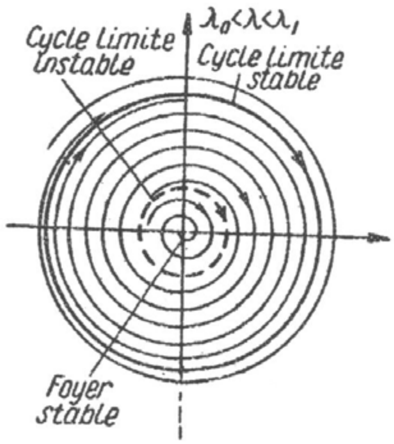

Fig. 15.

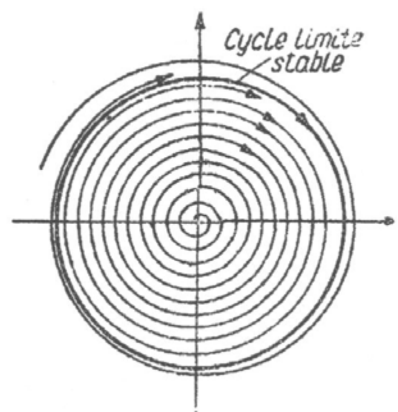

Fig. 16.

Causing the parameter to vary in reverse, we observe that, on "return", the oscillator takes a different path than when "going". Indeed, the representative point will stay with the limiting cycle until the instant at which $\lambda = \lambda_0$. At this instant, the two cycles are merged, compelling the representative point towards the state of equilibrium. The fact that the latter becomes stable when $\lambda = \lambda_1$, produces no effect on the movement of the representative point since, at the instant in which $\lambda = \lambda_1$, the characteristic of the cycle used as its path does not change.

The diagram I^2, λ (Figure 17) emphasizes a discontinuous ("abrupt") variation of the amplitude, a variation which, owing to its irreversibility, recalls hysteresis phenomena.

This phenomenon of abrupt excitation, which is quite interesting for the radio engineer, finds its natural and proper interpretation in the language of singular points, limiting cycles, and the bifurcation values of the parameter. For this reason, it can be seen immediately that when $\lambda_0 < \lambda < \lambda_1$, the representative point can be "thrown" from one stationary mode of operation into another by a sufficiently strong impulse. This is a claim that we suspect can be demonstrated intelligibly by quasi-linear theory.

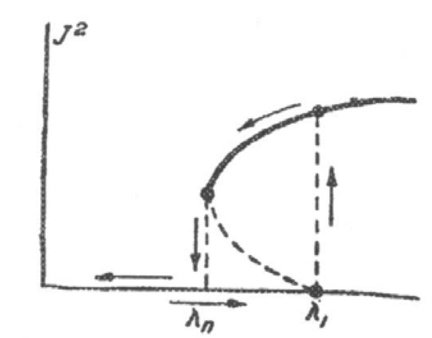

Fig. 17.

At this point, we should like to discuss a concept that would have no significance in the case of a conservative system, but which, in the case of a nonconservative system, has a great deal of interest. Let a path be stable. On the phase plane we can mark off a region containing the initial position of the representative point from which the latter, ultimately (when $t \to +\infty$) rejoins this path. This region is called the "large scale region of stability" or "region of attraction" of the stationary movement under consideration. Figure 18 and 19 represent two examples.

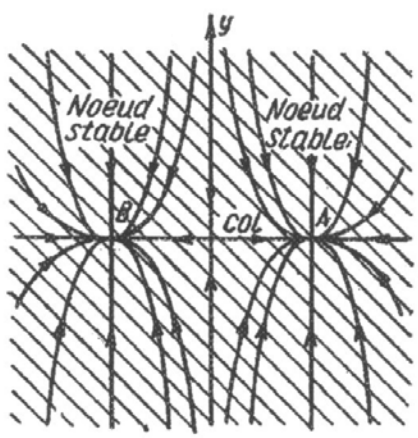

Fig. 18.

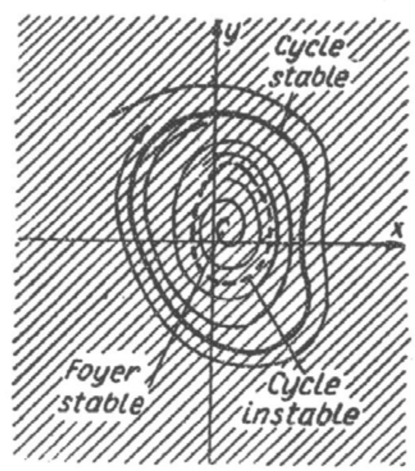

Fig. 19.

The phase plane of Figure 18 has three singular points: two stable nodes and one saddle. The range of attraction of node A is in the right semiplane, whereas that of node B is in the left semiplane. Let us now consider (Figure 19) a stable singular point surrounded by an unstable cycle, which is, in turn, contained within a stable cycle. The range of stability on a large scale of the position of equilibrium is the portion of the plane contained within the unstable cycle. The remainder of the plane is the range of attraction of the stable cycle.

Without providing more complicated examples, let us point out that the sharing of the extension in phase by the fields of attraction comes up against some obstacles, even in the case of one degree of freedom. It appears that considerations of probability must be taken into consideration here.

Let us note, in conclusion, that the concept of stability on a large scale imparts the true physical interpretation to the unstable limiting cycles and separating curves. These are the boundary motions similar to what is called, in geography, the tidal line: according to which, at the initial instant, the representative point is placed on one side or the other of these components and takes its direction towards differing destinations.

Section 4: Autonomous Systems

At the forefront of the many nonlinear problems is the one of autoexcitation, i.e. those oscillations created by the oscillating system itself without the participation of any external forces varying with time and at the expense of a constant source of energy (for example a storage battery). The qualitative questions and, partially, the quantitative questions that are concerned directly with the study of an autonomous nonlinear system, can be solved by methods that we have described above. We have already shown examples of the application of these methods to the problems relating to soft and abrupt excitation, and to transient phenomena. As we have said, quantitative methods of approximation allow us to find the amplitude of autoexcitations and the correct frequency in the case of almost sinusoidal oscillations. Permit us to comment that, at least insofar as only the zero-order approximation is concerned, in such problems the methods placed previously known results solidly upon a mathematical basis. In this way, it is possible to precisely demonstrate the existence of periodic solutions in certain cases, and establish their stability. We estimate that these demonstrations of existence have a high degree of usefulness, and the following is why.

When we set up any problem of physics as a differential equation, we are always forced to simplify it. We do not write the equation of the problem that is given to us, but rather that of a simplified and idealized problem. Now, how can we be certain that we have not disregarded any of the essential features of the real problem? The situation changes if we have demonstrated the existence of a periodic solution that has been verified by experiment. This demonstration is an argument to maintain that we have not omitted essential features (assuming as the latter those that make available the production of stationary oscillations). But these are only indirect data. The same will not be true if it can be demonstrated that the differential equations do not have a periodic solution, whereas the system that they claim to describe does possess such solutions. Most certainly we have then disregarded some essential feature, and we must attempt to recapture it.

Practical experience recognizes cases where experimental investigations demonstrate the existence of a solution, which has suggested the means for correcting omissions of this type and placing further discussion on the right track.

The following is one of the most elementary examples. Everyone knows how many technical manuals – chiefly among the older ones – describe the theory of the bell or electric buzzer. The armature, in the state of equilibrium, closes the circuit of the electromagnet. When a battery is placed in the circuit the electromagnet attracts the armature, the current is interrupted, the magnet loses the force of attraction, the spring pulls the armature back into its original position, and as the old expression goes – the game continues. If this reasoning is translated into differential equations, it is possible to show very easily that they do not permit the game to continue and that they allow only a periodic solution. Some essential point has therefore been disregarded. In reality, the theory of buzzers is less simple than it appears at first glance. Self-induction is necessary for oscillations to be possible at all. Mr. Leontovitch [17] has successfully discussed the problem of the buzzer and has clearly shown that not only is self-induction necessary for the existence of the phenomenon, but that it becomes a factor in the period of the oscillations, this period differing from that of the tuning fork or armature return spring. Other examples could be quoted showing the actual utility of experimental investigations devoted to questions of the existence of solutions.

Radio engineering often reveals conditions in which oscillators are practically sinusoidal[7]. However, during the last two years, and to a great extent owing again to the works of Van der Pol, interest has grown concerning systems executing oscillations which greatly differ from a sinusoidal shape and may be termed “relaxation oscillations”. The characteristic of these oscillations is, essentially, a function of the resistance or parameters equivalent to it. This implies that an equation of the type:

$$L\ddot{q} + \frac{q}{C} = f\left(q,\dot{q}\right) - R\dot{q} = \varphi(q,\dot{q}) \tag{8}$$

describes the autoexcitation system. The function $\varphi(q,\dot{q})$ is not limited to small values, as in the case of “almost sinusoidal” oscillations, but, as Van der Pol has pointed out, it may take on substantial values. Since, it is assumed here that we are concerned with periodic solutions, this case is entirely within the purview of the qualitative theory of Poincaré. The singular unstable point is a node, and the periodic solution corresponds to a limiting cycle.

According to the simplest hypothesis, when the characteristic of the tube can be modeled as a cubic parabola, the equation can be described in the form:

$$v - \varepsilon(1-v^2)\dot{v} + v = 0 \tag{9}$$

[7]There is one very simple mechanical system that allows production of autoexcitations that are almost sinusoidal. This is the Froude pendulum, which has been studied by Strelkov [18].

whereupon, assuming $\dot{v} = y$, it becomes:

$$\frac{dy}{dv} - \varepsilon(1 - v^2) + \frac{v}{y} = 0 . \tag{10}$$

If ε << 1 we have oscillations which are almost sinusoidal ("Thomsonian" oscillations). However, the qualitative theory is applied to the general case no matter what the value of ε may be. When ε << 1, the singular point is a focus, but when ε >> 1 it is a node; in both cases there is a limiting cycle.

Nevertheless, in quantitative studies of relaxation oscillations it is possible to proceed in another way. It is possible to idealize the problem by setting L = 0 and replacing the equation of the second order (9) by the equation of the first order:

$$\dot{v} = \frac{v}{\varepsilon(1 - v^2)} \tag{11}$$

which is easily integrated. Clearly, this equation does not allow for a periodic solution. At the close of a finite time duration, the velocity (or the electric current, or its derivative), represented by v, becomes infinite. After having idealized the problem in this way, in order to take the physical phenomenon into account in an approximate manner, a new condition is required to be introduced, which in our case assumes that at a certain instant the current undergoes a discontinuity whereas the voltage at the condenser terminals remains constant. This assumption or "condition of discontinuity" is physically justified by the fact that the energy cannot vary discontinuously. It is possible to provide it with another form by explicitly requiring conservation of energy. This "discontinuous" theory, together with the condition of discontinuity, permits evidence of a "discontinuous" periodic motion and finding its amplitude as well as its period.

Without being identical, this mode of treating relaxation oscillations, which is applicable to electrical and mechanical systems, is analogous to methods employed in mechanics to analyze elastic collisions. It is assumed that at the moment of collision the velocity changes discontinuously. The conservation of energy and momentum allows a velocity decrease after the collision from that existing beforehand. In principle, this method excludes the possibility of studying what is happening during the extremely short duration of the collision. The results that it gives are often sufficient since the collision is quite brief. However, if we wish to follow the phenomenon of the collision itself, the problem becomes extremely complicated. It is enough to merely recall the investigations of Hertz. Likewise, in our theory of relaxation oscillations, we can simplify the mathematical description of the problem by idealizing it. As a consequence we do not see how the system can "leap" from one state to the other.

We have applied this method to the study of electrical systems with one degree of freedom in which self-induction plays a secondary role [19], as well as to mechanical autoexcitation systems with a small mass and a high degree of damping [20].

In this case it should be noted that by taking the "parasitic" self-induction into account, nothing is obtained which is of physical interest. There still remains the parasitic capacitance of conductors etc. to be considered. Now, it is impossible to take into consideration all of the parasitic parameters. Our idealization has the advantage that it allows us to study relaxation systems that are relatively complicated, such as the multivibrator of Abraham-Bloch, a system with two degrees of freedom. This system has already been studied by Van der Pol [21], but with one essential restriction: he assumed the phenomenon to be completely symmetrical, and rejected the consideration that there were transitory phenomena following an initial asymmetrical state. This restriction allowed him to produce an equation of the second order. Nevertheless, in the general case, he would have obtained two equations of the second order, and this would have greatly complicated the problem. For the general case, our idealization provides two equations of the first order, which can easily be studied by the methods described above [22]. Thus, we have been able to study not only the stationary mode of operation (by calculating the amplitude and the period) but also the transitory asymmetrical phenomena in the multivibrator of Abraham-Bloch. These results were experimentally verified.

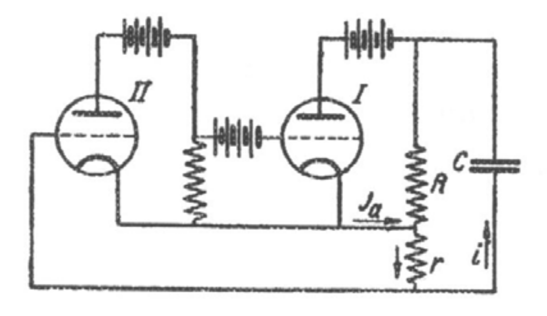

Fig. 20.

The discontinuous theory is not merely applicable to the case in which one of the parameters is small. Even in a circuit with capacitance and self-induction there are cases that can occur where, in certain regions, the velocity becomes so great that it is possible to replace the very rapid variation of state of the system by a discontinuity and to determine the final result of the motion in this region by using the "condition of discontinuity". Removing, in this way, those regions where quantitative research is most difficult, we can solve (with a precision satisfactory for all practical purposes) a whole series of questions that involve oscillations whose form is essentially nonsinusoidal. It is possible to apply this method, for example, to the configuration depicted in Figure 20.

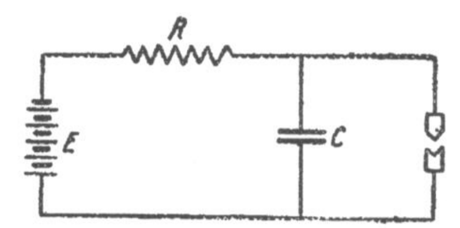

Fig. 21.

