III. *On Faraday's Lines of Force.* *By* J. Clerk Maxwell, B.A. *Fellow of Trinity College, Cambridge.*

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The present state of electrical science seems peculiarly unfavourable to speculation. The laws of the distribution of electricity on the surface of conductors have been analytically deduced from experiment; some parts of the mathematical theory of magnetism are established, while in other parts the experimental data are wanting; the theory of the conduction of galvanism and that of the mutual attraction of conductors have been reduced to mathematical formulæ, but have not fallen into relation with the other parts of the science. No electrical theory can now be put forth, unless it shews the connexion not only between electricity at rest and current electricity, but between the attractions and inductive effects of electricity in both states. Such a theory must accurately satisfy those laws, the mathematical form of which is known, and must afford the means of calculating the effects in the limiting cases where the known formulæ are inapplicable. In order therefore to appreciate the requirements of the science, the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress. The first process therefore in the effectual study of the science, must be one of simplification and reduction of the results of previous investigation to a form in which the mind can grasp them. The results of this simplification may take the form of a purely mathematical formula or of a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.

In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies. By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. Thus all the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers. Passing from the most universal of all analogies to a very partial one, we find the same resemblance in mathematical form between two different phenomena giving rise to a physical theory of light.

The changes of direction which light undergoes in passing from one medium to another, are identical with the deviations of the path of a particle in moving through a narrow space in which intense forces act. This analogy, which extends only to the direction, and not to the velocity of motion, was long believed to be the true explanation of the refraction of light; and we still find it useful in the solution of certain problems, in which we employ it without danger, as an artificial method. The other analogy, between light and the vibrations of an elastic medium, extends much farther, but, though its importance and fruitfulness cannot be over-estimated, we must recollect that it is founded only on a resemblance *in form* between the laws of light and those of vibrations. By stripping it of its physical dress and reducing it to a theory of "transverse alternations," we might obtain a system of truth strictly founded on observation, but probably deficient both in the vividness of its conceptions and the fertility of its method. I have said thus much on the disputed questions of Optics, as a preparation for the discussion of the almost universally admitted theory of attraction at a distance.

We have all acquired the mathematical conception of these attractions. We can reason about them and determine their appropriate forms or formulæ. These formulæ have a distinct mathematical significance, and their results are found to be in accordance with natural phenomena. There is no formula in applied mathematics more consistent with nature than the formula of attractions, and no theory better established in the minds of men than that of the action of bodies on one another at a distance. The laws of the conduction of heat in uniform media appear at first sight among the most different in their physical relations from those relating to attractions. The quantities which enter into them are *temperature, flow of heat, conductivity.* The word *force* is foreign to the subject. Yet we find that the mathematical laws of the uniform motion of heat in homogeneous media are identical in form with those of attractions varying inversely as the square of the distance. We have only to substitute *source of heat* for *centre of attraction, flow of heat* for *accelerating effect of attraction* at any point, and *temperature* for *potential*, and the solution of a problem in attractions is transformed into that of a problem in heat.

This analogy between the formulæ of heat and attraction was, I believe, first pointed out by Professor William Thomson in the *Cambridge Math. Journal,* Vol. III.

Now the conduction of heat is supposed to proceed by an action between contiguous parts of a medium, while the force of attraction is a relation between distant bodies, and yet, if we knew nothing more than is expressed in the mathematical formulæ, there would be nothing to distinguish between the one set of phenomena and the other.

It is true, that if we introduce other considerations and observe additional facts, the two subjects will assume very different aspects, but the mathematical resemblance of some of their laws will remain, and may still be made useful in exciting appropriate mathematical ideas.

It is by the use of analogies of this kind that I have attempted to bring before the mind, in a convenient and manageable form, those mathematical ideas which are necessary to the study of the phenomena of electricity. The methods are generally those suggested by the processes of reasoning which are found in the researches of Faraday *, and which, though they have been interpreted mathematically by Prof. Thomson and others, are very generally supposed to be of an indefinite and unmathematical character, when compared with those employed by the professed mathematicians. By the method which I adopt, I hope to render it evident that I am not attempting to establish any physical theory of a science in which I have hardly made a single experiment, and that the limit of my design is to shew how, by a strict application of the ideas and methods of Faraday, the connexion of the very different orders of phenomena which he has discovered may be clearly placed before the mathematical mind. I shall therefore avoid as much as I can the introduction of anything which does not serve as a direct illustration of Faraday's methods, or of the mathematical deductions which may be made from them. In treating the simpler parts of the subject I shall use Faraday's mathematical methods as well as his ideas. When the complexity of the subject requires it, I shall use analytical notation, still confining myself to the development of ideas originated by the same philosopher.

I have in the first place to explain and illustrate the idea of "lines of force."

When a body is electrified in any manner, a small body charged with positive electricity, and placed in any given position, will experience a force urging it in a certain direction. If the small body be now negatively electrified, it will be urged by an equal force in a direction exactly opposite.

The same relations hold between a magnetic body and the north or south poles of a small magnet. If the north pole is urged in one direction, the south pole is urged in the opposite direction.

In this way we might find a line passing through any point of space, such that it represents the direction of the force acting on a positively electrified particle, or on an elementary north pole, and the reverse direction of the force on a negatively electrified particle or an elementary south pole. Since at every point of space such a direction may be found, if we commence at any point and draw a line so that, as we go along it, its direction at any point shall always coincide with that of the resultant force at that point, this curve will indicate the direction of that force for every point through which it passes, and might be called on that account a *line of force*. We might in the same way draw other lines of force, till we had filled all space with curves indicating by their direction that of the force at any assigned point.

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* See especially Series XXXVIII. of the *Experimental Researches*, and *Phil. Mag.* 1852.

We should thus obtain a geometrical model of the physical phenomena, which would tell us the *direction* of the force, but we should still require some method of indicating the *intensity* of the force at any point. If we consider these curves not as mere lines, but as fine tubes of variable section carrying an incompressible fluid, then, since the velocity of the fluid is inversely as the section of the tube, we may make the velocity vary according to any given law, by regulating the section of the tube, and in this way we might represent the intensity of the force as well as its direction by the motion of the fluid in these tubes. This method of representing the intensity of a force by the velocity of an imaginary fluid in a tube is applicable to any conceivable system of forces, but it is capable of great simplification in the case in which the forces are such as can be explained by the hypothesis of attractions varying inversely as the square of the distance, such as those observed in electrical and magnetic phenomena. In the case of a perfectly arbitrary system of forces, there will generally be interstices between the tubes; but in the case of electric and magnetic forces it is possible to arrange the tubes so as to leave no interstices. The tubes will then be mere surfaces, directing the motion of a fluid filling up the whole space. It has been usual to commence the investigation of the laws of these forces by at once assuming that the phenomena are due to attractive or repulsive forces acting between certain points. We may however obtain a different view of the subject, and one more suited to our more difficult inquiries, by adopting for the definition of the forces of which we treat, that they may be represented in magnitude and direction by the uniform motion of an incompressible fluid.

I propose, then, first to describe a method by which the motion of such a fluid can be clearly conceived; secondly to trace the consequences of assuming certain conditions of motion, and to point out the application of the method to some of the less complicated phenomena of electricity, magnetism, and galvanism; and lastly to shew how by an extension of these methods, and the introduction of another idea due to Faraday, the laws of the attractions and inductive actions of magnets and currents may be clearly conceived, without making any assumptions as to the physical nature of electricity, or adding anything to that which has been already proved by experiment.

By referring everything to the purely geometrical idea of the motion of an imaginary fluid, I hope to attain generality and precision, and to avoid the dangers arising from a premature theory professing to explain the cause of the phenomena. If the results of mere speculation which I have collected are found to be of any use to experimental philosophers, in arranging and interpreting their results, they will have served their purpose, and a mature theory, in which physical facts will be physically explained, will be formed by those who by interrogating Nature herself can obtain the only true solution of the questions which the mathematical theory suggests.

I. *Theory of the Motion of an incompressible Fluid.*

(1) The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of motion and resistance to compression. It is not

even a hypothetical fluid which is introduced to explain actual phenomena. It is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used. The use of the word "Fluid" will not lead us into error, if we remember that it denotes a purely imaginary substance with the following property:

The portion of fluid which at any instant occupied a given volume, will at any succeeding instant occupy an equal volume.

This law expresses the incompressibility of the fluid, and furnishes us with a convenient measure of its quantity, namely its volume. The unit of quantity of the fluid will therefore be the unit of volume.

(2) The direction of motion of the fluid will in general be different at different points of the space which it occupies, but since the direction is determinate for every such point, we may conceive a line to begin at any point and to be continued so that every element of the line indicates by its direction the direction of motion at that point of space. Lines drawn in such a manner that their direction always indicates the direction of fluid motion are called *lines of fluid motion.*

If the motion of the fluid be what is called *steady motion,* that is, if the direction and velocity of the motion at any fixed point be independent of the time, these curves will represent the paths of individual particles of the fluid, but if the motion be variable this will not generally be the case. The cases of motion which will come under our notice will be those of steady motion.

(3) If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw a line of motion, these lines of motion will generate a tubular surface which we may call a *tube of fluid motion.* Since this surface is generated by lines in the direction of fluid motion no part of the fluid can flow across it, so that this imaginary surface is as impermeable to the fluid as a real tube.

(4) The quantity of fluid which in unit of time crosses any fixed section of the tube is the same at whatever part of the tube the section be taken. For the fluid is incompressible, and no part runs through the sides of the tube, therefore the quantity which escapes from the second section is equal to that which enters through the first.

If the tube be such that unit of volume passes through any section in unit of time it is called a *unit tube of fluid motion.*

(5) In what follows, various units will be referred to, and a finite number of lines or surfaces will be drawn, representing in terms of those units the motion of the fluid. Now in order to define the motion in every part of the fluid, an infinite number of lines would have to be drawn at indefinitely small intervals; but since the description of such a system of lines would involve continual reference to the theory of limits, it has been thought better to suppose

the lines drawn at intervals depending on the assumed unit, and afterwards to assume the unit as small as we please by taking a small submultiple of the standard unit.

(6) To define the motion of the whole fluid by means of a system of unit tubes.

Take any fixed surface which cuts all the lines of fluid motion, and draw upon it any system of curves not intersecting one another. On the same surface draw a second system of curves intersecting the first system, and so arranged that the quantity of fluid which crosses the surface within each of the quadrilaterals formed by the intersection of the two systems of curves shall be unity in unit of time. From every point in a curve of the first system let a line of fluid motion be drawn. These lines will form a surface through which no fluid passes. Similar impermeable surfaces may be drawn for all the curves of the first system. The curves of the second system will give rise to a second system of impermeable surfaces, which, by their intersection with the first system, will form quadrilateral tubes, which will be tubes of fluid motion. Since each quadrilateral of the cutting surface transmits unity of fluid in unity of time, every tube in the system will transmit unity of fluid through any of its sections in unit of time. The motion of the fluid at every part of the space it occupies is determined by this system of unit tubes; for the direction of motion is that of the tube through the point in question, and the velocity is the reciprocal of the area of the section of the unit tube at that point.

(7) We have now obtained a geometrical construction which completely defines the motion of the fluid by dividing the space it occupies into a system of unit tubes. We have next to shew how by means of these tubes we may ascertain various points relating to the motion of the fluid.

A unit tube may either return into itself, or may begin and end at different points, and these may be either in the boundary of the space in which we investigate the motion, or within that space. In the first case there is a continual circulation of fluid in the tube, in the second the fluid enters at one end and flows out at the other. If the extremities of the tube are in the bounding surface, the fluid may be supposed to be continually supplied from without from an unknown source, and to flow out at the other into an unknown reservoir; but if the origin of the tube or its termination be within the space under consideration, then we must conceive the fluid to be supplied by a *source* within that space, capable of creating and emitting unity of fluid in unity of time, and to be afterwards swallowed up by a *sink* capable of receiving and destroying the same amount continually.

There is nothing self-contradictory in the conception of these sources where the fluid is created, and sinks where it is annihilated. The properties of the fluid are at our disposal, we have made it incompressible, and now we suppose it produced from nothing at certain points and reduced to nothing at others. The places of production will be called *sources*, and their numerical value will be the number of units of fluid which they produce in unit of time. The places of reduction will, for want of a better name, be called *sinks*, and will be estimated by the number of units of fluid absorbed in unit of time. Both places will sometimes be called sources, a source being understood to be a sink when its sign is negative.