The parasitic parameters again give rise to the following observation. By studying an oscillatory system, we always idealize it, disregarding certain parameters (for example, the self-induction of a condenser or the capacitance of a coil). Nevertheless, we often obtain good results. The cause for this rests in the rather extensive properties of the mathematical approach, concerning which we will not go into detail. However, there are cases where, by disregarding certain parameters, no matter how small, we qualitatively change the overall picture of the phenomena. Let us consider the system shown in Figure 21.

The circuit is formed by a battery E, a resistance R, a capacitance C, and an electric arc. By using the characteristic of the arc, we produce three stationary values for the current using the customary graphical method. Analyzing the stability of these values by well-known methods, we find that two of them, A and B, are stable (Figure 22).

However, if we introduce an arbitrarily small self-induction L (Figure 23), the state of equilibrium becomes unstable. And so, in reality, it does not exist.

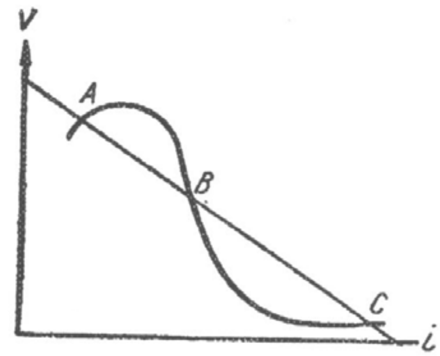

Fig. 22.

It is easy to see how cases of this kind can occur. If the order of the differential equation from which we draw conclusions concerning stability or instability (in the case of equilibrium this will be an equation of the n-th linear order with constant coefficients) does not rise as a result of the introduction of the parasitic parameter, the latter, if it is sufficiently small, is not able to change anything. On the other hand, if this parameter appears in the equations and increases their order, it can render equilibrium states (that would be considered stable by disregarding it) unstable. The physical meaning of this result is clear. In composing a system of equations of the n-th order, forming a system with n dimensions, we only assume initial conditions. And when we raise the order of the differential equations by taking into account the parasitic parameter, we allow by this a greater diversity of initial conditions. It is then possible that among the initial states newly allowed, the conditions are right for the system to diverge from the state of equilibrium. Therefore, a certain caution is necessary in idealized models.

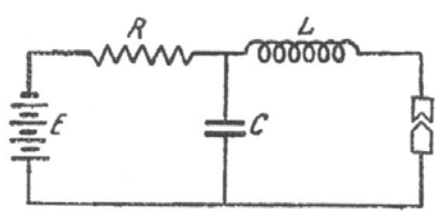

Fig. 23.

Some words are in order concerning *self-exciting systems with distributed parameters* that play an important role in radio engineering and in mechanics (oscillators containing antennas or Lecher wires, tubes whose grid makes up an oscillating system resulting in very high frequency waves (which were studied by Grechowa), telegraphic wires emitting a sound owing to the effect of wind, vibrating airplane wings, bowed musical instruments, organ pipes etc.). There still does not exist a precise mathematical theory for these phenomena. Nevertheless, without speaking rigorously, it is rather easy to create a possible theory for some of them by analogy with those produced under rather precise conditions - those systems existing with a finite number of degrees of freedom. This theory allows the calculation of amplitudes, the solution of questions of stability, etc. [23]. However, since this theory does not have a strict mathematical basis, it is necessary to use its results with care. It takes into account most characteristic phenomena occurring in distributed self-exciting systems. It anticipates that oscillatory modes of operation at different frequencies can be established under the same operating conditions. (The same is true in the self-exciting systems having a finite number of degrees of freedom). The production of such-and-such an operating mode is a function of the initial conditions or the history of the system. These phenomena have been produced and studied experimentally by Bendrikov and Brailo at Moscow, as well as by Gaponov at Gorki. It is possible to cause the disappearance of a particular oscillating wavelength in a Lecher wire oscillator by touching them with a finger. The system then begins to oscillate at another wavelength. Under certain conditions, when the finger is taken away the system will not return to the original wavelength, but continues oscillating at the new wavelength. This phenomenon can likewise be produced by capturing the energy using a resonant circuit. Apparently, similar phenomena occur in bowed musical instruments. Strelkov has performed similar experiments with a string vibrating under the effect of a jet of water or air. These quite simple experiments allow the observation of phenomena that are characteristic for distributed self-exciting systems.

Given that, under the same operating conditions different oscillating modes can occur, the question may be asked as to which of them is produced when the system is triggered. This question is often within the capacity of the theory of probabilities. It has not yet been solved theoretically for distributed systems. The statistical phenomena are easy to observe experimentally in the Lecher wire oscillator. They also take place in organ pipes.

Section 5: Effects of an External Force on a Self-Exciting System

One characteristic property of self-exciting systems, which is quite important for the entire realm of radio engineering, is the appearance of the phenomenon of forced or automatic synchronization, or frequency locking. This phenomenon, previously noted by Huygens for clocks hanging from the same wall, was first observed in radio engineering by G. Moeller [24] and Vincent [25]. It gave rise to many experimental and theoretical investigations, among which special mention should be made of those by Van der Pol [6]. As is well known, the simplest feature of this phenomenon consists in the following. When an external force of frequency ω acts on a self-exciting system of frequency ω_0, no beats are observed, just as would be the case in an undamped linear system when the difference frequency, $\omega - \omega_0$, is sufficiently small. The system is automatically synchronized to the frequency of the external force.

A similar phenomenon occurs in a system subjected to the effect of a force which is not periodic, but only quasiperiodic (which can be depicted by a sum of terms of incommensurable frequencies).

The locking likewise occurs, as observed by Van der Pol and Van der Mark [26] in relaxation oscillator systems, and as seen by Koga [27] in ordinary oscillators when one of the frequencies of the external force is close to a multiple of the frequency of the system. When the out-of-tune condition exceeds a certain value, which can be termed a locking limit, the appearance of beats may be ascertained. Since the force is sinusoidal, when the misalignment or out-of-tune condition far exceeds the locking limit, it can be broadly stated that two oscillations exist in the system, one in response to the frequency of the external force, and the other characteristic of the system.

Nevertheless, if the frequency misalignment only slightly exceeds the locking limit, this last frequency is shifted towards the frequency impressed from the outside, and, as we shall see, the whole phenomenon becomes complicated.

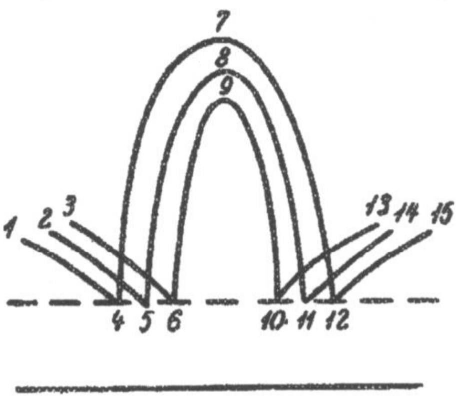

Fig. 24.

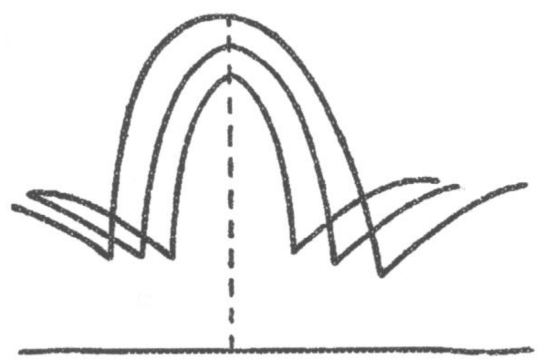

Fig. 25.

The theoretical study of the locking phenomenon consists in searching for stationary solutions of the differential equation:

$$\ddot{y} + \omega_0^2 \, y \;=\; \mu \, f(y, \dot{y}) + \Sigma \, A_s \cos\left(\omega_s \, t + \delta_s\right) \tag{12}$$

In the simplest case in which an electromotive force acts upon a system in a soft mode of operation with almost sinusoidal oscillations, the “curves of amplitude” have, as is known, the appearance shown in Figure 24. The portions of the curves labeled 4-7-12, 5-8-11, etc. belong to the synchronization mode. The parts 1-4-12-15 belong to the beat mode. Theoretically, these curves have a symmetrical appearance, but, most often, experience shows asymmetrical curves (for example those of Figure 25). This deformation is probably owing to the presence of a grid current. This is what appears to be confirmed by the recent results of Bakoulov (Moscow).

In his classical study, Van der Pol had studied a self-exciting system with a soft mode, modeling the characteristic of the tube as a cubic parabola, and he was able to detail the principal characteristics of the phenomena of synchronization and frequency shift. Nevertheless, there are still several questions remaining. In synchronization phenomena, as assumed by Ollendorff [28], it still remains to be explained whether there exists a threshold for the amplitude of the applied electromotive force (EMF).

Employing the “truncated” equations that Van der Pol set up for this problem:

$$\left.\begin{aligned}\frac{dx}{dt} &= -ay + x(1-r^2)\\ \frac{dy}{dt} &= A + ax + y(1-r^2)\end{aligned}\right\} \qquad (13)$$

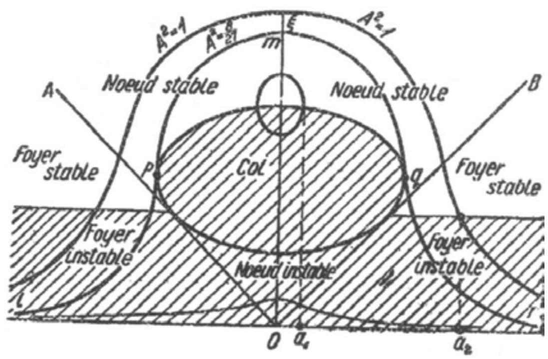

Fig. 26.

[where a = 2(ω_0 - ω)/α is the tuning misalignment, ω_0 is the frequency of the self-excitations, A = ($B\omega_0$)/(αa_0). B is the amplitude of the force used, α is a constant depending on the two parameters, a_0 is the amplitude of the self-oscillations, τ = at/2 and finally t represents time], it was possible to show that, in the topographical analysis of Poincaré, a threshold did not exist [29]. This has also been established experimentally [30].

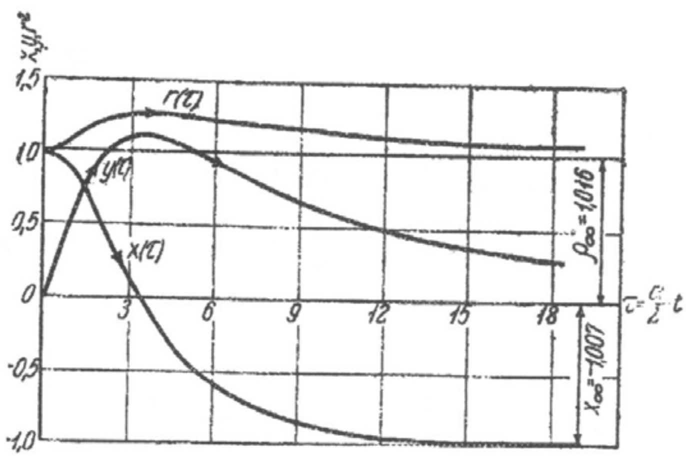

Fig. 27. The integral curves of the differential equations.

$$\begin{aligned}\frac{dx}{dt} &= -y + x(1-r^2)\\ \frac{dy}{dt} &= 0.303 + 0.3x + y(1-r^2)\end{aligned}$$

for the initial conditions $\tau = 0$, $x_o = 1$, $y_o = 0$, $A = \sqrt{\dfrac{8}{27}}$

Consequently, in the weak signal case it was easy to quantitatively demonstrate that the normalized width of the synchronization band is a function of the ratio of the amplitude of the signal to that of the self-oscillations ($\omega_0 - \omega$)/ ω.

This is what has permitted the application to weak signals of the method of field intensity measurement by the width of the synchronization band, as suggested by Appleton [31], [32]. A similar method, based on the phenomenon of synchronization of acoustic self-exciters, has been used to measure the intensity of sound [33], [34]. The phenomena of acoustic locking leads to an interesting problem that we are preparing to study, which is that of the automatic synchronization of woodwind and bowed orchestra instruments.

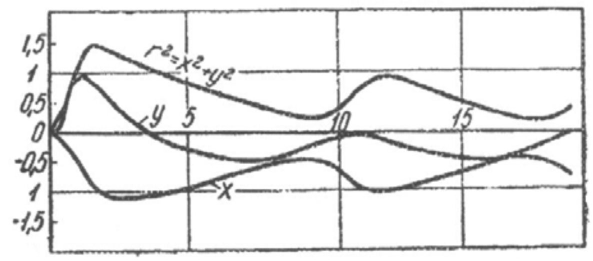

Fig. 28.

Let us return to the analysis of equation (13). If the square of the amplitude A^2 is plotted on the ordinate, and detuning, a, is plotted on the abscissa, Figure 26 is produced. The resonance curves are those of Van der Pol, but our figure shows the fields corresponding to the various types of transitory phenomena. One might be interested in what is happening when an electromotive force is applied to the oscillator and it becomes active.

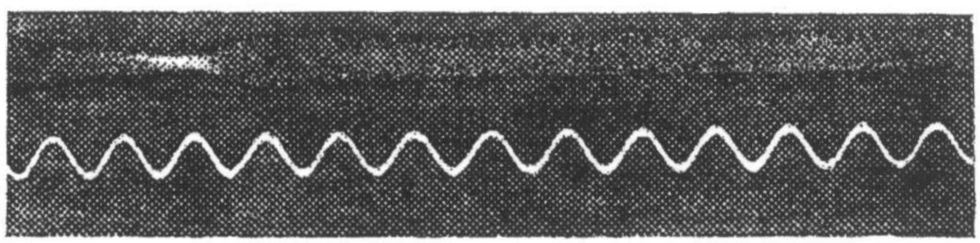

Fig. 29.

It is also possible to observe what occurs when the oscillator is triggered after the electromotive force has been applied. Let us examine the first case, which is physically the most interesting. The theoretical study of transient phenomena consists of discussing the nonstationary solutions of the equations for amplitudes (13), i.e. in following the

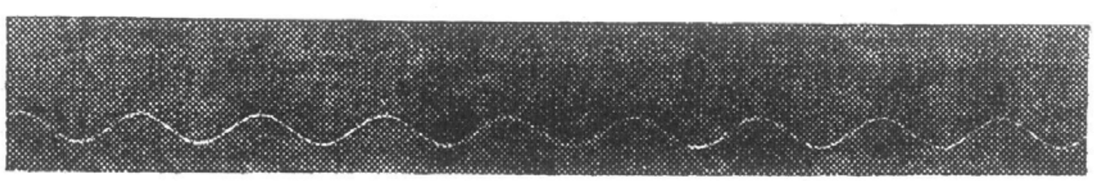

Fig. 30.

amplitude fluctuations, which are components x and y as a function of time. These results are summarized in the diagram of Figure 26. In order to know how the stationary oscillations are set up, with the variance and amplitude of the applied EMF being given, it is necessary to get the resonance curve corresponding to this EMF, and, on the latter, the data corresponding to the given variance. If this point is located in the domain of a stable node, the establishment of the mode is accomplished aperiodically: the coefficients with slow variation in the Van der Pol solution tend aperiodically toward constant values. If this point is in the domain of a stable focus, the phenomenon is oscillating. Finally, if the point falls within the range of instability, then there are no stable periodic solutions. These results have been experimentally verified in the works of Riazine [35] who

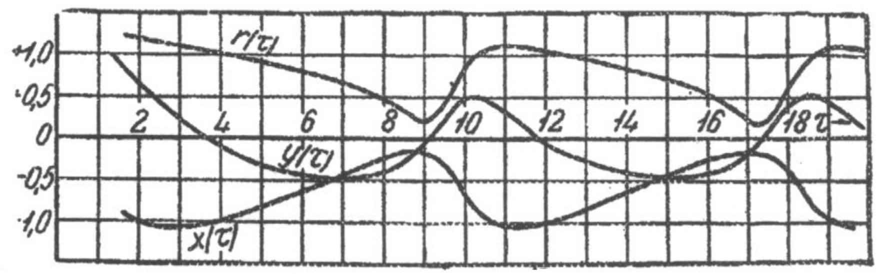

Fig. 31.

made oscillographs of various types of transient phenomena. He calculated the solutions of equation (13) by methods of numerical integration and confirmed the theoretical results by low frequency oscillograms. The calculated curves of aperiodic transitions are shown in Figure 27, those of oscillating transitions are shown in Figure 28, and the respective oscillograms are given in Figures 29 and 30. A pure sinusoid appears before the application of the signal, but the oscillograms do not show it.

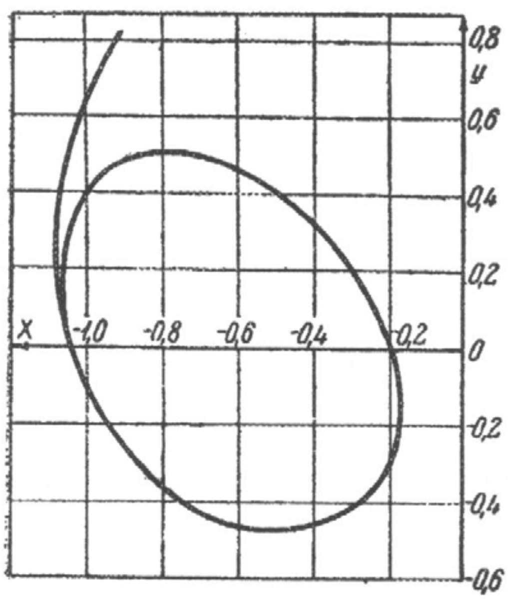

Fig. 32.

The same method of numerical integration was used to study what occurs, to a minor degree, outside of the synchronization band. In this case, the nonstationary solution of equation (13) on the x, y plane, does not tend toward a singular point, as in the domain of entrainment, but is coiled around a limiting cycle. The curves showing current as a function of time and limiting cycle are depicted in Figures 31 and 32. The theoretical results agree closely with the experimental findings (Figure 33). With regard to beats, the theoretical curves and oscillograms harmoniously indicate that their amplitude increases at a perceptibly greater rate than it decreases.

In order to establish the spectral composition of the beats, Riazine performed a harmonic analysis of the curves shown in Figure 31. The spectrum so obtained (Figure 34) shows that, in the vicinity of the synchronization region, the application of an electromotive force causes the appearance of a self-oscillation in the spectrum, that has equidistant combinations of frequencies clearly delineated. Oscillograms were produced for beats in the vicinity of the synchronization bands for ratios of frequencies characteristic of the applied frequency, which were approximately equal to 1:2, 1:3, 1:4 and 1:5 (Figures 35, 36, 37, 38). It became obvious from these oscillograms that we have a pulsation of amplitude in the region of the limit of forced synchronization, just as in the case of the 1:1 ratio. The envelope of the beats always shows practically the same characteristic:

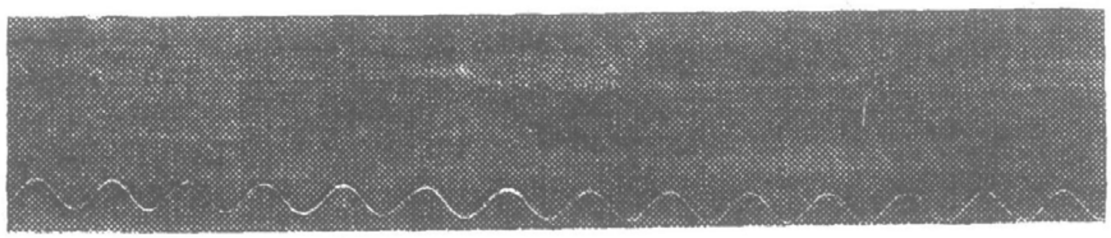

Fig. 33.

a rapid increase and then a slow decrease of amplitude. Let us again emphasize that in the vicinity of the synchronization band, the oscillations cannot be expressed by the linear superposition of two sinusoidal terms, as has been done up until now. There are at least three oscillations of amplitude that are almost equal.

The study, likewise, was concerned with the phenomena that occur in the case of abrupt modes (H. Sekerska [36]). In this case the so-called domain of potential self-oscillations has a special interest. In this region the resonance curves end in a peak, recalling those that we produced in the resonance phenomena of the n-th class (Section 6). There still exist synchronization phenomena in the frequency combinations given by several electromotive forces. As an example, we shall give a special case that we observed some time ago: synchronization "in the middle". The oscillator is tuned approximately to the frequency $(\omega_1 + \omega_2)/2$, with ω_1 and ω_2 being the frequencies of the two electromotive forces. Synchronization phenomena are then clearly observed, and especially so when there exists a simple relationship between the frequency of the oscillator and those of the electromotive forces. It is understood that this phenomenon is important for reception of a signal without a carrier (DSB). In reality, when the carrier frequency is produced at the transmitter site, for practical purposes it is quite difficult to arrive exactly at the center of the sideband frequencies, this being, moreover, absolutely necessary. The phenomenon that has just been described lends its assistance: the frequency of the oscillator is automatically located in the middle of the sideband frequencies.

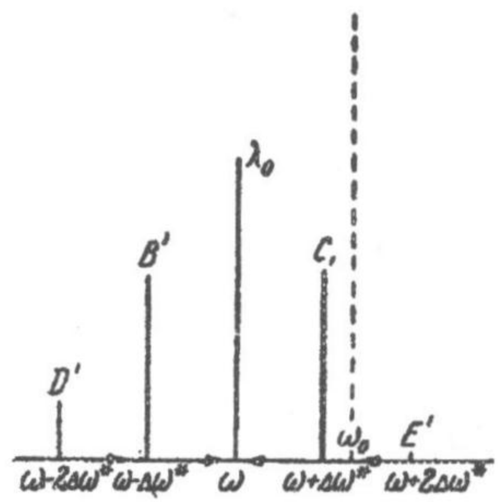

Fig. 34.

For this purpose, it is sufficient for the oscillator to only be approximately tuned. Similar phenomena occur when the oscillator is tuned to the frequency $(\omega_1 + \omega_2)/4$, and to other

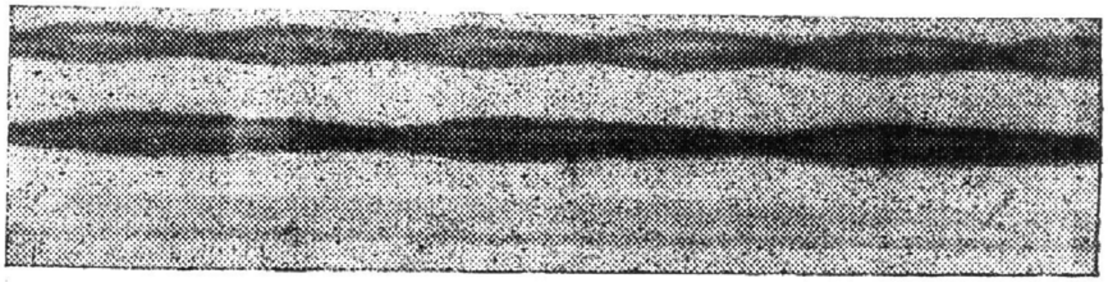

Fig. 35.

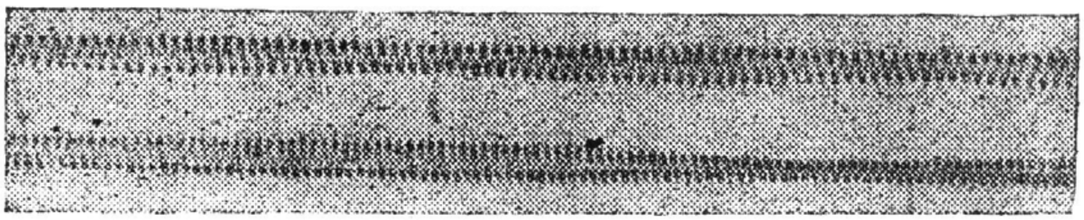

Fig. 36.

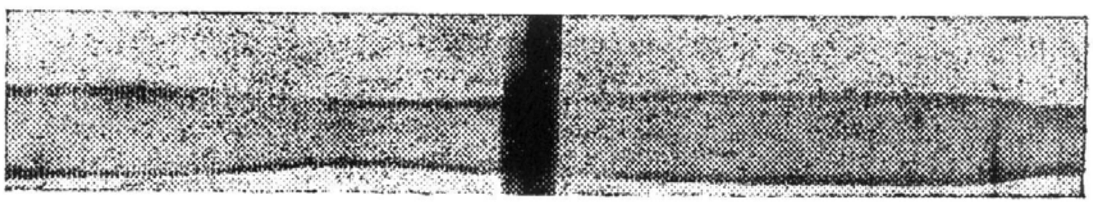

Fig. 37.

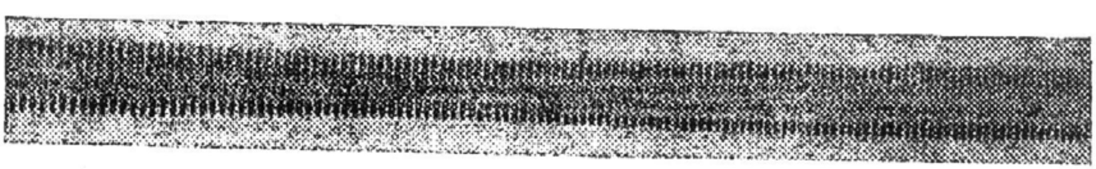

Fig. 38.

combinations of frequencies and their submultiples. The phenomenon of "in the center" synchronization has been studied theoretically by Goldstein and Petrossian[8].

[8] This publication is in preparation.

Section 6: Resonance Phenomena of the n-th Class

We shall devote a special paragraph to the phenomena that can be termed "resonance of the n-th class". The mathematical theory of these phenomena is based on the general results provided in the well known works of Poincaré [2] without any relationship to the physical applications concerning us. Poincaré shows that in nonlinear systems there can be periodic oscillations whose period is a multiple of that of the applied force ("periodic solutions of the second class"). It is useful for the discussion that follows to call any non-self-exciting system that becomes self-oscillating if the regenerative feedback is sufficiently increased, or if the value of any parameter is suitably modified, a potentially self-exciting system.

In Section 5 we referred to the phenomenon of forced or automatic synchronization that occurs in self-exciting systems subjected to the effect of a sinusoidal force that has a frequency close to the eigen-frequency. Koga [27] as well as Van der Pol and Van der Mark [26] have observed similar phenomena in self-exciting systems subjected to the effect of a force whose frequency is a multiple of the eigen-frequency. As for potentially self-exciting systems, in this case it is observed that when the frequency ω of the applied EMF is equal, or approximately equal, to a multiple of the eigen-frequency, the mode of operation being suitably selected, there is a special phenomenon of synchronous excitation. As long as its eigen-period is not close to a multiple of that of the applied force, a potential self-exciting system is the source of very weak "forced" oscillations. Nevertheless, when the system's eigen-frequency is sufficiently close to (ω/n), n being an integer, there can appear intense oscillations of frequency exactly equal to (ω/n).[9] This is the phenomenon of resonance of the n-th class [15], [37].

There still exists, in potentially self-exciting systems, another phenomenon which should be mentioned here. The resonance of the n-th class requires a well-defined mode of operation of the tube. If, starting from this mode of operation, the regeneration is increased, quite slightly in order that the system does not become self-exciting, under certain conditions there may be seen to appear, intense oscillations almost identical to the system's own oscillations, no matter what the period of the EMF may be. To these intense oscillations very weak "forced" oscillations are added in such a way that the system's own phenomenon is almost periodic. It can then be termed asynchronous excitation [39], [40], [41].