(8) It is evident that the amount of fluid which passes any fixed surface is measured by the number of unit tubes which cut it, and the direction in which the fluid passes is determined by that of its motion in the tubes. If the surface be a closed one, then any tube whose terminations lie on the same side of the surface must cross the surface as many times in the one direction as in the other, and therefore must carry as much fluid out of the surface as it carries in. A tube which begins within the surface and ends without it will carry out unity of fluid; and one which enters the surface and terminates within it will carry in the same quantity. In order therefore to estimate the amount of fluid which flows out of the closed surface, we must subtract the number of tubes which end within the surface from the number of tubes which begin there. If the result is negative the fluid will on the whole flow inwards.

If we call the beginning of a unit tube a unit source, and its termination a unit sink, then the quantity of fluid produced within the surface is estimated by the number of unit sources minus the number of unit sinks, and this must flow out of the surface on account of the incompressibility of the fluid.

In speaking of these unit tubes, sources and sinks, we must remember what was stated in (5) as to the magnitude of the unit, and how by diminishing their size and increasing their number we may distribute them according to any law however complicated.

(9) If we know the direction and velocity of the fluid at any point in two different cases, and if we conceive a third case in which the direction and velocity of the fluid at any point is the resultant of the velocities in the two former cases at corresponding points, then the amount of fluid which passes a given fixed surface in the third case will be the algebraic sum of the quantities which pass the same surface in the two former cases. For the rate at which the fluid crosses any surface is the resolved part of the velocity normal to the surface, and the resolved part of the resultant is equal to the sum of the resolved parts of the components.

Hence the number of unit tubes which cross the surface outwards in the third case must be the algebraical sum of the numbers which cross it in the two former cases, and the number of sources within any closed surface will be the sum of the numbers in the two former cases. Since the closed surface may be taken as small as we please, it is evident that the distribution of sources and sinks in the third case arises from the simple superposition of the distributions in the two former cases.

II. *Theory of the uniform motion of an imponderable incompressible fluid through a resisting medium.*

(10) The fluid is here supposed to have no inertia, and its motion is opposed by the action of a force which we may conceive to be due to the resistance of a medium through which the fluid is supposed to flow. This resistance depends on the nature of the medium, and will in general depend on the direction in which the fluid moves, as well as on its velocity. For the present we may restrict ourselves to the case of a uniform medium, whose resistance is the same in all directions. The law which we assume is as follows.

Any portion of the fluid moving through the resisting medium is directly opposed by a retarding force proportional to its velocity.

If the velocity be represented by v, then the resistance will be a force equal to kv acting on unit of volume of the fluid in a direction contrary to that of motion. In order, therefore, that the velocity may be kept up, there must be a greater pressure behind any portion of the fluid than there is in front of it, so that the difference of pressures may neutralise the effect of the resistance. Conceive a cubical unit of fluid (which we may make as small as we please, by (5)), and let it move in a direction perpendicular to two of its faces. Then the resistance will be kv, and therefore the difference of pressures on the first and second faces is kv, so that the pressure diminishes in the direction of motion at the rate of kv for every unit of length measured along the line of motion; so that if we measure a length equal to h units, the difference of pressure at its extremities will be kvh.

(11) Since the pressure is supposed to vary continuously in the fluid, all the points at which the pressure is equal to a given pressure p will lie on a certain surface which we may call the *surface* (p) *of equal pressure.* If a series of these surfaces be constructed in the fluid corresponding to the pressures 0, 1, 2, 3 &c., then the number of the surface will indicate the pressure belonging to it, and the surface may be referred to as the surface 0, 1, 2 or 3. The unit of pressure is that pressure which is produced by unit of force acting on unit of surface. In order therefore to diminish the unit of pressure as in (5) we must diminish the unit of force in the same proportion.

(12) It is easy to see that these surfaces of equal pressure must be perpendicular to the lines of fluid motion; for if the fluid were to move in any other direction, there would be a resistance to its motion which could not be balanced by any difference of pressures. (We must remember that the fluid here considered has no inertia or mass, and that its properties are those only which are formally assigned to it, so that the resistances and pressures are the only things to be considered.) There are therefore two sets of surfaces which by their intersection form the system of unit tubes, and the system of surfaces of equal pressure cuts both the others at right angles. Let h be the distance between two consecutive surfaces of equal pressure measured along a line of motion, then since the difference of pressures = 1,

$$kvh = 1,$$

which determines the relation of v to h, so that one can be found when the other is known. Let s be the sectional area of a unit tube measured on a surface of equal pressure, then since by the definition of a unit tube

$$vs = 1,$$

we find by the last equation

$$s = kh.$$

(13) The surfaces of equal pressure cut the unit tubes into portions whose length is h and section s. These elementary portions of unit tubes will be called *unit cells*. In each of them unity of volume of fluid passes from a pressure p to a pressure $(p-1)$ in unit of time, and therefore overcomes unity of resistance in that time. The work spent in overcoming resistance is therefore unity in every cell in every unit of time.

(14) If the surfaces of equal pressure are known, the direction and magnitude of the velocity of the fluid at any point may be found, after which the complete system of unit tubes may be constructed, and the beginnings and endings of these tubes ascertained and marked out as the sources whence the fluid is derived, and the sinks where it disappears. In order to prove the converse of this, that if the distribution of sources be given, the pressure at every point may be found, we must lay down certain preliminary propositions.

(15) If we know the pressures at every point in the fluid in two different cases, and if we take a third case in which the pressure at any point is the sum of the pressures at corresponding points in the two former cases, then the velocity at any point in the third case is the resultant of the velocities in the other two, and the distribution of sources is that due to the simple superposition of the sources in the two former cases.

For the velocity in any direction is proportional to the rate of decrease of the pressure in that direction; so that if two systems of pressures be added together, since the rate of decrease of pressure along any line will be the sum of the combined rates, the velocity in the new system resolved in the same direction will be the sum of the resolved parts in the two original systems. The velocity in the new system will therefore be the resultant of the velocities at corresponding points in the two former systems.

It follows from this, by (9), that the quantity of fluid which crosses any fixed surface is, in the new system, the sum of the corresponding quantities in the old ones, and that the sources of the two original systems are simply combined to form the third.

It is evident that in the system in which the pressure is the difference of pressure in the two given systems the distribution of sources will be got by changing the sign of all the sources in the second system and adding them to those in the first.

(16) If the pressure at every point of a closed surface be the same and equal to p, and if there be no sources or sinks within the surface, then there will be no motion of the fluid within the surface, and the pressure within it will be uniform and equal to p.

For if there be motion of the fluid within the surface there will be tubes of fluid motion, and these tubes must either return into themselves or be terminated either within the surface or at its boundary. Now since the fluid always flows from places of greater pressure to places of less pressure, it cannot flow in a re-entering curve; since there are no sources or sinks within the surface, the tubes cannot begin or end except on the surface; and since the pressure at all points of the surface is the same, there can be no motion in tubes having both extremities on the surface. Hence there is no motion within the surface, and therefore no difference of pressure which would cause motion, and since the pressure at the bounding surface is p, the pressure at any point within it is also p.

(17) If the pressure at every point of a given closed surface be known, and the distribution of sources within the surface be also known, then only one distribution of pressures can exist within the surface.

For if two different distributions of pressures satisfying these conditions could be found, a third distribution could be formed in which the pressure at any point should be the difference of the pressures in the two former distributions. In this case, since the pressures at the surface and the sources within it are the same in both distributions, the pressure at the surface in the third distribution would be zero, and all the sources within the surface would vanish, by (15).

Then by (16) the pressure at every point in the third distribution must be zero; but this is the difference of the pressures in the two former cases, and therefore these cases are the same, and there is only one distribution of pressure possible.

(18) Let us next determine the pressure at any point of an infinite body of fluid in the centre of which a unit source is placed, the pressure at an infinite distance from the source being supposed to be zero.

The fluid will flow out from the centre symmetrically, and since unity of volume flows out of every spherical surface surrounding the point in unit of time, the velocity at a distance r from the source will be

$$v = \frac{1}{4\pi r^2}.$$

The rate of decrease of pressure is therefore kv or $\frac{k}{4\pi r^2}$, and since the pressure $= 0$ when r is infinite, the actual pressure at any point will be $p = \frac{k}{4\pi r}$.

The pressure is therefore inversely proportional to the distance from the source.

It is evident that the pressure due to a unit sink will be negative and equal to

$$-\frac{k}{4\pi r}.$$

If we have a source formed by the coalition of S unit sources, then the resulting pressure will be $p = \frac{kS}{4\pi r}$, so that the pressure at a given distance varies as the resistance and number of sources conjointly.

(19) If a number of sources and sinks coexist in the fluid, then in order to determine the resultant pressure we have only to add the pressures which each source or sink produces. For by (15) this will be a solution of the problem, and by (17) it will be the only one. By this method we can determine the pressures due to any distribution of sources, as by the method of (14) we can determine the distribution of sources to which a given distribution of pressures is due.

(20) We have next to shew that if we conceive any imaginary surface as fixed in space and intersecting the lines of motion of the fluid, we may substitute for the fluid on one side of this surface a distribution of sources upon the surface itself without altering in any way the motion of the fluid on the other side of the surface.

For if we describe the system of unit tubes which defines the motion of the fluid, and wherever a tube enters through the surface place a unit source, and wherever a tube goes out through the surface place a unit sink, and at the same time render the surface impermeable to the fluid, the motion of the fluid in the tubes will go on as before.

(21) If the system of pressures and the distribution of sources which produce them be known in a medium whose resistance is measured by k, then in order to produce the same system of pressures in a medium whose resistance is unity, the rate of production at each source must be multiplied by k. For the pressure at any point due to a given source varies as the rate of production and the resistance conjointly; therefore if the pressure be constant, the rate of production must vary inversely as the resistance.

(22) *On the conditions to be fulfilled at a surface which separates two media whose coefficients of resistance are* k *and* k′.

These are found from the consideration, that the quantity of fluid which flows out of the one medium at any point flows into the other, and that the pressure varies continuously from one medium to the other. The velocity normal to the surface is the same in both media, and therefore the rate of diminution of pressure is proportional to the resistance. The direction of the tubes of motion and the surfaces of equal pressure will be altered after passing through the surface, and the law of this refraction will be, that it takes place in the plane passing through the direction of incidence and the normal to the surface, and that the tangent of the angle of incidence is to the tangent of the angle of refraction as k' is to k.

(23) Let the space within a given closed surface be filled with a medium different from that exterior to it, and let the pressures at any point of this compound system due to a given distribution of sources within and without the surface be given; it is required to determine a distribution of sources which would produce the same system of pressures in a medium whose coefficient of resistance is unity.

Construct the tubes of fluid motion, and wherever a unit tube enters either medium place a unit source, and wherever it leaves it place a unit sink. Then if we make the surface impermeable all will go on as before.

Let the resistance of the exterior medium be measured by k, and that of the interior by k'. Then if we multiply the rate of production of all the sources in the exterior medium (including those in the surface), by k, and make the coefficient of resistance unity, the pressures will remain as before, and the same will be true of the interior medium if we multiply all the sources in it by k', including those in the surface, and make its resistance unity.

Since the pressures on both sides of the surface are now equal, we may suppose it permeable if we please.

We have now the original system of pressures produced in a uniform medium by a combination of three systems of sources. The first of these is the given external system multiplied by k, the second is the given internal system multiplied by k', and the third is the system of sources and sinks on the surface itself. In the original case every source in the external medium had an equal sink in the internal medium on the other side of the surface, but now the source is multiplied by k and the sink by k', so that the result is for every external unit source on the surface, a source $= (k - k')$. By means of these three systems of sources the original system of pressures may be produced in a medium for which $k = 1$.

(24) Let there be no resistance in the medium within the closed surface, that is, let $k' = 0$, then the pressure within the closed surface is uniform and equal to p, and the pressure at the surface itself is also p. If by assuming any distribution of pairs of sources and sinks within the surface in addition to the given external and internal sources, and by supposing the medium the same within and without the surface, we can render the pressure at the surface uniform, the pressures so found for the external medium, together with the uniform pressure p in the internal medium, will be the true and only distribution of pressures which is possible.