In the simplest case of n = 2, if the EMF is $E = E_0 \sin \omega t$ and acts upon an oscillating circuit interposed between the plate and the filament of a tube whose characteristic

[9] The capability for exciting a potential self-exciting system on a frequency equal to that of half of that of the EMF has been likewise reported by Groszkowski.

can be represented by a polynomial of the third degree, then the equation of the phenomenon (the grid current being assumed nonexistent) can be described in the form:

$$CL\,\frac{d^2i}{dt^2}+CR\,\frac{di}{dt}+i=f_0\left(\frac{di}{dt}\right)+CE_0\,\omega\cos\omega t \tag{14}$$

in which

$$f_0\left(\frac{di}{dt}\right)=i_{a_0}+\alpha_0\,\frac{di}{dt}+\beta_0\left(\frac{di}{dt}\right)^2+\gamma_0\left(\frac{di}{dt}\right)^3.$$

We assume that $\gamma_0 < 0$. Using suitable transformations and notations:

$$\tau=\frac{\omega t}{2},\quad 2\vartheta=\frac{2R}{\omega L},\quad \xi=\frac{\omega^2-4\omega_0^2}{4\omega_0^2}, \tag{15}$$

$$q=\frac{4M}{4}\,\frac{E_0}{v_0},\quad x=M\,\frac{di}{dt}\ (v_0 - \text{ saturation voltage })$$

The equation can be rephrased in the form:

$$\ddot{x}+x=\mu f(x,\dot{x})-3q\sin 2\tau \tag{16}$$

in which

$$\mu f(x,\dot{x})=\frac{d}{d\tau}\left[\frac{1}{1+\xi}\,f_1(\tau)-2\vartheta x\right]+$$

$$+\frac{\xi}{1+\xi}\,x=\mu\left[(k+2x+\gamma_1 x^2)\dot{x}+\frac{\xi}{\beta}\,x\right]; \tag{17}$$

$$\gamma_1=\frac{3\gamma}{\beta}<0,\quad \mu=\frac{\beta}{1+\xi}$$

and the regeneration factor

$$k=\frac{\alpha-2\vartheta(1+\xi)}{\beta}$$

Applying the Van der Pol method to equation (16), by the transformation:

$$\begin{aligned}x&=u\sin\tau-v\cos\tau+q\sin 2\tau;\\ \dot{x}&=u\cos\tau+v\sin\tau+2q\cos 2\tau;\end{aligned} \tag{18}$$

we produce the system of truncated equations:

$$\dot{u}=\frac{\mu}{2}\left\{\left[k+\frac{\gamma_1}{4}\,(z+2q^2)\right]u-v\left(q+\frac{\xi}{\beta}\right)\right\}; \tag{19}$$

$$\dot{v}=\frac{\mu}{2}\left\{\left[k+\frac{\gamma_1}{4}\,(z+2q^2)\right]v-u\left(q-\frac{\xi}{\beta}\right)\right\} \tag{20}$$

where $z = u^2 + v^2$ is the square of the instantaneous amplitude.

These equations may be easily solved if $\psi = u/v$ and z [42] are selected as variables. Their initial values will be designated by ψ_0 and z_0. Taking:

$$m = \sqrt{\frac{q + \frac{\xi}{\beta}}{q - \frac{\xi}{\beta}}}, \quad 2p = \mu \sqrt{q^2 - \frac{\xi^2}{\beta^2}}, \quad M = \mu\left(k + \frac{\gamma_1 q^2}{2}\right)$$

$$\sigma = \frac{\xi}{pq}, \quad z_{st} = \frac{4\,(M + 2p)}{\mu\,|\gamma_1|}; \tag{21}$$

we obtain the solution in the following form:

$$\psi = m\,\frac{m + \psi_0 - (m - \psi_0)\,e^{-2p\tau}}{m + \psi_0 + (m - \psi_0)\,e^{-2p\tau}}; \tag{22}$$

$$z = z_{st}\frac{C^2 - 2\sigma C e^{-2p\tau} + e^{-4p\tau}}{\left[\frac{z_{st}}{z_0}\left(C^2 - 2\sigma C + 1\right) - C^2 - 2\sigma\,\frac{M + 2p}{M}\,C + \frac{M + 2p}{M - 2p}\right] e^{-(M + 2p)\tau} + \ldots} \rightarrow$$

$$\rightarrow \ldots \frac{C^2 - 2\sigma C e^{-2p\tau} + e^{-4p\tau}}{+\left[C^2 - 2\sigma\,\frac{M + 2p}{M}\,C e^{-2p\tau} + \frac{M + 2p}{M - 2p}\,e^{-4p\tau}\right]} \tag{23}$$

Equations (22) and (23) approximately describe what occurs beginning from any initial conditions whatsoever as well as the case when the system is not excited ($k < 0$) and also when it has self-exciting ($k > 0$). If $q = 0$, $\varepsilon = 0$, $p = 0$, i.e. if there is no applied force, we resort to the Van der Pol solution for autonomous systems [6].

It is evident from equation (23) that z only tends toward a constant value z_{st} different from zero when p is real and $M + 2p > 0$, i.e. when

$$q^2 > \frac{\xi^2}{\beta^2} \tag{24}$$

and

$$k + \frac{\gamma_1 q^2}{2} + \sqrt{q^2 - \frac{\xi^2}{\beta^2}} > 0; \tag{25}$$

these are the existence conditions for constant solutions in the case of u and v. When these conditions are satisfied, periodic oscillations whose period is double that of the

EMF will establish themselves in the system (c.f. Equation 18). These oscillations with double period only appear in a limited frequency interval that is characteristic of the system and, by its appearance, the phenomenon calls to mind certain resonance phenomena. The term “resonance of the second class” (and more generally of the n-th class) recalls that this theory is closely connected to the existence of periodic solutions of the second class of Poincaré [2].

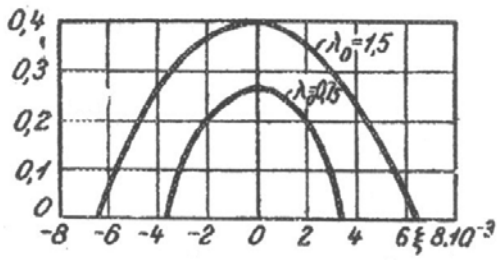

Fig. 39. Les courbes de résonance de la seconde espèce (théoriques).

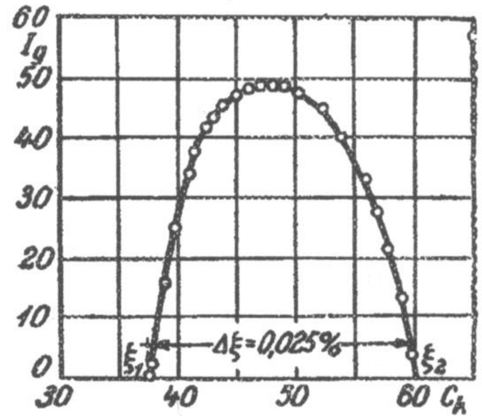

Fig. 40. La courbe de résonance de la seconde espèce (experimentale).

$\omega = 13{,}9 \cdot 10^5$.

Given that $\gamma_1 < 0$, condition (25) cannot be satisfied in the case of a potentially self-oscillating system ($k < 0$) except when the value of q is included within a certain interval $q_{min} < q < q_{max}$. We state that there is a threshold and a ceiling for the value of EMF that is capable of exciting a potentially self-oscillating system with double period oscillations. In the case of self-oscillating systems, condition (25) is satisfied, no matter how small q may be. There is, therefore, no threshold for the automatic synchronization of a self-oscillating system with a period that is double that of the applied EMF.

The square of the amplitude of the stationary oscillations, with a period double that of the EMF, which have been excited by resonance of the second class, is provided by the formula:

$$z_{st} = \frac{4}{|\gamma_1|}\left[k + \frac{\gamma_1 q^2}{2} + \sqrt{q^2 - \frac{\xi^2}{\beta^2}}\right] . \qquad (26)$$

According to this formula, the stationary amplitude is a function of the detuning parameter, ξ, quite apart from the case of ordinary resonance. The curves which provide z as a function of ξ -- they can be termed resonance curves of the second class – are shown in Figure 39 (theoretical) and in Figure 40 (experimental). Formula (26) likewise provides the stationary amplitude as a function of the value of the applied EMF. This function (“the characteristic amplitude”) is depicted in Figure 41 (theoretical) and in Figure 42 (experimental). Note that the “excitation band” (i.e. the frequency interval in which the second class resonance occurs) is equal to zero in the case of $q = q_{min}$, and becomes wider at first as q increases, then decreases and again drops to zero in the case of $q = q_{max}$.

The increase of oscillations up to the stationary amplitude has quite another characteristic than in the case of ordinary resonance as shown in Figure 43. The latter depicts the

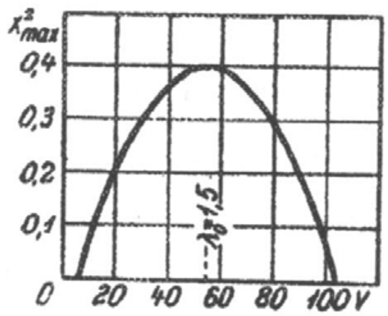

Fig. 41. Caractétistique d'amplitude (théorique).

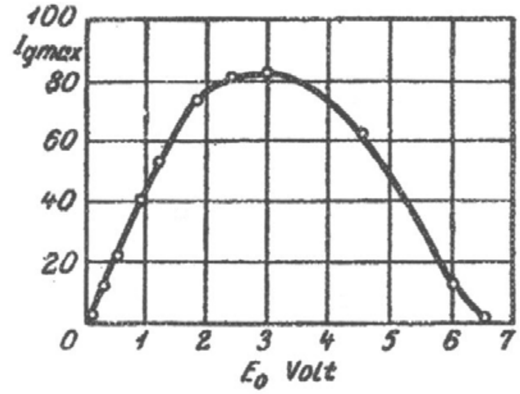

Fig. 42. Caractéristique d'amplitude (expérimentale).

variation of amplitude as a function of time, beginning from the instant at which the external force becomes a factor in the resonance of second class (curve 2) and in ordinary resonance (curve 1).

Let us note the similarity between curve 1 and the curve expressing the increase in amplitude of self-oscillating phenomena. This similarity is not accidental. In ordinary resonance, the excitation of oscillations takes place no matter what the initial conditions may be and can start more particularly beginning from absolute equilibrium ($i = 0$, $di/dt = 0$). In resonance of the second class pulses, whether they are very small or not, are necessary to cause the system to deviate from the initial state $z_0 = 0$. Under the effect of an external force, the position of equilibrium of a potentially self-oscillating system that satisfies resonance conditions of the second class becomes an unstable focus enclosed in a stable limiting cycle.

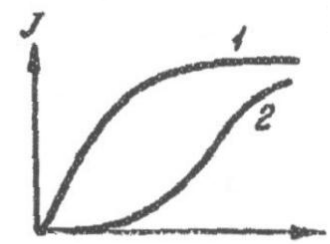

Fig. 43.

The special characteristic of the curve of oscillation growth, in resonance of the second class, can be used advantageously for practical ends (c.f. below)

When the system is in an abrupt mode of operation, resonance of the second class shows certain special features. Thus, at the boundaries of the regions of excitation, "resistance" phenomena can be observed owing to the partial superposition of different ranges of dynamic stability. If the characteristic of the tube is expressed by a fifth degree polynomial, which was done successfully by Appleton and Van der Pol [43] as well as other authors [44], then, by applying the methods of Poincaré (c.f. Section 2) to the case of self-oscillations, it is possible to provide an approximate theory of the phenomena. Figure 44 gives resonance curves of the abrupt mode of operation calculated in this manner. These agree closely with experimental results (Figure 45).

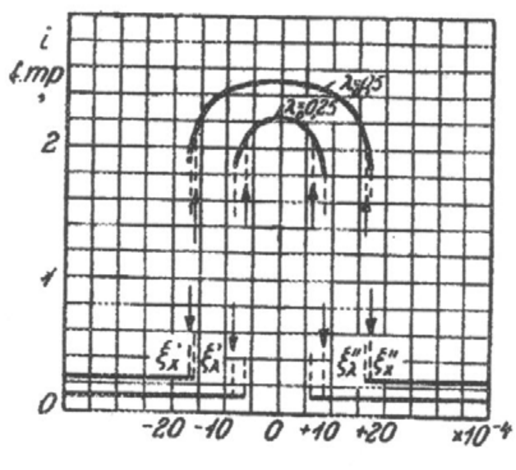

Fig. 44.

When an experimental study is made of the resonance of the second class in systems with abrupt excitations, certain precautions should be taken in order to avoid the phenomena of asynchronous excitation as defined above. In one of our laboratories (Central Radio Laboratory), it has been shown theoretically [39] and confirmed experimentally, by E. Roubtchinski [45]), that asynchronous excitation is only possible if the mode of operation is abrupt, and if the values of the regenerative feedback and the amplitude of the EMF are each included within a specific range.

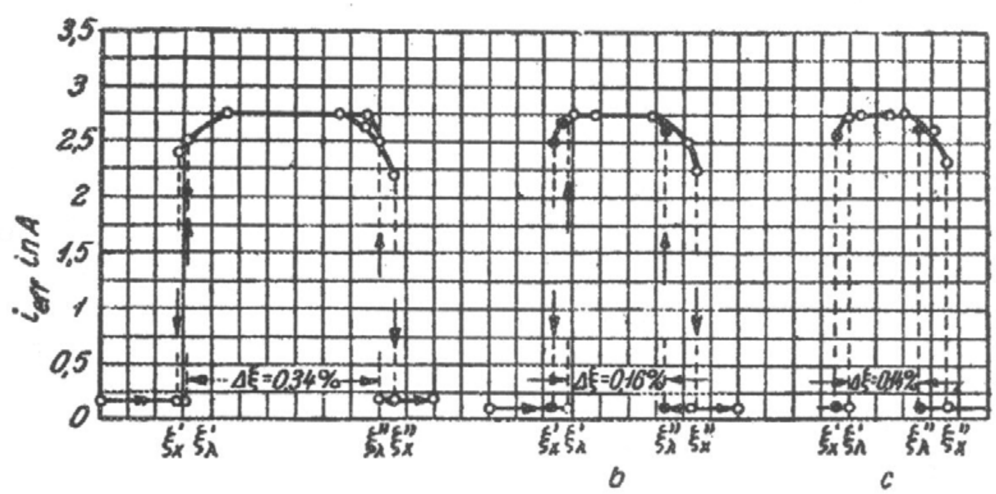

Fig. 45. a $E_0 = 163\ v_{eff}$; b $E_0 = 101\ v_{eff}$; c $E_0 = 79\ v_{eff}$.
$E_a = 1\,660\ v$, $E_g = 130\ v$, $V_f = 8{,}34\ v$, pour le tube M 28, $\omega = 1{,}047 \cdot 10^6$

Figure 46 shows the modes of operation corresponding to the different values of the feedback factor k. The region to the right of zero is that of spontaneous excitation. The region of resistance is found between zero and A ($0 > k > \gamma^2/(8|\xi_1|)$). Assuming the circuit is suitably tuned and that the EMF has a suitable value, it is possible to produce resonance of the second class to the left of A. In addition, phenomena of asynchronous excitation can occur in the shaded portion between A and B ($-\gamma^2_1/(8|\xi_1|) > k > -\gamma^2_1/(6|\xi_1|)$). In order to produce resonance of the second class, in the pure state, it is necessary to work to the left of B. This is a very important matter for receiving stations making use of this phenomenon.

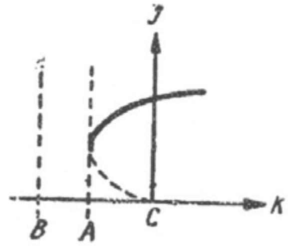

Fig. 46.