For if two such distributions could be found by taking different imaginary distributions of pairs of sources and sinks within the medium, then by taking the difference of the two for a third distribution, we should have the pressure of the bounding surface constant in the new system and as many sources as sinks within it, and therefore whatever fluid flows in at any point of the surface, an equal quantity must flow out at some other point.

In the external medium all the sources destroy one another, and we have an infinite medium without sources surrounding the internal medium. The pressure at infinity is zero, that at the surface is constant. If the pressure at the surface is positive, the motion of the fluid must be outwards from every point of the surface; if it be negative, it must flow inwards towards the surface. But it has been shewn that neither of these cases is possible, because if any fluid enters the surface an equal quantity must escape, and therefore the pressure at the surface is zero in the third system.

The pressure at all points in the boundary of the internal medium in the third case is therefore zero, and there are no sources, and therefore the pressure is everywhere zero, by (16).

The pressure in the bounding surface of the internal medium is also zero, and there is no resistance, therefore it is zero throughout; but the pressure in the third case is the difference of pressures in the two given cases, therefore these are equal, and there is only one distribution of pressure which is possible, namely, that due to the imaginary distribution of sources and sinks.

(25) When the resistance is infinite in the internal medium, there can be no passage of fluid through it or into it. The bounding surface may therefore be considered as impermeable to the fluid, and the tubes of fluid motion will run along it without cutting it.

If by assuming any arbitrary distribution of sources within the surface in addition to the given sources in the outer medium, and by calculating the resulting pressures and velocities as in the case of a uniform medium, we can fulfil the condition of there being no velocity across the surface, the system of pressures in the outer medium will be the true one. For since no fluid passes through the surface, the tubes in the interior are independent of those outside, and may be taken away without altering the external motion.

(26) If the extent of the internal medium be small, and if the difference of resistance in the two media be also small, then the position of the unit tubes will not be much altered from what it would be if the external medium filled the whole space.

On this supposition we can easily calculate the kind of alteration which the introduction of the internal medium will produce; for wherever a unit tube enters the surface we must conceive a source producing fluid at a rate $\frac{k'-k}{k}$, and wherever a tube leaves it we must place a sink annihilating fluid at the rate $\frac{k'-k}{k}$, then calculating pressures on the supposition that the resistance in both media is k the same as in the external medium, we shall obtain the true distribution of pressures very approximately, and we may get a better result by repeating the process on the system of pressures thus obtained.

(27) If instead of an abrupt change from one coefficient of resistance to another we take a case in which the resistance varies continuously from point to point, we may treat the medium as if it were composed of thin shells each of which has uniform resistance. By properly assuming a distribution of sources over the surfaces of separation of the shells, we may treat the case as if the resistance were equal to unity throughout, as in (23). The sources will then be distributed continuously throughout the whole medium, and will be positive whenever the motion is from places of less to places of greater resistance, and negative when in the contrary direction.

(28) Hitherto we have supposed the resistance at a given point of the medium to be the same in whatever direction the motion of the fluid takes place; but we may conceive a case in which the resistance is different in different directions. In such cases the lines of motion will not in general be perpendicular to the surfaces of equal pressure. If a, b, c be the components of the velocity at any point, and α, β, γ the components of the resistance at the same point, these quantities will be connected by the following system of linear equations, which may be called "*equations of conduction*," and will be referred to by that name.

$$a = P_1\alpha + Q_3\beta + R_2\gamma,$$
$$b = P_2\beta + Q_1\gamma + R_3\alpha,$$
$$c = P_3\gamma + Q_2\alpha + R_1\beta.$$

In these equations there are nine independent coefficients of conductivity. In order to simplify the equations, let us put

$$Q_1 + R_1 = 2S_1, \quad Q_1 - R_1 = 2lT,$$
$$\text{............\&c.} \quad \text{............\&c.}$$

where $$4T^2 = (Q_1 - R_1)^2 + (Q_2 - R_2)^2 + (Q_3 - R_3)^2,$$

and l, m, n are direction cosines of a certain fixed line in space.

The equations then become

$$a = P_1\alpha + S_3\beta + S_2\gamma + (n\beta - m\gamma)T,$$
$$b = P_2\beta + S_1\gamma + S_3\alpha + (l\gamma - n\alpha)T,$$
$$c = P_3\gamma + S_2\alpha + S_1\beta - (m\alpha - l\beta)T.$$

By the ordinary transformation of coordinates we may get rid of the coefficients marked S. The equations then become

$$a = P_1'\alpha + (n'\beta - m'\gamma)T,$$
$$b = P_2'\beta + (l'\gamma - n'\alpha)T,$$
$$c = P_3'\gamma + (m'\alpha - l'\beta)T,$$

where l', m', n' are the direction cosines of the fixed line with reference to the new axes. If we make

$$\alpha = \frac{dp}{dx}, \quad \gamma = \frac{dp}{dy}, \quad \text{and } z = \frac{dp}{dz},$$

the equation of continuity

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0,$$

becomes

$$P_1'\frac{d^2p}{dx^2} + P_2'\frac{d^2p}{dy^2} + P_3'\frac{d^2p}{dz^2} = 0,$$

and if we make $x = \sqrt{P_1'}\xi, \quad y = \sqrt{P_2'}\eta, \quad z = \sqrt{P_3'}\zeta,$

then

$$\frac{d^2p}{d\xi^2} + \frac{d^2p}{d\eta^2} + \frac{d^2p}{d\zeta^2} = 0,$$

the ordinary equation of conduction.

It appears therefore that the distribution of pressures is not altered by the existence of the coefficient T. Professor Thomson has shewn how to conceive a substance in which this coefficient determines a property having reference to an axis, which unlike the axes of P_1, P_2, P_3 is *dipolar*.

For further information on the equations of conduction, see Professor Stokes *On the Conduction of Heat in Crystals* (*Cambridge and Dublin Math. Journ.*), and Professor Thomson *on the Dynamical Theory of Heat*, Part V. (*Transactions of Royal Society of Edinburgh*, Vol. XXI. Part I.)

It is evident that all that has been proved in (14), (15), (16), (17), with respect to the superposition of different distributions of pressure, and there being only one distribution of pressures corresponding to a given distribution of sources, will be true also in the case in which the resistance varies from point to point, and the resistance at the same point is different in different directions. For if we examine the proof we shall find it applicable to such cases as well as to that of a uniform medium.

(29) We now are prepared to prove certain general propositions which are true in the most general case of a medium whose resistance is different in different directions and varies from point to point.

We may by the method of (28), when the distribution of pressures is known, construct the surfaces of equal pressure, the tubes of fluid motion, and the sources and sinks. It is evident that since in each cell into which a unit tube is divided by the surfaces of equal pressure unity of fluid passes from pressure p to pressure $(p-1)$ in unit of time, unity of work is done by the fluid in each cell in overcoming resistance.

The number of cells in each unit tube is determined by the number of surfaces of equal pressure through which it passes. If the pressure at the beginning of the tube be p and at the end p', then the number of cells in it will be $p - p'$. Now if the tube had extended from the source to a place where the pressure is zero, the number of cells would have been p, and if the tube had come from the sink to zero, the number would have been p', and the true number is the difference of these.

Therefore if we find the pressure at a source S from which S tubes proceed to be p, Sp is the number of cells due to the source S; but if S' of the tubes terminate in a sink at a pressure p', then we must cut off $S'p'$ cells from the number previously obtained. Now if we denote the source of S tubes by S, the sink of S' tubes may be written $-S'$, sinks always being reckoned negative, and the general expression for the number of cells in the system will be $\Sigma\,(Sp)$.

(30) The same conclusion may be arrived at by observing that unity of work is done on each cell. Now in each source S, S units of fluid are expelled against a pressure p, so that the work done by the fluid in overcoming resistance is Sp. At each sink in which S' tubes terminate, S' units of fluid sink into nothing under pressure p'; the work done upon the fluid by the pressure is therefore $S'p'$. The whole work done by the fluid may therefore be expressed by

$$W = \Sigma Sp - \Sigma S'p',$$

or more concisely, considering sinks as negative sources,

$$W = \Sigma(Sp).$$

(31) Let S represent the rate of production of a source in any medium, and let p be the pressure at any given point due to that source. Then if we superpose on this another equal source, every pressure will be doubled, and thus by successive superposition we find that a source nS would produce a pressure np, or more generally the pressure at any point due to a given source varies as the rate of production of the source. This may be expressed by the equation

$$p = RS,$$

where R is a coefficient depending on the nature of the medium and on the positions of the source and the given point. In a uniform medium whose resistance is measured by k,

$$p = \frac{kS}{4\pi r}, \quad \therefore R = \frac{k}{4\pi r},$$

R may be called the coefficient of resistance of the medium between the source and the given point. By combining any number of sources we have generally

$$p = \Sigma(RS).$$

(32) In a uniform medium the pressure due to a source S

$$p = \frac{k}{4\pi}\,\frac{S}{r}.$$

At another source S' at a distance r we shall have

$$S'p = \frac{k}{4\pi}\,\frac{SS'}{r} = Sp',$$

if p' be the pressure at S due to S'. If therefore there be two systems of sources $\Sigma(S)$ and $\Sigma(S')$, and if the pressures due to the first be p and to the second p', then

$$\Sigma(S'p) = \Sigma(Sp').$$

For every term $S'p$ has a term Sp' equal to it.

(33) Suppose that in a uniform medium the motion of the fluid is everywhere parallel to one plane, then the surfaces of equal pressure will be perpendicular to this plane. If we take two parallel planes at a distance equal to k from each other, we can divide the space between these planes into unit tubes by means of cylindric surfaces perpendicular to the planes, and these together with the surfaces of equal pressure will divide the space into cells of which the length is equal to the breadth. For if h be the distance between consecutive surfaces of equal pressure and s the section of the unit tube, we have by (13) $s = kh$.

But s is the product of the breadth and depth; but the depth is k, therefore the breadth is h and equal to the length.

If two systems of plane curves cut each other at right angles so as to divide the plane into little areas of which the length and breadth are equal, then by taking another plane at distance k from the first and erecting cylindric surfaces on the plane curves as bases, a system of cells will be formed which will satisfy the conditions whether we suppose the fluid to run along the first set of cutting lines or the second *.

Application of the Idea of Lines of Force.

I have now to shew how the idea of lines of fluid motion as described above may be modified so as to be applicable to the sciences of statical electricity, permanent magnetism, magnetism of induction, and uniform galvanic currents, reserving the laws of electro-magnetism for special consideration.

I shall assume that the phenomena of statical electricity have been already explained by the mutual action of two opposite kinds of matter. If we consider one of these as positive electricity and the other as negative, then any two particles of electricity repel one another with a force which is measured by the product of the masses of the particles divided by the square of their distance.

Now we found in (18) that the velocity of our imaginary fluid due to a source S at a distance r varies inversely as r^2. Let us see what will be the effect of substituting such a source for every particle of positive electricity. The velocity due to each source would be proportional to the attraction due to the corresponding particle, and the resultant velocity due to all the sources would be proportional to the resultant attraction of all the particles. Now we may find the resultant pressure at any point by adding the pressures due to the given sources, and therefore we may find the resultant velocity in a given direction from the rate of decrease of pressure in that direction, and this will be proportional to the resultant attraction of the particles resolved in that direction.

Since the resultant attraction in the electrical problem is proportional to the decrease of pressure in the imaginary problem, and since we may select any values for the constants in the imaginary problem, we may assume that the resultant attraction in any direction is numerically equal to the decrease of pressure in that direction, or

$$X = -\frac{pd}{dx}.$$

* See *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 286.

By this assumption we find that if V be the potential,

$$dV = Xdx + Ydy + Zdz = -dp,$$

or since at an infinite distance $V = 0$ and $p = 0$, $V = -p$.

In the electrical problem we have

$$V = -\Sigma\left(\frac{dm}{r}\right).$$

In the fluid $p = \Sigma\left(\frac{k}{4\pi}\frac{S}{r}\right)$;

$$\therefore S = \frac{4\pi}{k} dm.$$

If k be supposed very great, the amount of fluid produced by each source in order to keep up the pressures will be very small.

The potential of any system of electricity on itself will be

$$\Sigma(pdm) = \frac{k}{4\pi}, \Sigma(pS) = \frac{k}{4\pi} W.$$

If $\Sigma(dm)$, $\Sigma(dm')$ be two systems of electrical particles and pp' the potentials due to them respectively, then by (32)

$$\Sigma(pdm') = \frac{k}{4\pi}, \quad \Sigma(pS') = \frac{k}{4\pi}, \quad \Sigma(p'S) = \Sigma(p'dm),$$

or the potential of the first system on the second is equal to that of the second system on the first.

So that in the ordinary electrical problems the analogy in fluid motion is of this kind:

$$V = -p,$$

$$X = -\frac{dp}{dx} = ku,$$

$$dm = \frac{k}{4\pi} S,$$

whole potential of a system $= -\Sigma V dm = \frac{k}{4\pi} W$, where W is the work done by the fluid in overcoming resistance.

The lines of force are the unit tubes of fluid motion, and they may be estimated numerically by those tubes.

Theory of Dielectrics.

The electrical induction exercised on a body at a distance depends not only on the distribution of electricity in the inductric, and the form and position of the inducteous body, but on the nature of the interposed medium, or dielectric. Faraday * expresses this by the conception

* Series XI.

of one substance having a *greater inductive capacity*, or conducting the lines of inductive action more freely than another. If we suppose that in our analogy of a fluid in a resisting medium the resistance is different in different media, then by making the resistance less we obtain the analogue to a dielectric which more easily conducts Faraday's lines.

It is evident from (23) that in this case there will always be an apparent distribution of electricity on the surface of the dielectric, there being negative electricity where the lines enter and positive electricity where they emerge. In the case of the fluid there are no real sources on the surface, but we use them merely for purposes of calculation. In the dielectric there may be no real charge of electricity, but only an apparent electric action due to the surface.

If the dielectric had been of less conductivity than the surrounding medium, we should have had precisely opposite effects, namely, positive electricity where lines enter, and negative where they emerge.

If the conduction of the dielectric is perfect or nearly so for the small quantities of electricity with which we have to do, then we have the case of (24). The dielectric is then considered as a conductor, its surface is a surface of equal potential, and the resultant attraction near the surface itself is perpendicular to it.

Theory of Permanent Magnets.

A magnet is conceived to be made up of elementary magnetized particles, each of which has its own north and south poles, the action of which upon other north and south poles is governed by laws mathematically identical with those of electricity. Hence the same application of the idea of lines of force can be made to this subject, and the same analogy of fluid motion can be employed to illustrate it.

But it may be useful to examine the way in which the polarity of the elements of a magnet may be represented by the unit cells in fluid motion. In each unit cell unity of fluid enters by one face and flows out by the opposite face, so that the first face becomes a unit sink and the second a unit source with respect to the rest of the fluid. It may therefore be compared to an elementary magnet, having an equal quantity of north and south magnetic matter distributed over two of its faces. If we now consider the cell as forming part of a system, the fluid flowing out of one cell will flow into the next, and so on, so that the source will be transferred from the end of the cell to the end of the unit tube. If all the unit tubes begin and end on the bounding surface, the sources and sinks will be distributed entirely on that surface, and in the case of a magnet which has what has been called a solenoidal or tubular distribution of magnetism, all the imaginary magnetic matter will be on the surface*.

Theory of Paramagnetic and Diamagnetic Induction.

Faraday† has shewn that the effects of paramagnetic and diamagnetic bodies in the magnetic field may be explained by supposing paramagnetic bodies to conduct the lines of force better,

* See Professor Thomson *On the Mathematical Theory of Magnetism*, Chapters III. & V. *Phil. Trans.* 1851.

† *Experimental Researches* (3292).

and diamagnetic bodies worse, than the surrounding medium. By referring to (23) and (26), and supposing sources to represent north magnetic matter, and sinks south magnetic matter, then if a paramagnetic body be in the neighbourhood of a north pole, the lines of force on entering it will produce south magnetic matter, and on leaving it they will produce an equal amount of north magnetic matter. Since the quantities of magnetic matter on the whole are equal, but the southern matter is nearest to the north pole, the result will be attraction. If on the other hand the body be diamagnetic, or a worse conductor of lines of force than the surrounding medium, there will be an imaginary distribution of northern magnetic matter where the lines pass into the worse conductor, and of southern where they pass out, so that on the whole there will be repulsion.

We may obtain a more general law from the consideration that the potential of the whole system is proportional to the amount of work done by the fluid in overcoming resistance. The introduction of a second medium increases or diminishes the work done according as the resistance is greater or less than that of the first medium. The amount of this increase or diminution will vary as the square of the velocity of the fluid.

Now, by the theory of potentials, the moving force in any direction is measured by the rate of decrease of the potential of the system in passing along that direction, therefore when k', the resistance within the second medium, is greater than k, the resistance in the surrounding medium, there is a force tending from places where the resultant force v is greater to where it is less, so that a diamagnetic body moves from greater to less values of the resultant force *.

In paramagnetic bodies k' is less than k, so that the force is now from points of less to points of greater resultant magnetic force. Since these results depend only on the relative values of k and k', it is evident that by changing the surrounding medium, the behaviour of a body may be changed from paramagnetic to diamagnetic at pleasure.

It is evident that we should obtain the same mathematical results if we had supposed that the magnetic force had a power of exciting a polarity in bodies which is in the *same* direction as the lines in paramagnetic bodies, and in the *reverse* direction in diamagnetic bodies †. In fact we have not as yet come to any facts which would lead us to choose any one out of these three theories, that of lines of force, that of imaginary magnetic matter, and that of induced polarity. As the theory of lines of force admits of the most precise, and at the same time least theoretic statement, we shall allow it to stand for the present.

Theory of Magnecrystallic Induction.

The theory of Faraday ‡ with respect to the behaviour of crystals in the magnetic field may be thus stated. In certain crystals and other substances the lines of magnetic force are

* *Experimental Researches* (2797), (2798). See Thomson, *Cambridge and Dublin Mathematical Journal*, May, 1847.

† *Exp. Res.* (2429), (3320). See Weber, Poggendorff, lxxxvii. p. 145. Prof. Tyndall, *Phil. Trans.* 1856, p. 237.

‡ *Exp. Res.* (2836), &c.

conducted with different facility in different directions. The body when suspended in a uniform magnetic field will turn or tend to turn into such a position that the lines of force shall pass through it with least resistance. It is not difficult by means of the principles in (28) to express the laws of this kind of action, and even to reduce them in certain cases to numerical formulæ. The principles of induced polarity and of imaginary magnetic matter are here of little use; but the theory of lines of force is capable of the most perfect adaptation to this class of phenomena.

Theory of the Conduction of Current Electricity.

It is in the calculation of the laws of constant electric currents that the theory of fluid motion which we have laid down admits of the most direct application. In addition to the researches of Ohm on this subject, we have those of M. Kirchhoff, *Ann. de Chim.* XLI. 496, and of M. Quincke, XLVII. 203, on the Conduction of Electric Currents in Plates. According to the received opinions we have here a current of fluid moving uniformly in conducting circuits, which oppose a resistance to the current which has to be overcome by the application of an electro-motive force at some part of the circuit. On account of this resistance to the motion of the fluid the pressure must be different at different points in the circuit. This pressure, which is commonly called electrical tension, is found to be physically identical with the *potential* in statical electricity, and thus we have the means of connecting the two sets of phenomena. If we knew what amount of electricity, measured statically, passes along that current which we assume as our unit of current, then the connexion of electricity of tension with current electricity would be completed *. This has as yet been done only approximately, but we know enough to be certain that the conducting powers of different substances differ only in degree, and that the difference between glass and metal is, that the resistance is a great but finite quantity in glass, and a small but finite quantity in metal. Thus the analogy between statical electricity and fluid motion turns out more perfect than we might have supposed, for there the induction goes on by conduction just as in current electricity, but the quantity conducted is insensible owing to the great resistance of the dielectrics †.

On Electro-motive Forces.

When a uniform current exists in a closed circuit it is evident that some other forces must act on the fluid besides the pressures. For if the current were due to difference of pressures, then it would flow from the point of greatest pressure in both directions to the point of least pressure, whereas in reality it circulates in one direction constantly. We

* See *Exp. Res.* (371).

† *Exp. Res.* Vol. III. p. 513.

must therefore admit the existence of certain forces capable of keeping up a constant current in a closed circuit. Of these the most remarkable is that which is produced by chemical action. A cell of a voltaic battery, or rather the surface of separation of the fluid of the cell and the zinc, is the seat of an electro-motive force which can maintain a current in opposition to the resistance of the circuit. If we adopt the usual convention in speaking of electric currents, the positive current is from the fluid through the platinum, the conducting circuit, and the zinc, back to the fluid again. If the electro-motive force act only in the surface of separation of the fluid and zinc, then the tension of electricity in the fluid must exceed that in the zinc by a quantity depending on the nature and length of the circuit and on the strength of the current in the conductor. In order to keep up this difference of pressure there must be an electro-motive force whose intensity is measured by that difference of pressure. If F be the electro-motive force, I the quantity of the current or the number of electrical units delivered in unit of time, and K a quantity depending on the length and resistance of the conducting circuit, then

$$F = IK = p - p',$$

where p is the electric tension in the fluid and p' in the zinc.

If the circuit be broken at any point, then since there is no current the tension of the part which remains attached to the platinum will be p, and that of the other will be p'. $p - p'$, or F affords a measure of the intensity of the current. This distinction of quantity and intensity is very useful*, but must be distinctly understood to mean nothing more than this:—The quantity of a current is the amount of electricity which it transmits in unit of time, and is measured by I the number of unit currents which it contains. The intensity of a current is its power of overcoming resistance, and is measured by F or IK, where K is the resistance of the whole circuit.

The same idea of quantity and intensity may be applied to the case of magnetism†. The quantity of magnetization in any section of a magnetic body is measured by the number of lines of magnetic force which pass through it. The intensity of magnetization in the section depends on the resisting power of the section, as well as on the number of lines which pass through it. If k be the resisting power of the material, and S the area of the section, and I the number of lines of force which pass through it, then the whole intensity throughout the section

$$= F = I\frac{k}{S}.$$

When magnetization is produced by the influence of other magnets only, we may put p for the magnetic tension at any point, then for the whole magnetic solenoid

$$F = I\int \frac{k}{S}\,dx = IK = p - p'.$$

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* *Exp. Res.* Vol. III. p. 519.

† *Exp. Res.* (2870), (3293).

When a solenoidal magnetized circuit returns into itself, the magnetization does not depend on difference of tensions only, but on some magnetizing force of which the intensity is F.

If i be the quantity of the magnetization at any point, or the number of lines of force passing through unit of area in the section of the solenoid, then the total quantity of magnetization in the circuit is the number of lines which pass through any section $I = \Sigma i dy dz$, where $dy dz$ is the element of the section, and the summation is performed over the whole section.

The intensity of magnetization at any point, or the force required to keep up the magnetization, is measured by $ki = f$, and the total intensity of magnetization in the circuit is measured by the sum of the local intensities all round the circuit,

$$F = \Sigma (f dx),$$

where dx is the element of length in the circuit, and the summation is extended round the entire circuit.

In the same circuit we have always $F = IK$, where K is the total resistance of the circuit, and depends on its form and the matter of which it is composed.

On the Action of closed Currents at a Distance.

The mathematical laws of the attractions and repulsions of conductors have been most ably investigated by Ampère, and his results have stood the test of subsequent experiments.

From the single assumption, that the action of an element of one current upon an element of another current is an attractive or repulsive force acting in the direction of the line joining the two elements, he has determined by the simplest experiments the mathematical form of the law of attraction, and has put this law into several most elegant and useful forms. We must recollect however that no experiments have been made on these elements of currents except under the form of closed currents either in rigid conductors or in fluids, and that the laws of closed currents only can be deduced from such experiments. Hence if Ampère's formulæ applied to closed currents give true results, their truth is not proved for *elements* of currents unless we assume that the action between two such elements must be along the line which joins them. Although this assumption is most warrantable and philosophical in the present state of science, it will be more conducive to freedom of investigation if we endeavour to do without it, and to assume the laws of closed currents as the ultimate datum of experiment.

Ampère has shewn that when currents are combined according to the law of the parallelogram of forces, the force due to the resultant current is the resultant of the forces due to the component currents, and that equal and opposite currents generate equal and opposite forces, and when combined neutralize each other.

He has also shewn that a closed circuit of any form has no tendency to turn a moveable circular conductor about a fixed axis through the centre of the circle perpendicular to its plane, and that therefore the forces in the case of a closed circuit render $Xdx + Ydy + Zdz$ a complete differential.