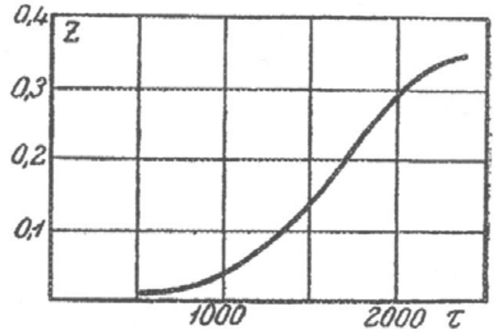

Fig. 47.

In an abrupt mode of operation, the transitory phenomena in resonance of the second class have been studied by A. Melikian [46]. His experiments show that the phase is set up much more quickly than the amplitude. In the case of a soft mode of operation, theory gives the same result, provided that customary numerical values of the parameters are used.

Assuming this condition to be true, as a hypothesis in his calculations, Melikian obtained a relatively simple set of theoretical formulae for the abrupt mode of operation. Figure 47 shows one of the theoretical curves of growth of oscillations. On these pages, we have reproduced some oscillograms produced by Melikian. The one shown in Figure 48, which

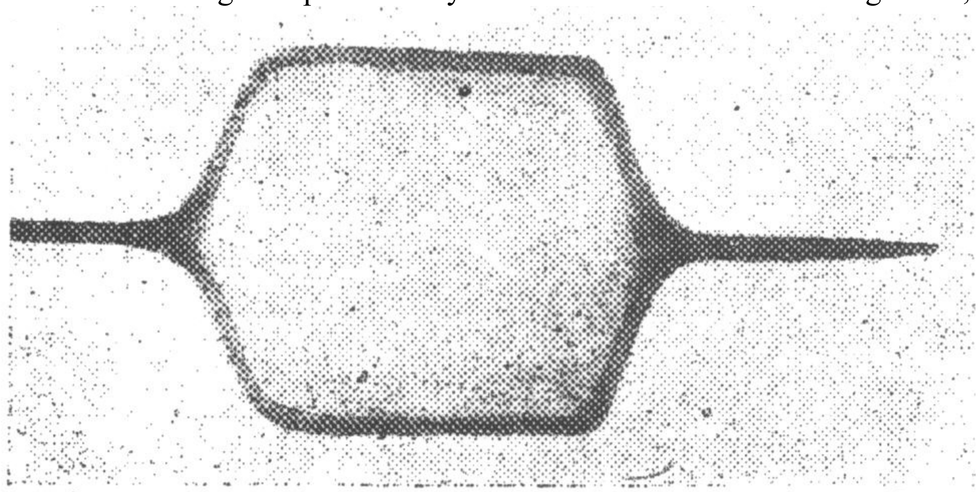

Fig. 48.

has been produced using an electronic oscillograph with an incandescent cathode requiring the synchronous repetition of the phenomenon, shows the effect of a rectangularly shaped signal. In Figures 49 and 50, the oscillograms shown illustrate the increase and decrease of oscillations in a triode transmitter, and in a potentially self-oscillating system, under the effect of a force of double the frequency compared to its own frequency. They were taken using an oscillograph with a cold cathode and with an internal photographic device which allowed the recording of a single event.

The theory developed for resonance of the n-th class likewise allows for the analysis of automatic synchronization phenomena of self-oscillating systems at a frequency equal to a submultiple of the frequency used [37]. Just as in ordinary locking (c.f. Section 5) there is no threshold for the EMF and the decrease of the latter only causes the synchronization band to become narrower. Theoretical and experimental data show that, above a certain limit, the amplitude of the "self-oscillations" decreases when the amplitude of the force

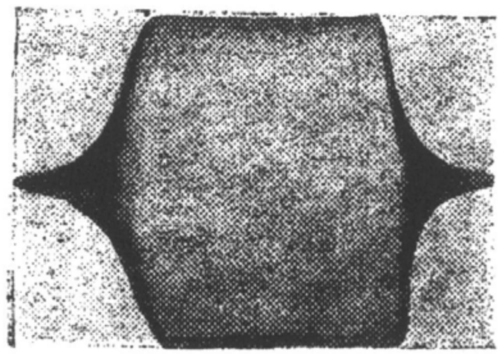

Fig. 49.

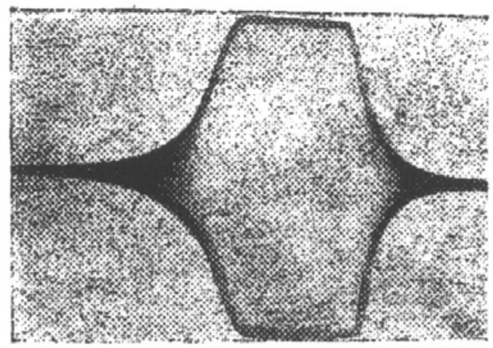

Fig. 50.

increases, and, beginning at a specific value of the latter, becomes equal to zero. The self-excitations are damped by the EMF, and only forced oscillations having the frequency of the EMF remain in the system.

Theoretical researches conducted by the Central Radio Laboratory have shown that it is impossible to produce oscillations whose resonance is of the third class in a potentially self-oscillating system with one degree of freedom and a soft mode of operation. Experimental data show (Tschikhatchov) that oscillations synchronized in a period three times that of the EMF or resulting from asynchronous excitation can be "driven" to the left of B (Figure 46). In potentially self-oscillating systems with two or more degrees of freedom, it is possible to excite oscillations whose resonance is of a higher order. Tschikhatchov produced resonances phenomena of the fourth class in the installation described in Figure 51.

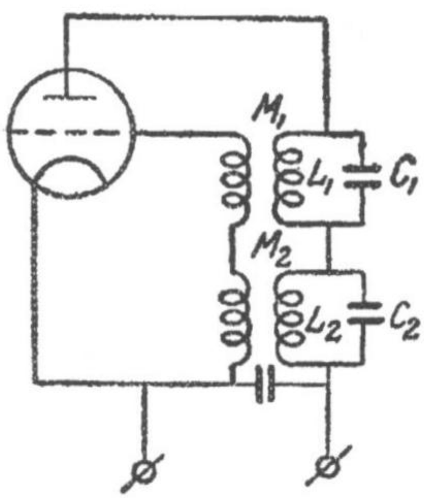

Fig. 51.

Resonance phenomena of the n-th class have a certain relationship with the excitation of oscillations by periodic variation of a system's parameters. This is the "parametric excitation" to be discussed in the following section (Section 7). Consequently, in reality, we can interpret the excitation by resonance of the n-th class in a purely qualitative manner. The external force acting on a potentially self-oscillating system causes "forced" oscillations to appear, which have the same period as the force. By reproducing the reasoning used to analyze the stability of motion by the methods of Poincaré and Liapounov, we can consider our system to be nonlinear and, in the vicinity of the forced oscillations, to be a linear system whose parameters are functions of the forced solution q sin nt. The properties of this linear system, with parameters varying periodically, are well known (Section 7). If it is located in one of the regions of instability, the forced oscillation will be unstable and the system will perform increasing oscillations whose frequency will be a submultiple of that of the force. This method of reasoning offers certain advantages, and it is often useful to consider the resonance of the n-th class as a sort of parametric excitation. In order to distinguish the parametric phenomena in their true sense, which occur when the parameters of an electrical or mechanical system are caused to vary indirectly with those of resonance of the n-th class, we shall call the first ones heteroparametric and the second ones self-parametric.

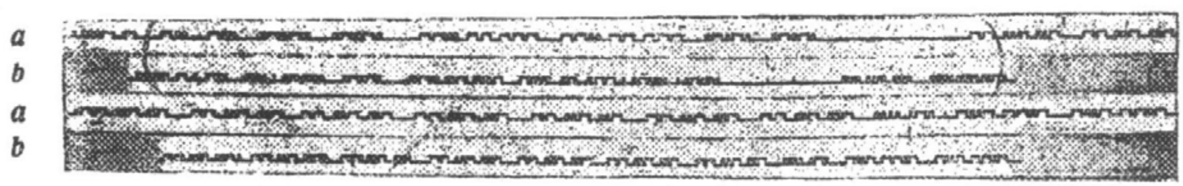

Fig. 52.

The phenomena that have just been described possess encouraging properties for certain advantages in their practical application. Experience has shown that resonance of the second class can be used successfully for frequency reduction as well as the generation of very high amplifications in cases where the frequency should remain very stable, as, for example, in the case of transmitters with independent excitation, and above all in the case of reception.

The resonance curves of the second class (with abrupt edges, the existence of a threshold, and a ceiling for the applied EMF) give rise to new methods for selective reception. Nevertheless, of course, it should not be forgotten that with receivers as well as with high-speed operating automatic transceivers, we are not only dealing with stationary phenomena, but with increasing and decreasing phenomena which, as we have seen (Figure 43), are basically different from those occurring in the case of ordinary resonance.

Experiments performed in 1930 and 1931 showed that, under practical conditions, a device with a resonance of the second class has applications in a radio receiver. The device was used as a selective filter ("self-parametric filter") and gave excellent results. Figure 52 shows a photograph of two simultaneous recordings of signals emitted by radio station WCI (wavelength 16,317 meters) made on 4 February 1931 at the radio receiving exchange of Boutovo near Moscow. Track A comes from a radio receiver furnished with a self-parametric filter. Track B comes from a radio receiver that has a crystal filter. The difference is obvious.

Prolonged tests on a radio receiver with a self-parametric filter were performed at Sagaredjo, near Tiflis, in a region that is subject to intense atmospheric disturbances, and these have shown that this filter is very effective in distinguishing an extended harmonic signal from a static crash.

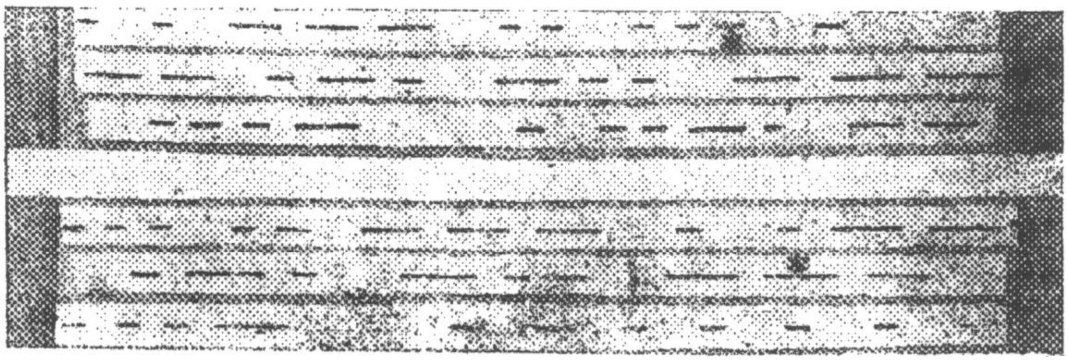

Fig. 53.

This property of systems with resonances of the second class is explained by the characteristics of the law according to which oscillations increase. In a linear filter, a brief impulse (with respect to time duration) that is sufficiently strong can cause oscillations whose amplitude is comparable or even greater than the desired signal. However, owing to the peculiar features of the curve of growth for a self-parametric filter, it only gives small oscillations. In this way, the auto-parametric (or self-parametric) filter practically suppresses atmospheric disturbances that have the form of short impulses, even though they may attain a considerable value. This insensitivity to atmospheric impulses remains when they are superimposed on a signal. However, very strong atmospheric discharges have the effect of "segmenting" the signal (as was observed by I. Borouchko and N. Weissbein in 1931). In recordings on tape there are certain signs (dots or dashes) that show discontinuities (1, 3 on Figure 53).

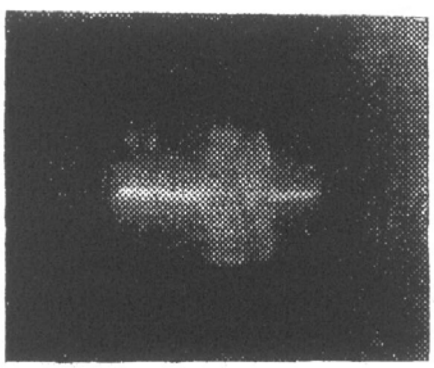

Fig. 54.

A. Melikian studied this last phenomenon in one of our laboratories (the Central Radio Laboratory). Using the oscillographic method he very skillful examined in detail the simultaneous effect of a signal plus short pulses (a train of damped oscillations) in a self-parametric system. He showed that "segmenting" (breaking up) occurs when the impulse arrives at an instant for which the double period oscillations, due to the signal, are already almost established (Figure 54). However, if the pulse arrives at a time when the double period oscillations are still weak, it accelerates their growth.

Figures 55 and 56 allow comparison of the oscillograms for growth and decay of a self-parametric system (Figure 55) with those of a linear system (Figure 56).

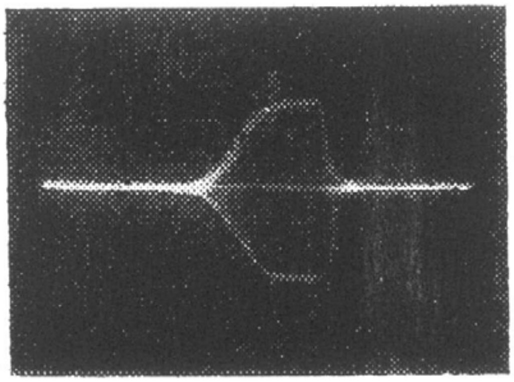

Fig. 55.

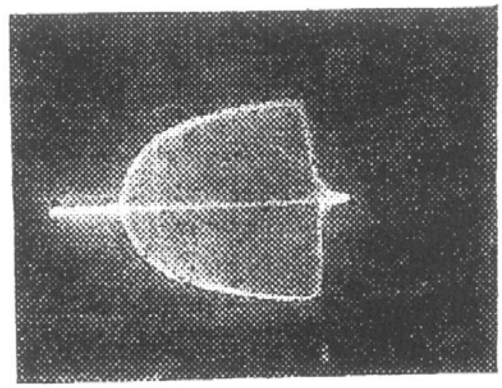

Fig. 56.