Finally, he has shewn that if there be two systems of circuits similar and similarly situated, the quantity of electrical current in corresponding conductors being the same, the resultant forces are equal, whatever be the absolute dimensions of the systems, which proves that the forces are, *cæteris paribus*, inversely as the square of the distance.

From these results it follows that the mutual action of two closed currents whose areas are very small is the same as that of two elementary magnetic bars magnetized perpendicularly to the plane of the currents.

The direction of magnetization of the equivalent magnet may be predicted by remembering that a current travelling round the earth from east to west as the sun appears to do, would be equivalent to that magnetization which the earth actually possesses, and therefore in the reverse direction to that of a magnetic needle when pointing freely.

If a number of closed unit currents in contact exist on a surface, then at all points in which two currents are in contact there will be two equal and opposite currents which will produce no effect, but all round the boundary of the surface occupied by the currents there will be a residual current not neutralized by any other; and therefore the result will be the same as that of a single unit current round the boundary of all the currents.

From this it appears that the external attractions of a shell uniformly magnetized perpendicular to its surface are the same as those due to a current round its edge, for each of the elementary currents in the former case has the same effect as an element of the magnetic shell.

If we examine the lines of magnetic force produced by a closed current, we shall find that they form closed curves passing round the current and *embracing* it, and that the total intensity of the magnetizing force all along the closed line of force depends on the quantity of the electric current only. The number of unit lines * of magnetic force due to a closed current depends on the form as well as the quantity of the current, but the number of unit cells † in each complete line of force is measured simply by the number of unit currents which embrace it. The unit cells in this case are portions of space in which unit of magnetic quantity is produced by unity of magnetizing force. The length of a cell is therefore inversely as the intensity of the magnetizing force, and its section is inversely as the quantity of magnetic induction at that point.

The whole number of cells due to a given current is therefore proportional to the strength of the current multiplied by the number of lines of force which pass through it. If by any change of the form of the conductors the number of cells can be increased, there will be a force tending to produce that change, so that there is always a force urging a conductor transverse to the lines of magnetic force, so as to cause more lines of force to pass through the closed circuit of which the conductor forms a part.

The number of cells due to two given currents is got by multiplying the number of lines of inductive magnetic action which pass through each by the quantity of the currents respectively. Now by (9) the number of lines which pass through the first current is the sum of its own lines and those of the second current which would pass through the first if the

* *Exp. Res.* (3122). See Art. (6) of this paper. † Art. (13).

second current alone were in action. Hence the whole number of cells will be increased by any motion which causes more lines of force to pass through either circuit, and therefore the resultant force will tend to produce such a motion, and the work done by this force during the motion will be measured by the number of new cells produced. All the actions of closed conductors on each other may be deduced from this principle.

On Electric Currents produced by Induction.

Faraday has shewn * that when a conductor moves transversely to the lines of magnetic force, an electro-motive force arises in the conductor, tending to produce a current in it. If the conductor is closed, there is a continuous current, if open, tension is the result. If a closed conductor move transversely to the lines of magnetic induction, then, if the number of lines which pass through it does not change during the motion, the electro-motive forces in the circuit will be in equilibrium, and there will be no current. Hence the electro-motive forces depend on the number of lines which are cut by the conductor during the motion. If the motion be such that a greater number of lines pass through the circuit formed by the conductor after than before the motion, then the electro-motive force will be measured by the increase of the number of lines, and will generate a current the reverse of that which would have produced the additional lines. When the number of lines of inductive magnetic action through the circuit is increased, the induced current will tend to diminish the nnmber of the lines, and when the number is diminished the induced current will tend to increase them.

That this is the true expression for the law of induced currents is shewn from the fact that, in whatever way the number of lines of magnetic induction passing through the circuit be increased, the electro-motive effect is the same, whether the increase take place by the motion of the conductor itself, or of other conductors, or of magnets, or by the change of intensity of other currents, or by the magnetization or demagnetization of neighbouring magnetic bodies, or lastly by the change of intensity of the current itself.

In all these cases the electro-motive force depends on the *change* in the number of lines of inductive magnetic action which pass through the circuit †.

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* *Exp. Res.* (3077), &c.

† The electro-magnetic forces, which tend to produce motion of the material conductor, must be carefully distinguished from the electro-motive forces, which tend to produce electric currents.

Let an electric current be passed through a mass of metal of any form. The distribution of the currents within the metal will be determined by the laws of conduction. Now let a constant electric current be passed through another conductor near the first. If the two currents are in the same direction the two conductors will be attracted towards each other, and would come nearer if not held in their positions. But though the material conductors are attracted, the currents (which are free to choose any course within the metal) will not alter their original distribution, or incline towards each other. For, since no change takes place in the system, there will be no electro-motive forces to modify the original distribution of currents.

In this case we have electro-magnetic forces acting on the material conductor, without any electro-motive forces tending to modify the current which it carries.

Let us take as another example the case of a linear conductor, not forming a closed circuit, and let it be made to traverse the lines of magnetic force, either by its own motion, or by changes in the magnetic field. An electro-motive force will act in the direction of the conductor, and, as it cannot produce a current, because there is no circuit, it will produce electric tension at the extremities. There will be no electro-magnetic attraction on the material conductor, for this attraction depends on the existence of the current within it, and this is prevented by the circuit not being closed.

Here then we have the opposite case of an electro-motive force acting on the electricity in the conductor, but no attraction on its material particles.

It is natural to suppose that a force of this kind, which depends on a change in the number of lines, is due to a change of state which is measured by the number of these lines. A closed conductor in a magnetic field may be supposed to be in a certain state arising from the magnetic action. As long as this state remains unchanged no effect takes place, but, when the state changes, electro-motive forces arise, depending as to their intensity and direction on this change of state. I cannot do better here than quote a passage from the first series of Faraday's *Experimental Researches*, Art. (60).

"While the wire is subject to either volta-electric or magneto-electric induction it appears to be in a peculiar state, for it resists the formation of an electrical current in it; whereas, if in its common condition, such a current would be produced; and when left uninfluenced it has the power of originating a current, a power which the wire does not possess under ordinary circumstances. This electrical condition of matter has not hitherto been recognised, but it probably exerts a very important influence in many if not most of the phenomena produced by currents of electricity. For reasons which will immediately appear (71) I have, after advising with several learned friends, ventured to designate it as the *electro-tonic* state." Finding that all the phenomena could be otherwise explained without reference to the electro-tonic state, Faraday in his second series rejected it as not necessary; but in his recent researches* he seems still to think that there may be some physical truth in his conjecture about this new state of bodies.

The conjecture of a philosopher so familiar with nature may sometimes be more pregnant with truth than the best established experimental law discovered by empirical inquirers, and though not bound to admit it as a physical truth, we may accept it as a new idea by which our mathematical conceptions may be rendered clearer.

In this outline of Faraday's electrical theories, as they appear from a mathematical point of view, I can do no more than simply state the mathematical methods by which I believe that electrical phenomena can be best comprehended and reduced to calculation, and my aim has been to present the mathematical ideas to the mind in an embodied form, as systems of lines or surfaces, and not as mere symbols, which neither convey the same ideas, nor readily adapt themselves to the phenomena to be explained. The idea of the electro-tonic state, however, has not yet presented itself to my mind in such a form that its nature and properties may be clearly explained without reference to mere symbols, and therefore I propose in the following investigation to use symbols freely, and to take for granted the ordinary mathematical operations. By a careful study of the laws of elastic solids and of the motions of viscous fluids, I hope to discover a method of forming a mechanical conception of this electro-tonic state adapted to general reasoning †.

Part II. *On Faraday's "Electro-tonic State."*

When a conductor moves in the neighbourhood of a current of electricity, or of a magnet, or when a current or magnet near the conductor is moved, or altered in intensity, then a force

* (3172) (3269).

† See Prof. W. Thomson *On a Mechanical Representation of Electric, Magnetic and Galvanic Forces. Camb. and Dub. Math. Jour.* Jan. 1847.

acts on the conductor and produces electric tension, or a continuous current, according as the circuit is open or closed. This current is produced only by *changes* of the electric or magnetic phenomena surrounding the conductor, and as long as these are constant there is no observed effect on the conductor. Still the conductor is in different states when near a current or magnet, and when away from its influence, since the removal or destruction of the current or magnet occasions a current, which would not have existed if the magnet or current had not been previously in action.

Considerations of this kind led Professor Faraday to connect with his discovery of the induction of electric currents, the conception of a state into which all bodies are thrown by the presence of magnets and currents. This state does not manifest itself by any known phenomena as long as it is undisturbed, but any change in this state is indicated by a current or tendency towards a current. To this state he gave the name of the "Electro-tonic State," and although he afterwards succeeded in explaining the phenomena which suggested it by means of less hypothetical conceptions, he has on several occasions hinted at the probability that some phenomena might be discovered which would render the electro-tonic state an object of legitimate induction. These speculations, into which Faraday had been led by the study of laws which he has well established, and which he abandoned only for want of experimental data for the direct proof of the unknown state, have not, I think, been made the subject of mathematical investigation. Perhaps it may be thought that the quantitative determinations of the various phenomena are not sufficiently rigorous to be made the basis of a mathematical theory; Faraday, however, has not contented himself with simply stating the numerical results of his experiments and leaving the law to be discovered by calculation. Where he has perceived a law he has at once stated it, in terms as unambiguous as those of pure mathematics; and if the mathematician, receiving this as a physical truth, deduces from it other laws capable of being tested by experiment, he has merely assisted the physicist in arranging his own ideas, which is confessedly a necessary step in scientific induction.

In the following investigation, therefore, the laws established by Faraday will be assumed as true, and it will be shewn that by following out his speculations other and more general laws can be deduced from them. If it should then appear that these laws, originally devised to include one set of phenomena, may be generalized so as to extend to phenomena of a different class, these mathematical connexions may suggest to physicists the means of establishing physical connexions; and thus mere speculation may be turned to account in experimental science.

On Quantity and Intensity as Properties of Electric Currents.

It is found that certain effects of an electric current are equal at whatever part of the circuit they are estimated. The quantities of water or of any other electrolyte decomposed at two different sections of the same circuit, are always found to be equal or equivalent, however different the material and form of the circuit may be at the two sections. The magnetic effect of a conducting wire is also found to be independent of the form or material of the wire in the same circuit. There is therefore an electrical effect which is equal at every section of the circuit. If we conceive of the conductor as the channel along which a fluid is constrained to move, then the quantity of fluid transmitted by each section will be the same, and we may define the *quantity* of an electric current to be the quantity of electricity which passes across a complete section of the current in unit of time. We may for the present measure quantity of electricity by the quantity of water which it would decompose in unit of time.

In order to express mathematically the electrical currents in any conductor, we must have a definition, not only of the entire flow across a complete section, but also of the flow at a given point in a given direction.

DEF. The quantity of a current at a given point and in a given direction is measured, when uniform, by the quantity of electricity which flows across unit of area taken at that point perpendicular to the given direction, and when variable by the quantity which would flow across this area, supposing the flow uniformly the same as at the given point.

In the following investigation, the quantity of electric current at the point (xyz) estimated in the directions of the axes x, y, z respectively will be denoted by $a_2\ b_2\ c_2$.

The quantity of electricity which flows in unit of time through the elementary area dS

$$= dS\,(la_2 + mb_2 + nc_2),$$

where lmn are the direction-cosines of the normal to dS.

This flow of electricity at any point of a conductor is due to the electro-motive forces which act at that point. These may be either external or internal.

External electro-motive forces arise either from the relative motion of currents and magnets, or from changes in their intensity, or from other causes acting at a distance.

Internal electro-motive forces arise principally from difference of electric tension at points of the conductor in the immediate neighbourhood of the point in question. The other causes are variations of chemical composition or of temperature in contiguous parts of the conductor.

Let p_2 represent the electric tension at any point, and $X_2\ Y_2\ Z_2$ the sums of the parts of all the electro-motive forces arising from other causes resolved parallel to the co-ordinate axes, then if $\alpha_2\ \beta_2\ \gamma_2$ be the effective electro-motive forces

$$\left.\begin{aligned} \alpha_2 &= X_2 - \frac{dp_2}{dx} \\ \beta_2 &= Y_2 - \frac{dp_2}{dy} \\ \gamma_2 &= Z_2 - \frac{dp_2}{dz} \end{aligned}\right\} \quad \text{(A)}$$

Now the quantity of the current depends on the electro-motive force and on the resistance of the medium. If the resistance of the medium be uniform in all directions and equal to k_2,

$$\alpha_2 = k_2 a_2, \qquad \beta_2 = k_2 b_2, \qquad \gamma_2 = k_2 c_2, \qquad \text{(B)}$$

but if the resistance be different in different directions, the law will be more complicated.

These quantities $\alpha_2\ \beta_2\ \gamma_2$ may be considered as representing the intensity of the electric action in the directions of xyz.

The intensity measured along an element $d\sigma$ of a curve

$$\epsilon = l\alpha + m\beta + n\gamma,$$

where lmn are the direction-cosines of the tangent.

The integral $\int\epsilon d\sigma$ taken with respect to a given portion of a curve line, represents the total intensity along that line. If the curve is a closed one, it represents the total intensity of the electro-motive force in the closed curve.

Substituting the values of $\alpha\beta\gamma$ from equations (A)

$$\int\epsilon d\sigma = \int(Xdx + Ydy + Zdz) - p + C.$$

If, therefore $(Xdx + Ydy + Zdz)$ is a complete differential, the value of $\int\epsilon d\sigma$ for a closed curve will vanish, and in all closed curves

$$\int\epsilon d\sigma = \int(Xdx + Ydy + Zdz),$$

the integration being effected along the curve, so that in a closed curve the total intensity of the effective electro-motive force is equal to the total intensity of the impressed electro-motive force.

The total *quantity* of conduction through any surface is expressed by

$$\int edS,$$

where

$$e = la + mb + nc,$$

lmn being the direction-cosines of the normal,

$$\therefore \quad \int edS = \iint adydz + \iint bdzdx + \iint cdxdy,$$

the integrations being effected over the given surface. When the surface is a closed one, then we may find by integration by parts

$$\int edS = \iiint\left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz}\right)dx\,dy\,dz.$$

If we make

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 4\pi\rho \text{............(C)}$$

$$\int edS = 4\pi\iiint\rho dxdydz,$$

where the integration on the right side of the equation is effected over every part of space within the surface. In a large class of phenomena, including all cases of uniform currents, the quantity ρ disappears.

Magnetic Quantity and Intensity.

From his study of the lines of magnetic force, Faraday has been led to the conclusion that in the tubular surface* formed by a system of such lines, the quantity of magnetic induction across any section of the tube is constant, and that the alteration of the character of these lines in passing from one substance to another, is to be explained by a difference of *inductive capacity* in the two substances, which is analogous to conductive power in the theory of electric currents.

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* *Exp. Res.* 3271, definition of "Sphondyloid."

In the following investigation we shall have occasion to treat of magnetic quantity and intensity in connexion with electric. In such cases the magnetic symbols will be distinguished by the suffix 1, and the electric by the suffix 2. The equations connecting a, b, c, k, α, β, γ, p, and ρ, are the same in form as those which we have just given. a, b, c are the symbols of magnetic induction with respect to quantity; k, denotes the resistance to magnetic induction, and may be different in different directions; α, β, γ, are the effective magnetizing forces, connected with a, b, c, by equations (B); p, is the magnetic tension or potential which will be afterwards explained; ρ denotes the density of *real magnetic matter* and is connected with a, b, c by equations (C). As all the details of magnetic calculations will be more intelligible after the exposition of the connexion of magnetism with electricity, it will be sufficient here to say that all the definitions of total quantity, with respect to a surface, and total intensity with respect to a curve, apply to the case of magnetism as well as to that of electricity.

Electro-magnetism.

Ampère has proved the following laws of the attractions and repulsions of electric currents:

I. Equal and opposite currents generate equal and opposite forces.

II. A crooked current is equivalent to a straight one, provided the two currents nearly coincide throughout their whole length.

III. Equal currents traversing similar and similarly situated closed curves act with equal forces, whatever be the linear dimensions of the circuits.

IV. A closed current exerts no force tending to turn a circular conductor about its centre.

It is to be observed, that the currents with which Ampère worked were constant and therefore re-entering. All his results are therefore deduced from experiments on closed currents, and his expressions for the mutual action of the elements of a current involve the assumption that this action is exerted in the direction of the line joining those elements. This assumption is no doubt warranted by the universal consent of men of science in treating of attractive forces considered as due to the mutual action of particles; but at present we are proceeding on a different principle, and searching for the explanation of the phenomena, not in the currents alone, but also in the surrounding medium.

The first and second laws shew that currents are to be combined like velocities or forces.

The third law is the expression of a property of all attractions which may be conceived of as depending on the inverse square of the distance from a fixed system of points; and the fourth shews that the electro-magnetic forces may always be reduced to the attractions and repulsions of imaginary matter properly distributed.

In fact, the action of a very small electric circuit on a point in its neighbourhood is identical with that of a small magnetic element on a point outside it. If we divide any given portion of a surface into elementary areas, and cause equal currents to flow in the same direction round all these little areas, the effect on a point not in the surface will be the same as that of a shell coinciding with the surface, and uniformly magnetized normal to its surface. But by the first law all the currents forming the little circuits will destroy one another, and leave a single current running round the bounding line. So that the magnetic effect of a uniformly magnetized shell is equivalent to that of an electric current round the edge of the shell. If the direction of the current coincide with that of the apparent motion of the sun, then the direction of magnetization of the imaginary shell will be the same as that of the real magnetization of the earth*.

The total intensity of magnetizing force in a closed curve passing through and embracing the closed current is constant, and may therefore be made a measure of the quantity of the current. As this intensity is independent of the form of the closed curve and depends only on the quantity of the current which passes through it, we may consider the elementary case of the current which flows through the elementary area *dydz*.

Let the axis of x point towards the west, z towards the south, and y upwards. Let xyz be the position of a point in the middle of the area *dydz*, then the total intensity measured round the four sides of the element is

$$+\left(B_1 + \frac{d\beta_1}{dz}\frac{dz}{2}\right)dy,$$

$$-\left(\gamma_1 + \frac{d\gamma_1}{dy}\frac{dy}{2}\right)dz,$$

$$-\left(\beta_1 - \frac{d\beta_1}{dz}\frac{dz}{2}\right)dy,$$

$$+\left(\gamma_1 - \frac{d\gamma_1}{dy}\frac{dy}{2}\right)dz,$$

$$\text{Total intensity} = \left(\frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}\right)dy\,dz.$$

The quantity of electricity conducted through the elementary area *dydz* is $a_2 dydz$, and therefore if we define the measure of an electric current to be the total intensity of magnetizing force in a closed curve embracing it, we shall have

$$a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy},$$

$$b_2 = \frac{d\gamma_1}{dx} - \frac{d\alpha_1}{dz},$$

$$c_2 = \frac{d\alpha_1}{dy} - \frac{d\beta_1}{dx}.$$

These equations enable us to deduce the distribution of the currents of electricity whenever we know the values of α, β, γ, the magnetic intensities. If α, β, γ be exact differentials of a function of xyz with respect to x, y and z respectively, then the values of a_2 b_2 c_2 disappear;

* See *Experimental Researches* (3265) for the relations between the electrical and magnetic circuit, considered as *mutually embracing* curves.

and we know that the magnetism is not produced by electric currents in that part of the field which we are investigating. It is due either to the presence of permanent magnetism within the field, or to magnetizing forces due to external causes.

We may observe that the above equations give by differentiation

$$\frac{da_2}{dx} + \frac{db_2}{dy} + \frac{dc_2}{dz} = 0,$$

which is the equation of continuity for closed currents. Our investigations are therefore for the present limited to closed currents; and in fact we know little of the magnetic effects of any currents which are not closed.

Before entering on the calculation of these electric and magnetic states it may be advantageous to state certain general theorems, the truth of which may be established analytically.

THEOREM I.

The equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0,$$

(where V and ρ are functions of xyz never infinite, and vanishing for all points at an infinite distance,) can be satisfied by one, and only one, value of V. See Art. (17) above.

THEOREM II.

The value of V which will satisfy the above conditions is found by integrating the expression

$$\iiint \frac{\rho\, dxdydz}{(\overline{x-x'}|^2 + \overline{y-y'}|^2 + \overline{z-z'}|^2)^{\frac{1}{2}}},$$

where the limits of xyz are such as to include every point of space where ρ is finite.

The proofs of these theorems may be found in any work on attractions or electricity, and in particular in Green's *Essay on the Application of Mathematics to Electricity*. See Arts. 18, 19 of this Paper. See also Gauss, *on Attractions*, translated in Taylor's *Scientific Memoirs*.

THEOREM III.

Let U and V be two functions of xyz, then

$$\iiint U\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right) dxdydz = -\iiint \left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right) dxdydz$$

$$= \iiint \left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right) V dx\, dy\, dz\,;$$

where the integrations are supposed to extend over all the space in which U and V have values differing from 0.—(Green, p. 10.)

This theorem shews that if there be two attracting systems the actions between them are equal and opposite. And by making $U = V$ we find that the potential of a system on itself is proportional to the integral of the square of the resultant attraction through all space; a result deducible from Art. (30), since the volume of each cell is inversely as the square of the velocity (Arts. 12, 13), and therefore the number of cells in a given space is directly as the square of the velocity.

THEOREM IV.

Let $\alpha, \beta, \gamma, \rho$ be quantities finite through a certain space and vanishing in the space beyond, and let k be given for all parts of space as a continuous or discontinuous function of xyz, then the equation in p

$$\frac{d}{dx}\frac{1}{k}\left(\alpha - \frac{dp}{dx}\right) + \frac{d}{dy}\frac{1}{k}\left(\beta - \frac{dp}{dy}\right) + \frac{d}{dz}\frac{1}{k}\left(\gamma - \frac{dp}{dz}\right) + 4\pi\rho = 0,$$

has one, and only one solution, in which p is always finite and vanishes at an infinite distance.

The proof of this theorem, by Prof. W. Thomson, may be found in the *Cambridge and Dublin Math. Journal*, Jan. 1848.

If $\alpha\beta\gamma$ be the electro-motive forces, p the electric tension, and k the coefficient of resistance, then the above equation is identical with the equation of continuity

$$\frac{da_2}{dx} + \frac{db_2}{dy} + \frac{dc_2}{dz} + 4\pi\rho = 0;$$

and the theorem shews that when the electro-motive forces and the rate of production of electricity at every part of space are given, the value of the electric tension is determinate.

Since the mathematical laws of magnetism are identical with those of electricity, as far as we now consider them, we may regard $\alpha\beta\gamma$ as magnetizing forces, p as *magnetic tension*, and ρ as *real magnetic density*, k being the coefficient of resistance to magnetic induction.

The proof of this theorem rests on the determination of the minimum value of

$$Q = \iiint\left\{\frac{1}{k}\left(\alpha - \frac{dp}{dx} - k\frac{dV}{dx}\right)^2 + \frac{1}{k}\left(\beta - \frac{dp}{dy} - k\frac{dV}{dy}\right)^2 + \frac{1}{k}\left(\gamma - \frac{dp}{dz} - k\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz;$$

where V is got from the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0,$$

and p has to be determined.

The meaning of this integral in electrical language may be thus brought out. If the presence of the media in which k has various values did not affect the distribution of forces, then the "quantity" resolved in x would be simply $\frac{dV}{dx}$ and the intensity $k\frac{dV}{dx}$. But the actual quantity and intensity are $\frac{1}{k}\left(\alpha - \frac{dp}{dx}\right)$ and $\alpha - \frac{dp}{dx}$, and the parts due to the distribution of media alone are therefore

$$\frac{1}{k}\left(\alpha - \frac{dp}{dx}\right) - \frac{dV}{dx} \text{ and } \alpha - \frac{dp}{dx} - k\frac{dV}{dx}.$$

Now the product of these represents the work done on account of this distribution of media, the distribution of sources being determined, and taking in the terms in y and z we get the expression Q for the total work done by that part of the whole effect at any point which is due to the distribution of conducting media, and not directly to the presence of the sources.

This quantity Q is rendered a minimum by one and only one value of p, namely, that which satisfies the original equation.

THEOREM V.