Section 7: Parametric Excitation

The phenomena incited in nonlinear circuits by an external stimulus (Section 6) are closely related to the excitation of oscillations by periodic variation of the parameters of an oscillating system. This effect, which can be called parametric excitation (for short), has been known to physicists for quite some time. (Melde [47], Rayleigh [48]) The great importance that it has in radio engineering is likewise known. However, although the possibility of parametric excitation of electrical oscillations has been known for a long time (Rayleigh [49], Poincaré [50], Brillouin [51] and later Van der Pol [52]), it is only in the last few years that the full value of this phenomenon was realized, and its systematic study undertaken. We should like to mention the experiments of Heegner [53] and Guenther-Winter [54] concerning the excitation of electrical oscillations at acoustic frequencies by alternately magnetizing the iron core of a self-induction coil, as well as the experiments of Guenther-Winter [55] and I. Wantanabe, T. Saito, and K. Kaito [56] on the excitation of electrical oscillations by mechanical periodic variation of the self-induction or capacitance of an electrical oscillating system.

We have also performed experiments on the parametric excitation of electric oscillations by mechanical periodic variation of the self-induction [57] of a circuit, but using very different devices from those used by Guenther-Winter and Wantanabe. In addition, we produced the parametric excitation of electric oscillations by periodic variation of the capacitance of a circuit[10] [58].

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[10] W.L. Barrow (Proceedings of the Institute of Radio Engineers, Volume 22, p. 210, 1934) wrongly assumed that his experiments showed the capability for the parametric excitation of an oscillating circuit by periodic variation of its capacitance. He caused the variation of not only the capacitance of an oscillating circuit, but also the ohmic resistance of a shunt containing a condenser. Now, the variation of a positive resistance can be carried out (and is carried out) without expenditure of energy. Therefore, by

Theoretically, the authors mentioned are not limited to the use of a linear differential equation with periodical coefficients, which provides the excitation conditions but can say nothing as to the capability and characteristic of a stationary mode of operation. Now, this question is no less important than the preceding one. This is why, in order to be complete, we began with a general overview of the theory of parametric excitation, a theory that should be supported by a nonlinear differential equation.

It is easy to show by energy considerations that it is possible to excite oscillations in the latter by causing the capacitance of a circuit to vary suitably. Let us assume that, at an initial instant in time, $t = 0$, when the current is equal to zero and the condenser possesses a charge q, we reduce its capacitance by a small quantity ΔC. Having done this, we supply the work $(\Delta C/2C^2)\, q^2$. Let us then allow the condenser to discharge, and at the instant $t = T/4$ (T being the period of the circuit) when all the energy is magnetic and the charge on the condenser is zero, let us restore its capacitance to its initial value. We do this with no expenditure of work. At the instant $t = T/2$ the current is reduced to zero and the condenser carries a charge that is greater or smaller than q, depending on whether the energy supplied to the system at reduced capacitance is bigger or smaller than the energy dissipated. At the instant $t = T/2$, the cycle of variation of capacitance is complete. Let us state this differently. The oscillations will gradually increase, *no matter how small the initial charge* (on the capacitor) provided that the following condition is fulfilled:

$$\frac{\Delta C}{2C^2} q^2 > \frac{1}{2} R_i \frac{\pi}{\omega} \quad , \qquad m > \frac{\varepsilon}{2}$$

in which

$$\varepsilon = \frac{\pi R}{\omega L}$$

ε being the mean logarithmic decrement of the system and

$$m = \frac{\Delta C}{2C} = \frac{C_{max} - C_{min}}{C_{max} + C_{min}}$$

being the modulation index of the varying parameter.

The initial charge q is always present, even in the absence of outside disturbed inductions (electric field lines, atmospheric discharges), owing to statistical fluctuations.

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this principle alone, the device does not allow supplying to the circuit (by mechanical work) the energy that is necessary for the excitation and maintenance of oscillations. There is no doubt that Barrow observed in his experiments not only the parametric excitation of oscillations by periodic variation of the capacitance, but also phenomena owing to the presence of an electronic tube and from regenerative feedback.

By causing a periodic variation of the circuit capacitance (by some mechanical process) at a frequency double that of the circuit's own frequency, we can thereby excite electrical oscillations without using any EMF. A similar reasoning is applicable in the case of mechanical variation of the circuit's self-induction.

This abridged discussion is enough to show that, in order to produce parametric excitation, two conditions should be satisfied:

1. The frequency of variation of the parameter should be suitably selected (in our example it is double the frequency of the circuit).
2. In the case of a given mean logarithmic decrement, the modulation index of the parameter should be sufficiently high.[11]

A more complete study of the initiation of oscillations in phenomena of parametric excitation leads, as is known, to the discussion of "unstable" solutions of linear differential equations with periodic coefficients. If for example, the capacitance varies according to the law:

$$\frac{1}{C} = \frac{1}{C_o}\left(1 + m\cos\nu t\right)$$

We have for $q = \int i \, dt$ the equation

$$L\frac{d^2 q}{dt} + R\frac{dq}{dt} + \frac{1}{C}\left(1 + m\cos\nu t\right)q = 0 \tag{28}$$

which leads to the form

$$\ddot{x} + \lambda^2\left(1 + m_1\cos 2\tau\right)x = 0 \tag{29}$$

(the equation of Mathieu), by taking

$$q = x\,e^{-\frac{R}{2L}t}, \qquad \tau = \frac{\nu t}{2}, \qquad \lambda^2 = \frac{2(\omega_o^2 - \delta^2)}{\nu^2}$$

$$\omega_o^2 = \frac{1}{LC_o}, \qquad 2\delta = \frac{R}{L}, \qquad m_1 = m\frac{\omega_o^2}{\omega_o^2 - \delta^2}$$

Equation (29) was discussed from a mathematical viewpoint by Mathieu, Hill, Poincaré, etc. It was also discussed, with respect to our present problem, by Rayleigh, then by Andronov and Leontovitch [59] and by Van der Pol and Strutt [60]. It is known that equations of the same type appear in a great number of problems of celestial mechanics, optics, elasticity, acoustics, etc. The general solution of equation (29) is in the form:

$$x = C_1 e^{h\tau}\chi_o(\tau) + C_2 e^{-h\tau}\chi(-\tau)$$

---

[11] In the case of sinusoidal variation of capacitance the condition $m > \varepsilon/2$ is replaced by $m > 2\varepsilon/\pi$.

$\chi(\tau)$ being a periodic function. In order for there to be parametric excitation, it is necessary for h to have a real part which has an absolute value that is greater than δ. This condition between the parameters λ and m defines the "unstable regions" of equation (28). They are located in the vicinity of $2\omega_1/\nu = n$, n being an integer. Their boundaries can be calculated by the approximate method of Rayleigh [61]. Thus, in the case of the first region of instability (n = 1), we have, approximately, with terms of order m^2:

$$\sqrt{1+\sqrt{\frac{m^2}{4}-4\vartheta^2}} \geq \frac{2\omega_1}{\nu} \geq \sqrt{1-\sqrt{\frac{m^2}{4}-4\vartheta^2}} \tag{30}$$

In order to find the second region (n = 2), terms on the order of m^4 should be taken into account. It follows that:

$$\sqrt{4+\frac{2}{3}m^2+\sqrt{m^4-64\vartheta^2}} \geq \frac{2\omega_1}{\nu} \geq \sqrt{4+\frac{2}{3}m^2-\sqrt{m^4-64\vartheta^2}} \tag{31}$$

The width of the regions of instability decreases as m^n.

As is shown by relations (4), (5) in order for the initiation to be possible, it is necessary that, in the case of n = 1:

$$m > 4\vartheta \tag{32}$$

and in the case of n = 2:

$$m > 2\sqrt{2\vartheta} \tag{33}$$

The modulation index required for initiation is therefore greater, the decrement remaining the same, in the case of n = 2, than for n = 1. Initiation becomes still more difficult in the case n = 3, 4, etc. This is why the case of n = 1 is, for practical purposes, the most interesting. It is the only one that we shall discuss here.

Once conditions (30) or (31) were satisfied, if the linear equation (29) was exact for any values of q, the amplitude of the oscillations would increase without bound. Therefore, in order for a system with periodically varying parameters to reach a stationary mode of operation and become a generator of alternating current, it is necessary that it conform to a nonlinear differential equation[12]. In this case, the linear equation (2) is only valid (approximately) in the case of sufficiently small amplitudes. It allows only the setting up of conditions (30), (31) which should confirm the parameters for which there was initiation.

As will be seen, our experiments confirm this manner of regarding the phenomenon. In order to obtain a permanent mode of operation, it is necessary to introduce into the circuit

[12] The problem of the frequency modulation of a triode oscillator, also a nonlinear one, has been studied by S. Rytov in an article carried in the journal Technical Physics USSR.

nonlinear components such as an iron cored coil, incandescent lamps, etc.; in the first case, the equation of the problem is:

$$\frac{d\varphi(\dot{q})}{dt} + R\dot{q} + \frac{1 + m\cos 2\omega t}{C_o} q = 0 \tag{34}$$

with the nonlinear dependence of the flux on the current, $\varphi(\dot{q})$, being given in the form of a polynomial, for example. The mathematical theory of the phenomena includes the search for periodic solutions of equation (34) and the discussion of their stability, in addition to investigating the condition in which the state of equilibrium becomes unstable (condition of initiation).

If the nonlinear part of the self-induction is small with respect to its linear part and if, in addition, m is small, it is possible to apply the methods of Section 3 to this equation. In the simplest hypothesis, in which

$$\dot{\varphi}(\dot{q}) = A + L_o\dot{q} + \beta\dot{q}^2 + \gamma\dot{q}^3 \tag{35}$$

by setting

$$\tau = \omega t, \qquad \omega_o^2 = \frac{1}{L_o C_o}, \qquad \xi = \frac{\omega^2 - \omega_o^2}{\omega^2} \quad \chi = \frac{q}{q_o}$$

it follows that

$$\ddot{\chi} + \chi = \mu f(\chi, \dot{\chi}, \tau)$$

whose solutions approximately satisfy the "truncated" equations

$$\dot{u} = -\frac{1}{2}\left\{\left[\frac{m}{2}(1-\xi) + 2\vartheta\right]u + \left(\xi + \frac{\gamma_1}{4}z\right)v\right\}$$

$$\dot{v} = \frac{1}{2}\left\{\left[\frac{m}{2}(1-\xi) + 2\vartheta\right]v + \left(\xi + \frac{\gamma_1}{4}z\right)u\right\}$$

which can be treated by the methods of Section 2. The stationary solution is:

$$\frac{\gamma_1}{4}z = -\xi + \operatorname{Sign}\gamma_1\sqrt{\frac{m^2(1-\xi)^2}{4} - 4\delta^2} \tag{36}$$

Whence the condition:

$$m(1-\xi) > 4\delta \tag{37}$$

which is practically identical to equation (32).

The curves of Figures (37) and (38), which can be called heteroparametric resonance curves (c.f. Section 5), provide z (the square of the amplitude) as a function of the "detuning" ξ. They differ essentially from the ordinary resonance curves and resonance curves of the second class. As shown by Figure 57, whereas:

$$\xi < -\sqrt{\frac{m^2}{4}(1-\xi)^2 - 4\delta^2} \tag{36}$$

(where $\gamma_1 < 0$)

no oscillations are observed. When:

$$\xi_1 \;=\; -\sqrt{\frac{m^2}{4}\,(1-\xi)^2 \;-\; 4\delta^2}$$

the oscillations begin, starting with small amplitudes and, z increasing, are gradually amplified. z increases linearly until, when the detuning assumes the value

$$\xi_1 \;>\; \sqrt{\frac{m^2}{4}\,(1-\xi)^2 \;-\; 4\delta^2} \quad ,$$

the oscillations stop abruptly.

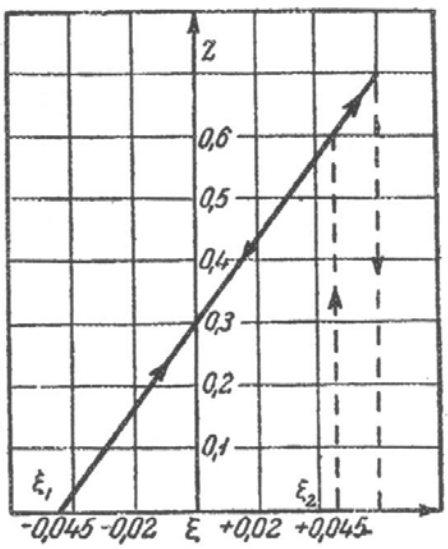

Fig. 57. $\gamma_1 = -0{,}15$; $m = 0{,}15$; $\vartheta_0 = 0{,}03$.

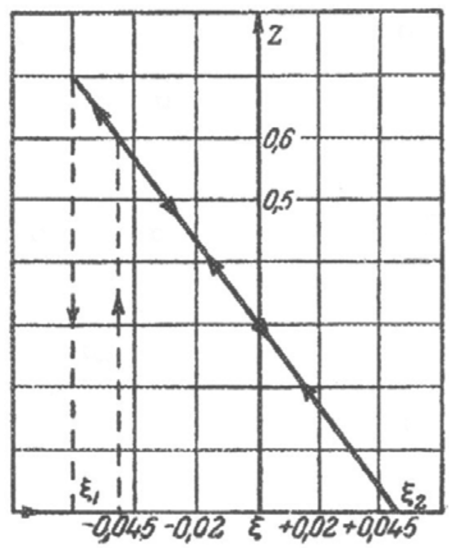

Fig. 58. $\gamma_1 = +0{,}15$; $m = 0{,}15$; $\vartheta_0 = 0{,}03$.

As shown by equation (10), the zero-order approximation theory is only limited on one side by the detuning ξ. Stable, finite amplitudes exist outside of the range of values of ξ in which the conditions of initiation are fulfilled. In other words, the parametric oscillations, once excited, can be "driven" into the regions where the equilibrium is stable. When ξ varies in a reverse direction, the oscillations appear when $\xi = +\xi_1$, then decrease and vanish when $\xi = -\xi_1$. Resistance, therefore, appears on only one side. In order to calculate the extent of the circuit resistance as well as to solve several other questions, μ will have to be used and the harmonics taken into consideration. Figure 58 shows that when $\gamma_1 > 0$ the phenomenon is reversed: z increases whereas ξ decreases, and the circuit resistance appears for positive values of $\xi = +\xi_1$.

These two cases were observed experimentally by W. Lazarew [66]. V. Gouliaev and V. Nigouline [62] and have shown that the same results are produced by expressing the flux, as Dreyfus [63] and Zenneck [64] have proposed, by the function:

$$\Phi(\dot{q}) \;=\; \Phi_o \,\mathrm{arc\,tg}\,(k\,\dot{q}) \;+\; L_2\,\dot{q}\,.$$

We set the production of the effect of parametric excitation and the verification of the theory described above as a goal in our experiments.