If a, b, c be three functions of x, y, z satisfying the equation

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0,$$

it is always possible to find three functions α, β, γ which shall satisfy the equations

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = a,$$

$$\frac{d\gamma}{dx} - \frac{d\alpha}{dz} = b,$$

$$\frac{d\alpha}{dy} - \frac{d\beta}{dx} = c.$$

Let $A = \int c dy$, where the integration is to be performed upon c considered as a function of y, treating x and z as constants. Let $B = \int a dz$, $C = \int b dx$, $A' = \int b dz$, $B' = \int c dx$, $C' = \int a dy$, integrated in the same way.

Then

$$\alpha = A - A' + \frac{d\psi}{dx},$$

$$\beta = B - B' + \frac{d\psi}{dy},$$

$$\gamma = C - C' + \frac{d\psi}{dz}$$

will satisfy the given equations; for

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = \int \frac{da}{dy}\, dz - \int \frac{dc}{dz}\, dx - \int \frac{db}{dy}\, dx + \int \frac{da}{dy} dy,$$

and

$$0 = \int \frac{da}{dx}\, dx + \int \frac{db}{dy}\, dx + \int \frac{dc}{dz}\, dx;$$

$$\therefore \frac{d\beta}{dz} - \frac{d\gamma}{dy} = \int \frac{da}{dx}\, dx + \int \frac{da}{dy}\, dy + \int \frac{da}{dz} dz$$

$$= a.$$

In the same way it may be shewn that the values of α, β, γ satisfy the other given equations. The function ψ may be considered at present as perfectly indeterminate.

The method here given is taken from Prof. W. Thomson's memoir on Magnetism (*Phil. Trans.* 1851, p. 283).

As we cannot perform the required integrations when a, b, c are discontinuous functions of x, y, z, the following method, which is perfectly general though more complicated, may indicate more clearly the truth of the proposition.

Let A, B, C be determined from the equations

$$\frac{d^2A}{dx^2} + \frac{d^2A}{dy^2} + \frac{d^2A}{dz^2} + a = 0,$$

$$\frac{d^2B}{dx^2} + \frac{d^2B}{dy^2} + \frac{d^2B}{dz^2} + b = 0,$$

$$\frac{d^2C}{dx^2} + \frac{d^2C}{dy^2} + \frac{d^2C}{dz^2} + c = 0,$$

by the methods of Theorems I. and II., so that A, B, C are never infinite, and vanish when x, y, or z is infinite.

Also let

$$\alpha = \frac{dB}{dz} - \frac{dC}{dy} + \frac{d\psi}{dx},$$

$$\beta = \frac{dC}{dx} - \frac{dA}{dz} + \frac{d\psi}{dy},$$

$$\gamma = \frac{dA}{dy} - \frac{dB}{dx} + \frac{d\psi}{dz},$$

then

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = \frac{d}{dx}\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) - \left(\frac{d^2A}{dx^2} + \frac{d^2A}{dy^2} + \frac{d^2A}{dz^2}\right)$$

$$= \frac{d}{dx}\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) + a.$$

If we find similar equations in y and z, and differentiate the first by x, the second by y, and the third by z, remembering the equation between a, b, c, we shall have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) = 0;$$

and since A, B, C are always finite and vanish at an infinite distance, the only solution of this equation is

$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = 0,$$

and we have finally

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = a,$$

with two similar equations, shewing that α, β, γ have been rightly determined.

The function ψ is to be determined from the condition

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi:$$

if the left-hand side of this equation be always zero, ψ must be zero also.

THEOREM VI.

Let a, b, c be any three functions of x, y, z, it is possible to find three functions α, β, γ and a fourth V, so that

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0,$$

and

$$a = \frac{d\beta}{dz} - \frac{d\gamma}{dy} + \frac{dV}{dx},$$

$$b = \frac{d\gamma}{dx} - \frac{d\alpha}{dz} + \frac{dV}{dy},$$

$$c = \frac{d\alpha}{dy} - \frac{d\beta}{dx} + \frac{dV}{dz}.$$

Let

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = -4\pi\rho,$$

and let V be found from the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho,$$

then

$$a' = a - \frac{dV}{dx},$$

$$b' = b - \frac{dV}{dy},$$

$$c' = c - \frac{dV}{dz},$$

satisfy the condition

$$\frac{da'}{dx} + \frac{db'}{dy} + \frac{dc'}{dz} = 0;$$

and therefore we can find three functions A, B, C, and from these α, β, γ, so as to satisfy the given equations.

THEOREM VII.

The integral throughout infinity

$$Q = \iiint (a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1)\,dxdydz,$$

where $a_1\, b_1\, c_1$, $\alpha_1\, \beta_1\, \gamma_1$ are any functions whatsoever, is capable of transformation into

$$Q = + \iiint \{4\pi p\rho_1 - (\alpha_0 a_2 + \beta_0 b_2 + \gamma_0 c_2)\}\, dx\,dy\,dz,$$

in which the quantities are found from the equations

$$\frac{da_1}{dx} + \frac{db_1}{dy} + \frac{dc_1}{dz} + 4\pi\rho_1 = 0,$$

$$\frac{d\alpha_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz} + 4\pi\rho_1{}' = 0;$$

$\alpha_0\, \beta_0\, \gamma_0\, V$ are determined from $a_1\, b_1\, c_1$ by the last theorem, so that

$$a_1 = \frac{d\beta_0}{dz} - \frac{d\gamma_0}{dy} + \frac{dV}{dx};$$

$a_2\, b_2\, c_2$ are found from $\alpha_1\, \beta_1\, \gamma_1$ by the equations

$$a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}\ \&c.,$$

and p is found from the equation

$$\frac{d^2p}{dx^2} + \frac{d^2p}{dy^2} + \frac{d^2p}{dz^2} + 4\pi\rho_1{}' = 0.$$

For, if we put a_1 in the form

$$\frac{d\beta_0}{dz} - \frac{d\gamma_0}{dy} + \frac{dV}{dx},$$

and treat b_1 and c_1 similarly, then we have by integration by parts through infinity, remembering that all the functions vanish at the limits,

$$Q = -\iiint \left\{ V\left(\frac{d\alpha_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz}\right) + \alpha_0\left(\frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}\right) + \beta_0\left(\frac{d\gamma_1}{dx} - \frac{d\alpha_1}{dz}\right) + \gamma_0\left(\frac{d\alpha_1}{dy} - \frac{d\beta_1}{dx}\right)\right\} dx\,dy\,dz,$$

$$\text{or} \qquad Q = + \iiint \{(4\pi V\rho') - (\alpha_0 a_2 + \beta_0 b_2 + \gamma_0 c_2)\}\, dx\,dy\,dz,$$

and by Theorem III.

$$\iiint V\rho'\, dx\,dy\,dz = \iiint p\rho\, dx\,dy\,dz,$$

so that finally

$$Q = \iiint \{4\pi p\rho - (\alpha_0 a_2 + \beta_0 b_2 = \gamma_0 c_2)\}\, dx\,dy\,dz.$$

If $a_1\, b_1\, c_1$ represent the components of magnetic quantity, and $\alpha_1\, \beta_1\, \gamma_1$ those of magnetic intensity, then ρ will represent the *real magnetic density*, and p the magnetic potential or tension. $a_2\, b_2\, c_2$ will be the components of quantity of electric currents, and $\alpha_0\, \beta_0\, \gamma_0$ will be three functions deduced from $a_1\, b_1\, c_1$, which will be found to be the mathematical expression for Faraday's Electro-tonic state.

Let us now consider the bearing of these analytical theorems on the theory of magnetism. Whenever we deal with quantities relating to magnetism, we shall distinguish them by the suffix ($_1$). Thus $a_1\, b_1\, c_1$ are the components resolved in the directions of x, y, z of the quantity of magnetic induction acting through a given point, and $\alpha_1\beta_1\gamma_1$ are the resolved intensities of magnetization at the same point, or, what is the same thing, the components of the force which would be exerted on a unit south pole of a magnet placed at that point without disturbing the distribution of magnetism.

The electric currents are found from the magnetic intensities by the equations

$$a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy} \text{ \&c.}$$

When there are no electric currents, then

$$\alpha_1 dx + \beta_1 dy + \gamma_1 dz = dp_1,$$

a perfect differential of a function of x, y, z. On the principle of analogy we may call p_1 the magnetic tension.

The forces which act on a mass m of south magnetism at any point are

$$-m\frac{dp_1}{dx},\ -m\frac{dp_1}{dy},\ \text{and}\ -m\frac{dp_1}{dz},$$

in the direction of the axes, and therefore the whole work done during any displacement of a magnetic system is equal to the decrement of the integral

$$Q = \iiint \rho_1 p_1 dx dy dz$$

throughout the system.

Let us now call Q the *total potential of the system on itself*. The increase or decrease of Q will measure the work lost or gained by any displacement of any part of the system, and will therefore enable us to determine the forces acting on that part of the system.

By Theorem III. Q may be put under the form

$$Q = +\frac{1}{4\pi}\iiint (a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1) dx dy dz,$$

in which $\alpha_1\,\beta_1\,\gamma_1$ are the differential coefficients of p_1 with respect to x, y, z respectively.

If we now assume that this expression for Q is true whatever be the values of $\alpha_1\,\beta_1\,\gamma_1$, we pass from the consideration of the magnetism of permanent magnets to that of the magnetic effects of electric currents, and we have then by Theorem VII.

$$Q = \iiint \left\{ p_1\rho_1 - \frac{1}{4\pi}\left(\alpha_0 a_2 + \beta_0 b_2 + \gamma_0 c_2\right)\right\} dx dy dz.$$

So that in the case of electric currents, the components of the currents have to be multiplied by the functions $\alpha_0\beta_0\gamma_0$ respectively, and the summations of all such products throughout the system gives us the part of Q due to those currents.

We have now obtained in the functions $\alpha_0\,\beta_0\,\gamma_0$ the means of avoiding the consideration of the quantity of magnetic induction which *passes through* the circuit. Instead of this artificial method we have the natural one of considering the current with reference to quantities existing in the same space with the current itself. To these I give the name of *Electro-tonic functions*, or *components of the Electro-tonic intensity*.

Let us now consider the conditions of the conduction of the electric currents within the medium during changes in the electro-tonic state. The method which we shall adopt is an application of that given by Helmholtz in his memoir on the Conservation of Force*.

Let there be some external source of electric currents which would generate in the conducting mass currents whose quantity is measured by $a_2\, b_2\, c_2$ and their intensity by $\alpha_2\, \beta_2\, \gamma_2$.

Then the amount of work due to this cause in the time dt is

$$dt\iiint (a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2)\,dxdydz$$

in the form of resistance overcome, and

$$\frac{dt}{4\pi}\frac{d}{dt}\iiint (a_2\alpha_0 + b_2\beta_0 + c_2\gamma_0)\,dxdydz$$

in the form of work done mechanically by the electro-magnetic action of these currents. If there be no external cause producing currents, then the quantity representing the whole work done by the external cause must vanish, and we have

$$dt\iiint (a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2)\,dxdydz + \frac{dt}{4\pi}\frac{d}{dt}\iiint (a_2\alpha_0 + b_2\beta_0 + c_2\gamma_0)\,dxdydz,$$

where the integrals are taken through any arbitrary space. We must therefore have

$$a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 = \frac{1}{4\pi}\frac{d}{dt}(a_2\alpha_0 + b_2\beta_0 + c_2\gamma_0)$$

for every point of space; and it must be remembered that the variation of Q is supposed due to variations of $\alpha_0\, \beta_0\, \gamma_0$, and not of $a_2\, b_2\, c_2$. We must therefore treat $a_2\, b_2\, c_2$ as constants, and the equation becomes

$$a_2\left(\alpha_2 + \frac{1}{4\pi}\frac{d\alpha_0}{dt}\right) + b_2\left(\beta_2 + \frac{1}{4\pi}\frac{d\beta_0}{dt}\right) + c_2\left(\gamma_2 + \frac{1}{4\pi}\frac{d\gamma_0}{dt}\right) = 0.$$

In order that this equation may be independent of the values of $a_2\, b_2\, c_2$, each of these coefficients must $= 0$; and therefore we have the following expressions for the electro-motive forces due to the action of magnets and currents at a distance in terms of the electro-tonic functions,

$$\alpha_2 = -\frac{1}{4\pi}\frac{d\alpha_0}{dt}, \qquad \beta_2 = -\frac{1}{4\pi}\frac{d\beta_0}{dt}, \qquad \gamma_2 = -\frac{1}{4\pi}\frac{d\gamma_0}{dt}.$$

It appears from experiment that the expression $\frac{d\alpha_0}{dt}$ refers to the change of electro-tonic state of a *given particle of the conductor*, whether due to change in the electro-tonic functions themselves or to the motion of the particle.