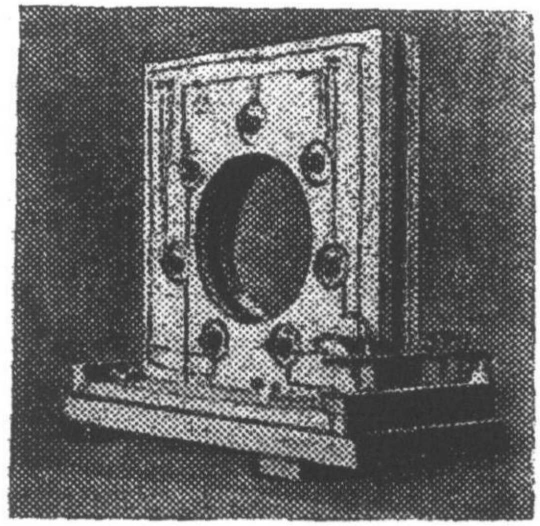

Fig. 59.

In the first experiments we cause the self-induction of a circuit to vary by using the device shown in Figures 59, 60, 61. The variable self-induction was composed of seven pairs of flat coils fastened face to face with two parallel discs on the periphery of two perimeters. A serrated metallic disc was able to rotate within the empty space fixed between the coils. The teeth, likewise, seven in number, were cut in such a fashion as to synchronously occupy and vacate the region simultaneous with the field of the coils. As the disc spins, the self induction of the circuit decreases when the teeth enter the field of the coils, and then increases when they leave it. By using a disc made from duraluminum, we were able to achieve a peripheral velocity of 220m/sec, and in this way attain a

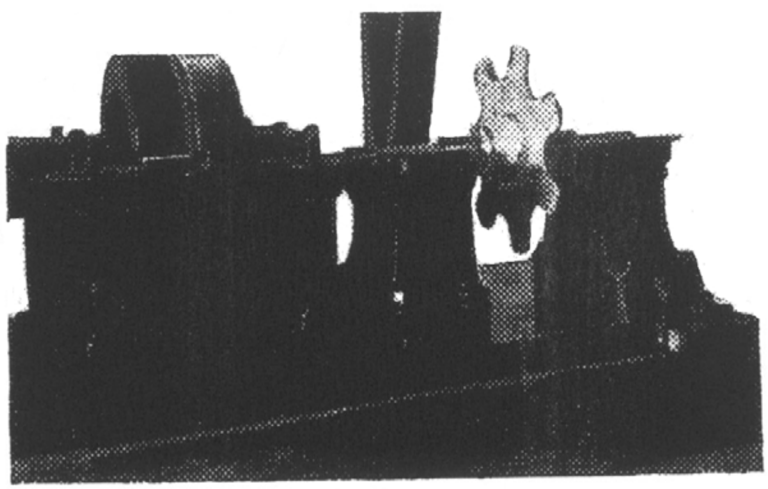

Fig. 60.

considerable rate of variation of the parameter (1700-2000 per/sec). The coils were supplied with iron cores divided in such a way as to concentrate the field and increase self-induction. The apparatus permitted production of the parametric excitation of rather powerful oscillations in the circuit shown in Figure 62, *which has no current or voltage source*. By tuning this circuit to a frequency approximately equal to $\omega/2$, ω being the frequency of variation of the self-induction, the presence of oscillations whose frequency is exactly equal to $\omega/2$ may be ascertained. The amplitude swiftly increased until the apparatus ruptured, either in the condenser or in the conductors of the circuit. In our experiments, the voltage went as high as 12,000-15,000V. As required by theory, it was necessary to introduce a nonlinear component into the system in order to

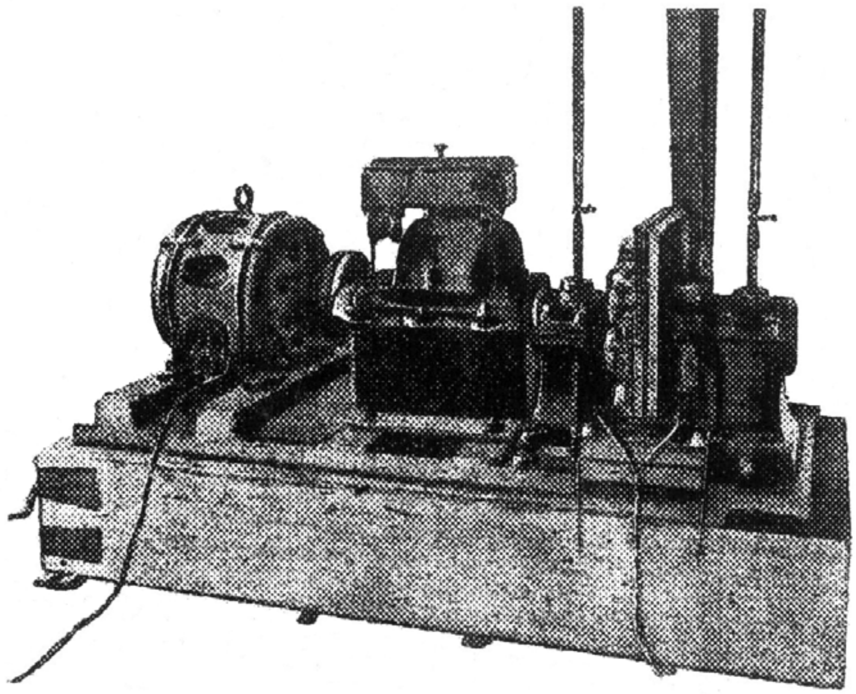

Fig. 61.

produce a stationary mode of operation. In the first experiments this was a bank of 100-watt incandescent lamps connected into the oscillating circuit.

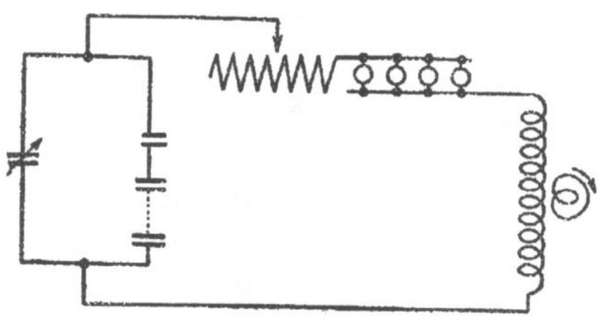

Fig. 62.

More detailed experiments were carried out in our laboratory at the Institute of Electrophysics of Leningrad by W. Lazarew [66] with an apparatus providing a greater modulation index (40% instead of 14% as in the first experiments) and a greater power (as high as 4kW). The device causing the self-induction to vary is depicted in Figures 63 and 64. The duraluminum rotor had eight teeth. The self-induction varied at a rate of approximately 1900 per/sec, which provided oscillations of approximately 950 per/sec. A stationary mode of operation was produced using nonlinear self-induction, either in the iron cores of the coils of the stator or of a coil with a special iron core possessing an auxiliary winding for direct current magnetization. By causing the intensity of the latter to vary, it is possible to shift the operating point of the iron on the curve of magnetic induction and modify the coefficients β and γ of formulae (35), (36).

By measuring the maximum damping occurring with the initiation of oscillations, we find that experiment agrees very satisfactorily with formula (4), as shown by the table below.

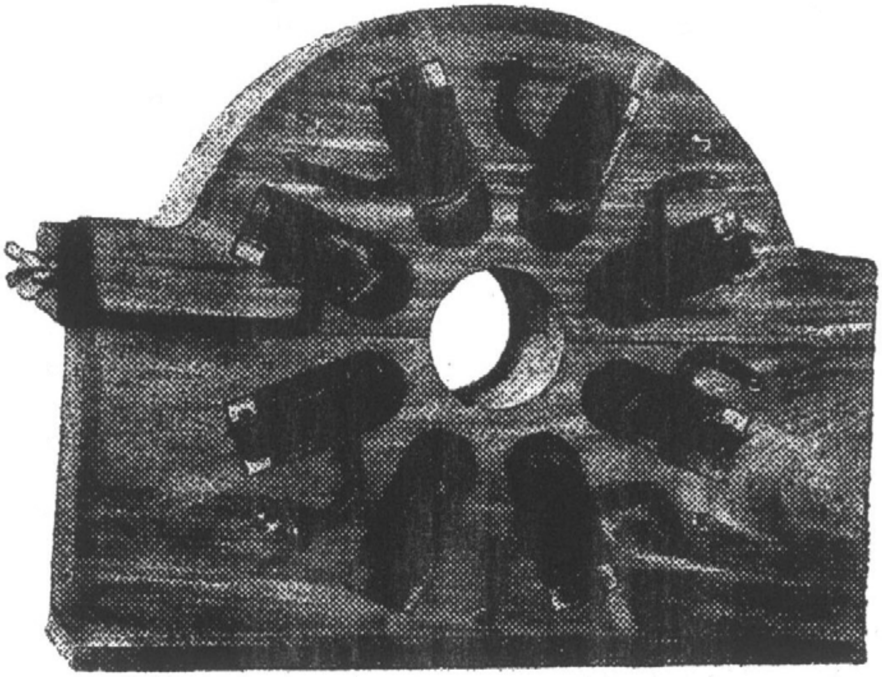

Fig. 63.

The experimental curves of "heteroparametric" resonance clearly have the features indicated below (Figures 65 and 66). They rise and fall according to whether the

TABLEAU 1

	m	δ	$\frac{2}{\pi}\delta$
I	0,144	0,23	0,146
II	0,35	0,57	0,365
III	0,384	0,595	0,375

permanent mode of operation occurs owing to the induction coils themselves or to a special coil magnetized by a continuous current. Figures 67 and 68 are oscillograms of the stationary current. Figure 69 is an oscillogram of the transient regime of operation.

A circuit diagram is provided in Figure 70 for the excitation of electrical oscillations by *the*

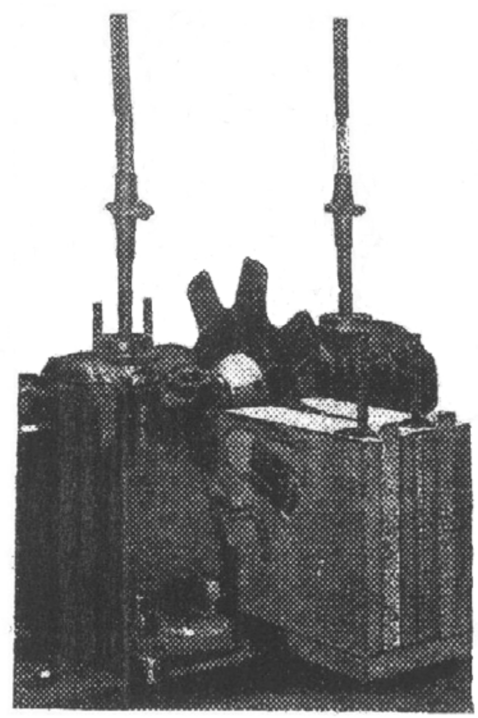

Fig. 64.

periodic mechanical variation of the capacitance of a device. The oscillation circuit is formed by condenser C, whose capacitance varies periodically, shunted with an oil condenser C (serving to tune the circuit) and a self-induction coil L (several sections of the secondary of a nonferrous core inductor). Condenser C (in Figure 71) includes

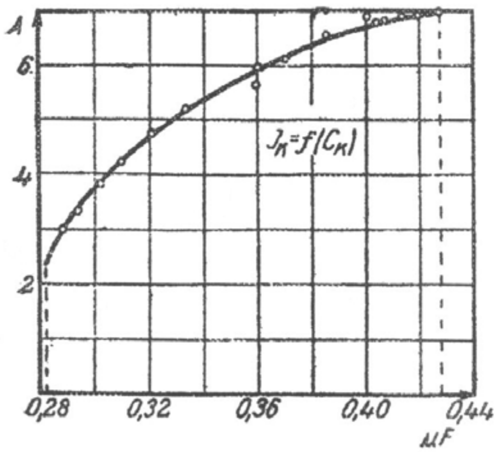

Fig. 65.

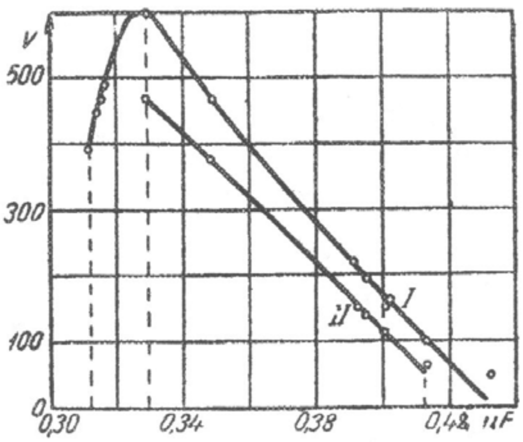

Fig. 66.

I. $v_k = f(C_k)$; $\vartheta_I = 0{,}414$; $i_d = 40$ mA
II. $v_k = f(C_k)$; $\vartheta_{II} = 0{,}47$.

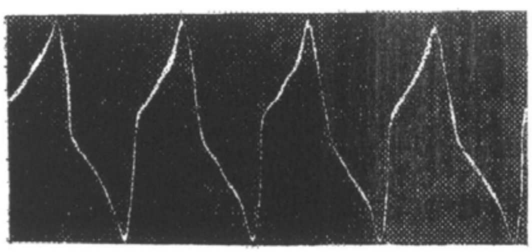

Fig. 67.

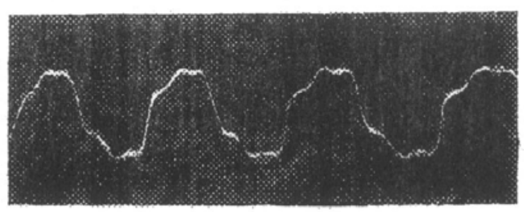

Fig. 68.

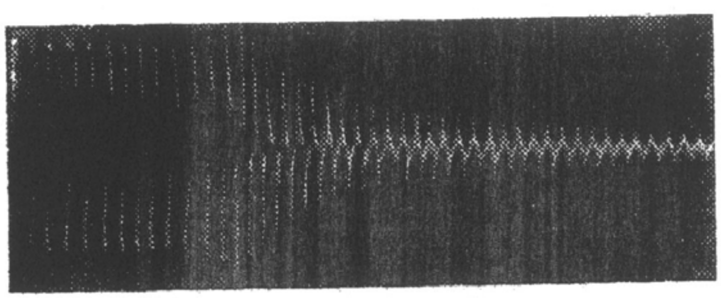

Fig. 69.

two systems of armatures, one stationary (stator) and the other rotating (rotor). The stator is made up of 26 square aluminum plates, each one having 14 radial grooves arranged symmetrically. The rotor is an assembly of 25 circular aluminum plates that are perforated in the same manner as those of the stator and actuated by a direct current motor with a maximum rotational speed of 4000 rpm. When the motor performs at n revolutions per second, the capacitance varies with a frequency of 14n per/sec.

Six 220V neon tubes in series and a Hartmann-Braun static voltmeter of 1200V allowed observation of the presence of oscillations and the evaluation of their intensity. Neon tubes were used to limit the growth of the oscillations.

Since the rotor revolves at a fixed rate, there is a range of values of c in which the voltmeter fluctuates and the neon tubes light up. This range corresponds to the frequencies characteristic of the circuit in the vicinity of n/2. Checking the frequency of oscillations by means of a tuning fork, we were able to ascertain that it is constant within the whole range of excitation and equal to 7n (n being measured with a tachometer).

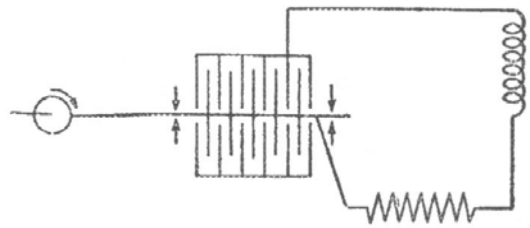

Fig. 70.

If the neon tubes are removed, the system becomes linear, and it is possible to predict that the oscillations will increase until the apparatus is ruptured. This is what actually occurs.

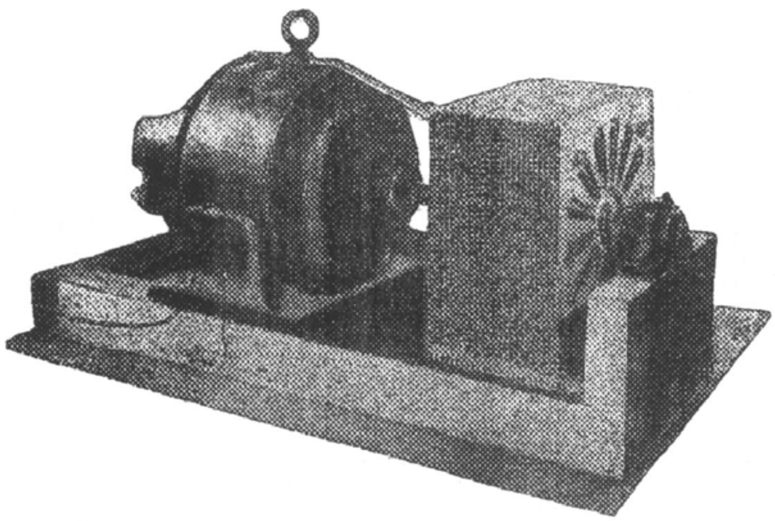

Fig. 71.

The voltage, which the neon tubes kept between 600-700V, increased in their absence until a spark was produced between the armatures of the condenser (at between 2000-3000V). The frequency at which the spark occurs decreased proportionally as the frequency characteristic of the circuit departs from the value of 7n. This observation is likewise supported by theoretical considerations. According to the latter, the increase in oscillations is reduced proportionally as one approaches the boundaries of the region of instability of linear equation (29).

In these experiments the modulation index of the capacitance was 0.175.

The experimental curves of Figure 72 show the amplitude of the voltage excited as a function of detuning and circuit damping. When the latter is increased, the range of parametric excitations is reduced. Its measured width is certainly in agreement with that provided by theoretical considerations.

In conclusion, permit us to add that H. Sekerska (Institute of Physics, Moscow) has provided the details of a new process allowing production of the Melde [47] phenomenon, i.e. the parametric excitation of a vibrating string[13]. A variable weight suspended by a metal wire permits tuning the normal modes of this latter to different frequencies. The wire completes the circuit of an alternating current at 50 per/sec. Consequently, the temperature and, therefore, the *tension* of the wire are caused to vary periodically at the rate of 100 per/sec. Provided the current strength is sufficient, when one of the normal modes of the wire is approximately tuned to a frequency of 50 per/sec, the parametric excitation of this mode is observed.

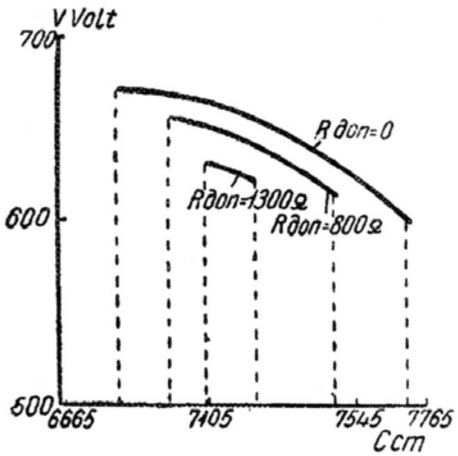

Fig. 72.

Section 8: Forced Oscillations of a System with Periodic Parameters - Parametric Coupling

As we have seen in Section 7, if a system with periodically varying parameters is found in an unstable region, the stationary state cannot be described by a linear equation. But, if it is located in a stable region, the stationary mode differs from equilibrium only owing to the effect of a periodic or quasi-periodic external force. Since the oscillations are small enough, the phenomenon can then be described by an inhomogeneous linear equation with periodic coefficients which, in the simplest case, is in the form:

$$\ddot{q} + 2\delta\dot{q} + \rho(t)q = f(t) \tag{38}$$

where δ is a damping coefficient, $\rho(t)$ a periodic function, and $f(t)$ a periodic or quasi-periodic function. Equation (38) represents a generalization of the well known phenomena of resonances produced by the action of a periodic, or quasi-periodic, force $f(t)$ on a linear system with constant parameters (harmonic resonator). G. Gorielik[14] subjected these phenomena of *generalized resonance* to a detailed theoretical study based on a few considerations of principles that were provided by one of our group [57].

The phenomena, which have their source in harmonic resonators, endow a special physical importance to sinusoidal functions and the harmonic analysis of an arbitrary function. It is the language of sinusoidal functions that is used by theory to treat resonance phenomena in systems with constant parameters.

[13] The publication is in process.
[14] The article is in this issue of Technical Physics of the USSR.

However, this language ceases to be able to handle systems with periodically varying parameters. The function f(t) should satisfy certain conditions in order for there to be resonance. The form of forced oscillations, which are expressed for a resonator with periodic parameters, involves selectivity principles using new periodic or quasi-periodic functions. These may be considered as generalizations of the sine and cosine functions, and are determined by the intrinsic properties of the resonator.

In linear systems with periodic parameters, resonance phenomena, in the absence of an external force and damping, present a character that changes, depending upon the regime of operation of the resonator, i.e. whether the ideal system described by the equation

$$\ddot{q} + \rho(t)q = 0$$

is in a stable region, at the boundary with an unstable region, or in an unstable region. In the first case, the forced oscillations of resonance are proportional to $1/\delta$ just as in the case of a harmonic resonator. In the second case, there are two kinds of resonance: a "strong" resonance in which the forced oscillations are proportional to $1/\delta^2$ and a "weak" resonance where they are proportional to $1/\delta$. If $f(t) = E\cos(\omega t + \varphi)$, it has been found that by varying the phase φ, it is possible to pass out of the weak resonance regime. In the third case the resonance becomes more pronounced as the modulation index increases; if the force is sinusoidal the nature of the phenomena is likewise a function of its phase.

The theory of resonators with periodically varying parameters takes into account certain phenomena that have some similarity to those occurring in regenerative receivers. In the latter, the coupling between the grid circuit and the plate circuit "regenerates". This allows partial restoration of the energy dissipated by the forced oscillations at the expense of the plate battery. The theory of "regeneration" can be made using a linear equation with constant coefficients by disregarding the nonlinear terms of the tube characteristic. The regeneration decreases the coefficient of the dissipating term.

A "regeneration effect" is produced in a similar manner. In other words, by means of utilizing a local source, it is possible to partially compensate for the losses of energy in a circuit performing forced oscillations if one of its parameters is caused to vary at a suitable frequency. The phenomenon is especially advantageous if the frequency characteristic of the circuit, that of the EMF, and that of the variation of the parameter are all in the ratio of 1:1:2. The modulation index plays a role analogous to that of the coupling coefficient in a regenerative receiver. An essential difference between conventional regeneration and this "parametric regeneration" is that this one depends primarily on the phase that the parameter variation has relative to the EMF.

The effect of "parametric regeneration" was observed and studied at the Central Radio Laboratory by Divilkovski and Rytov, as well as by Roubtchinski[15].

The work of our laboratories has revealed new effects of parametric coupling between oscillating systems. They differ essentially from the well-known phenomena that take place in linear coupled systems.

Let us take, for example, a mass suspended by a string with one fixed point. This is the elastic pendulum studied by G. Gorelik and a member of our group [67] regarding a question in optics[16]. When the mass oscillated vertically, the length of the pendulum underwent a periodic variation. If the frequency of the elastic oscillations is double that of the angular oscillations, there will be *parametric excitation* of the latter by the former. (Whence the term "parametric coupling".) With respect to those phenomena discussed in Section 7, the phenomenon has this difference: the variation of the parameter itself is a function of the oscillation that it excites. In effect, the angular oscillation causes the appearance of a centrifugal force at the frequency of the elastic oscillations, and consequently it reacts on the latter by ordinary resonance. The naturally autonomous coupling is expressed in the differential equations of the system by nonlinear terms. Parametric coupling can likewise be seen in self-exciting systems with two degrees of freedom: for example that of Figure 51, which was studied by Tourbovitsch[17]. Since the operating point is selected in such a manner that the polynomial expressing the characteristic of the tube has a term of the second degree, which is clearly indicated, the equations of the system will be in the form:

$$\ddot{x} + \omega^2 x = 2\beta \dot{x} \dot{y} + \ldots$$

$$\ddot{y} + 4\omega^2 y = \beta \dot{y}^2 + \ldots$$

(We have written only the most significant terms.) It can be seen that oscillation y, at frequency 2ω, causes a variation of circuit "resistance" at the critical frequency ω (parametric action) and that, in return, oscillation x generates energy at frequency 2ω, which reacts by resonance on oscillation y.

A. Tscharakhtschian[18] studied the action of a sinusoidal force on two circuits with parametric coupling forming a "parametric transformer"; in this system, the variation of the current in the primary circuit causes the induction coil of the secondary circuit to vary by modifying the magnetization of the iron-core coils. This allowed the production of parametric excitation phenomena.

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[15] The publications are in preparation

[16] This model allows providing a standard qualitative table of certain anomalies of combinative diffusion (Raman effect) with CO_2 molecules, which Fermi (Zeitschrift für Physik, Vol. 71, 1931, p. 250) treated suitably by quantum mechanical methods.

[17] The publication is in preparation.

[18] The publication is in preparation.

Section 9: The Role of Statistics in Dynamic Systems

We shall conclude with a few words on some questions whose theoretical and experimental study has just begun at our laboratories[19], and which relate the theory of oscillations to statistical theories.

Even in the simplest case of oscillation initiation in a triode oscillator, the role of statistics can be clearly seen in the behavior of the system [69], [70]. Even without a deviation from the normal, if, at the initial instant, the system is found in a state of equilibrium, it will always diverge owing to random pulses [71] (produced, for example by random fluctuations). Now, the time for the system to arrive at a stationary state is a function of the magnitude of the initial perturbation (this, of course, concerns the time necessary for the state of the system to arrive at a value differing from the stationary state by a given value). However, in a triode oscillator, the oscillating circuit levels the pulses to a “mean”. This is why their influence is always shown by the formation of small oscillations characteristic of the circuit, whose magnitude is a function of the spectral intensity of the pulses.

This leveling of the pulses will not have time to be carried out if the system is very “fast” one. The instantaneous values of the current and voltage will then be random. The effect of various initial conditions will be directly observed in systems that have a small variation of initial conditions in the establishment of such and such a final state. This is the case, for example, for rocker-relays (Kipp relays) which possess a saddle (at the origin of the coordinates) and two stable nodes (one on the right and the other on the left of the saddle). It is possible to produce a relay having two stable nodes located on the phase plane, to the right and to the left of saddle O and symmetrical with respect to the latter. Let us assume that when the relay is triggered, the representative point is located at O. It is clear that, if the initial perturbations are distributed according to the laws of chance, deviations to the right and to the left will likewise be probable and if, consequently, the system is triggered without applying an external pulse, it will travel either towards the right node or towards the left node, according to a statistical law. This statistical law will be altered if steady pulses are used. By comparing the effect of steady pulses with those of random pulses, it is possible to evaluate the magnitude of the latter. Experiments of this kind were performed in one of our laboratories. The amplified current fluctuations of a vacuum tube were used as a source of random pulses. The magnitude of the fluctuations determined in this way agrees satisfactorily with the well-known theoretical and experimental findings of Schottky.

[19] The experiments are still in progress.

It is possible to raise another question concerning the transition of a system from one state to another owing to the effect of random pulses. This problem was treated theoretically using the Fokker equation. In particular, L. Pontryagin mathematically calculated the expected duration of the transition from one state to the other. Using the results obtained it is possible to compute the duration of the transition from one stationary state to another, thus allowing discussion of the mean duration that the system remained in such-and-such a stationary state. Naturally, this is a function of the amplitude of the random pulses. We were able to experimentally ascertain the existence of these "spontaneous" transitions from one stationary state to the other.

By observing how long the system remained in such-and-such a stationary state, it is possible to determine, the magnitude of the random pulses by using some plausible supplementary postulates. Note that, in principle, the existence of the random pulses limits the precision with which it is possible to attribute a definite period to an oscillatory phenomenon.

Section 10: Closing Remarks on Oscillations

Our desire was to provide a short overview of some research carried out during the last few years in the laboratory of the Institute of Physics of the University of Moscow, the Central Radio Laboratory (Leningrad), the Laboratory of Nonlinear Oscillations of the Institute of Electrophysics (Leningrad) and the University of Gorki. In order to not encumber our report, we have omitted a whole series of questions relating, for example, to systems with several degrees of freedom[20]. And, within the topics that we have discussed, we have had to confine ourselves to the most essential topics. In this way a great number of interesting details have often been sacrificed. References are provided below. Soon a number of works, so far published only in Russian, will appear in other languages, too.

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