If α_0 be expressed as a function of x, y, z, and t, and if x, y, z be the co-ordinates of a moving article, then the electro-motive force measured in the direction of x is

$$\alpha_2 = -\frac{1}{4\pi}\left(\frac{d\alpha_0}{dx}\frac{dx}{dt} + \frac{d\alpha_0}{dy}\frac{dy}{dt} + \frac{d\alpha_0}{dz}\frac{dz}{dt} + \frac{d\alpha_0}{dt}\right).$$

* Translated in Taylor's *New Scientific Memoirs*, Part II.

The expressions for the electro-motive forces in y and z are similar. The distribution of currents due to these forces depends on the form and arrangement of the conducting media and on the resultant electric tension at any point.

The discussion of these functions would involve us in mathematical formulæ, of which this paper is already too full. It is only on account of their physical importance as the mathematical expression of one of Faraday's conjectures that I have been induced to exhibit them at all in their present form. By a more patient consideration of their relations, and with the help of those who are engaged in physical inquiries both in this subject and in others not obviously connected with it, I hope to exhibit the theory of the electro-tonic state in a form in which all its relations may be distinctly conceived without reference to analytical calculations.

Summary of the Theory of the Electro-tonic State.

We may conceive of the electro-tonic state at any point of space as a quantity determinate in magnitude and direction, and we may represent the electro-tonic condition of a portion of space by any mechanical system which has at every point some quantity, which may be a velocity, a displacement, or a force, whose direction and magnitude correspond to those of the supposed electro-tonic state. This representation involves no physical theory, it is only a kind of artificial notation. In analytical investigations we make use of the three components of the electro-tonic state, and call them electro-tonic functions. We take the resolved part of the electro-tonic intensity at every point of a closed curve, and find by integration what we may call the *entire electro-tonic intensity round the curve.*

Prop. I. *If on any surface a closed curve be drawn, and if the surface within it be divided into small areas, then the entire intensity round the closed curve is equal to the sum of the intensities round each of the small areas, all estimated in the same direction.*

For, in going round the small areas, every boundary line between two of them is passed along twice in opposite directions, and the intensity gained in the one case is lost in the other. Every effect of passing along the interior divisions is therefore neutralized, and the whole effect is that due to the exterior closed curve.

Law I. *The entire electro-tonic intensity round the boundary of an element of surface measures the quantity of magnetic induction which passes through that surface, or, in other words, the number of lines of magnetic force which pass through that surface.*

By Prop. I. it appears that what is true of elementary surfaces is true also of surfaces of finite magnitude, and therefore any two surfaces which are bounded by the same closed curve will have the same quantity of magnetic induction through them.

Law II. *The magnetic intensity at any point is connected with the quantity of magnetic induction by a set of linear equations, called the equations of conduction*.*

* See Art. (28).

LAW III. *The entire magnetic intensity round the boundary of any surface measures the quantity of electric current which passes through that surface.*

LAW IV. *The quantity and intensity of electric currents are connected by a system of equations of conduction.*

By these four laws the magnetic and electric quantity and intensity may be deduced from the values of the electro-tonic functions. I have not discussed the values of the units, as that will be better done with reference to actual experiments. We come next to the attraction of conductors of currents, and to the induction of currents within conductors.

LAW V. *The total electro-magnetic potential of a closed current is measured by the product of the quantity of the current multiplied by the entire electro-tonic intensity estimated in the same direction round the circuit.*

Any displacement of the conductors which would cause an increase in the potential will be assisted by a force measured by the rate of increase of the potential, so that the mechanical work done during the displacement will be measured by the increase of potential.

Although in certain cases a displacement in direction or alteration of intensity *of the current* might increase the potential, such an alteration would not itself produce work, and there will be no tendency towards this displacement, for alterations in the current are due to electro-motive force, not to electro-magnetic attractions, which can only act on the conductor.

LAW VI. *The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.*

The electro-motive force in a closed conductor is measured by the rate of change of the entire electro-tonic intensity round the circuit referred to unit of time. It is independent of the nature of the conductor, though the current produced varies inversely as the resistance; and it is the same in whatever way the change of electro-tonic intensity has been produced, whether by motion of the conductor or by alterations in the external circumstances.

In these six laws I have endeavoured to express the idea which I believe to be the mathematical foundation of the modes of thought indicated in the *Experimental Researches.* I do not think that it contains even the shadow of a true physical theory; in fact, its chief merit as a temporary instrument of research is that it does not, even in appearance, *account for* anything.

There exists however a professedly physical theory of electro-dynamics, which is so elegant, so mathematical, and so entirely different from anything in this paper, that I must state its axioms, at the risk of repeating what ought to be well known. It is contained in M. W. Weber's *Electro-dynamic Measurements*, and may be found in the Transactions of the Leibnitz Society, and of the Royal Society of Sciences of Saxony *. The assumptions are,

(1) That two particles of electricity when in motion do not repel each other with the same force as when at rest, but that the force is altered by a quantity depending on the relative motion of the two particles, so that the expression for the repulsion at distance r is

* When this was written, I was not aware that part of M. Weber's Memoir is translated in Taylor's *Scientific Memoirs*, Vol. V. Art. XIV. The value of his researches, both experimental and theoretical, renders the study of his theory necessary to every electrician.

$$\frac{ee'}{r^2}\left(1 + a\,\overline{\frac{dr}{dt}}\right|^2 + br\,\frac{d^2r}{dt^2}\right).$$

(2) That when electricity is moving in a conductor, the velocity of the positive fluid *relatively to the matter of the conductor* is equal and opposite to that of the negative fluid.

(3) The total action of one conducting element on another is the resultant of the mutual actions of the masses of electricity of both kinds which are in each.

(4) The electro-motive force at any point is the difference of the forces acting on the positive and negative fluids.

From these axioms are deducible Ampère's laws of the attraction of conductors, and those of Neumann and others, for the induction of currents. Here then is a really physical theory, satisfying the required conditions better perhaps than any yet invented, and put forth by a philosopher whose experimental researches form an ample foundation for his mathematical investigations. What is the use then of imagining an electro-tonic state of which we have no distinctly physical conception, instead of a formula of attraction which we can readily understand? I would answer, that it is a good thing to have two ways of looking at a subject, and to admit that there *are* two ways of looking at it. Besides, I do not think that we have any right at present to understand the action of electricity, and I hold that the chief merit of a temporary theory is, that it shall guide experiment, without impeding the progress of the true theory when it appears. There are also objections to making any ultimate forces in nature depend on the velocity of the bodies between which they act. If the forces in nature are to be reduced to forces acting between particles, the principle of the Conservation of Force requires that these forces should be in the line joining the particles and functions of the distance only. The experiments of M. Weber on the reverse polarity of diamagnetics, which have been recently repeated by Professor Tyndall, establish a fact which is equally a consequence of M. Weber's theory of electricity and of the theory of lines of force.

With respect to the history of the present theory, I may state that the recognition of certain mathematical functions as expressing the "electro-tonic state" of Faraday, and the use of them in determining electro-dynamic potentials and electro-motive forces, is, as far as I am aware, original; but the distinct conception of the possibility of the mathematical expressions arose in my mind from the perusal of Prof. W. Thomson's papers "On a Mechanical Representation of Electric, Magnetic and Galvanic Forces," *Cambridge and Dublin Mathematical Journal*, January, 1847, and his "Mathematical Theory of Magnetism," *Philosophical Transactions*, Part I. 1851, Art. 78, &c. As an instance of the help which may be derived from other physical investigations, I may state that after I had investigated the Theorems of this paper Professor Stokes pointed out to me the use which he had made of similar expressions in his "Dynamical Theory of Diffraction," Section 1, *Cambridge Transactions*, Vol. IX. Part 1. Whether the theory of these functions, considered with reference to electricity, may lead to new mathematical ideas to be employed in physical research, remains to be seen. I propose in the rest of this paper to discuss a few electrical and magnetic problems with reference to spheres. These are intended merely as concrete examples of the methods of which the theory has been given; I reserve the detailed investigation of cases chosen with special reference to experiment till I have the means of testing their results.

EXAMPLES.

I. *Theory of Electrical Images.*

The method of Electrical Images, due to Prof. W. Thomson*, by which the theory of spherical conductors has been reduced to great geometrical simplicity, becomes even more simple when we see its connexion with the methods of this paper. We have seen that the pressure at any point in a uniform medium, due to a spherical shell (radius $= a$) giving out fluid at the rate of $4\pi Pa^2$ units in unit of time, is $kP\frac{a^2}{r}$ outside the shell, and kPa inside it, where r is the distance of the point from the centre of the shell.

If there be two shells, one giving out fluid at a rate $4\pi Pa^2$, and the other absorbing at the rate $4\pi P'a'^2$, then the expression for the pressure will be, outside the shells,

$$p = 4\pi P\frac{a^2}{r} - 4\pi P'\frac{a'^2}{r'},$$

where r and r' are the distances from the centres of the two shells. Equating this expression to zero we have, as the surface of no pressure, that for which

$$\frac{r'}{r} = \frac{P'a'^2}{Pa^2}.$$

Now the surface, for which the distances to two fixed points have a given ratio, is a sphere of which the centre O is in the line joining the centres of the shells CC' produced, so that

$$C'O = CC'\frac{\overline{P'a'^2}|^2}{\overline{Pa^2}|^2 - \overline{P'a'^2}|^2}$$

and its radius

$$= CC'\frac{Pa^2 . P'a'^2}{\overline{Pa^2}|^2 - \overline{P'a'^2}|^2}.$$

If at the centre of this sphere we place another source of the fluid, then the pressure due to this source must be added to that due to the other two; and since this additional pressure depends only on the distance from the centre, it will be constant at the surface of the sphere, where the pressure due to the two other sources is zero.

We have now the means of arranging a system of sources within a given sphere, so that when combined with a given system of sources outside the sphere, they shall produce a given constant pressure at the surface of the sphere.

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* See a series of papers "On the Mathematical Theory of Electricity," in the *Cambridge and Dublin Math. Jour.*, beginning March, 1848.

Let a be the radius of the sphere, and p the given pressure, and let the given sources be at distances b_1 b_2 &c. from the centre, and let their rates of production be $4\pi P_1$, $4\pi P_2$ &c.

Then if at distances $\frac{a^2}{b_1}$, $\frac{a^2}{b_2}$ &c. (measured in the same direction as b_1 b_2 &c. from the centre) we place negative sources whose rates are

$$-4\pi P_1 \frac{a}{b_1}, \quad -4\pi P_2 \frac{a}{b_2} \text{ \&c.}$$

the pressure at the surface $r = a$ will be reduced to zero. Now placing a source $4\pi \frac{pa}{k}$ at the centre, the pressure at the surface will be uniform and equal to p.

The whole amount of fluid emitted by the surface $r = a$ may be found by adding the rates of production of the sources within it. The result is

$$4\pi a \left\{\frac{p}{k} - \frac{P_1}{b_1} - \frac{P_2}{b_2} - \text{\&c.}\right\}.$$

To apply this result to the case of a conducting sphere, let us suppose the external sources $4\pi P_1$, $4\pi P_2$ to be small electrified bodies, containing e_1 e_2 of positive electricity. Let us also suppose that the whole charge of the conducting sphere is $= E$ previous to the action of the external points. Then all that is required for the complete solution of the problem is, that the surface of the sphere shall be a surface of equal potential, and that the total charge of the surface shall be E.

If by any distribution of imaginary sources within the spherical surface we can effect this, the value of the corresponding potential outside the sphere is the true and only one. The potential inside the sphere must really be constant and equal to that at the surface.

We must therefore find the *images* of the external electrified points, that is, for every point at distance b from the centre we must find a point on the same radius at a distance $\frac{a^2}{b_1}$, and at that point we must place a quantity $= -e\,\frac{a}{b_1}$ of imaginary electricity.

At the centre we must put a quantity E' such that

$$E' = E + e_1 \frac{a}{b_1} + e_2 \frac{a}{b_2} + \text{\&c.};$$

then if R be the distance from the centre, $r_1 r_2$ &c. the distances from the electrified points, and $r'_1 r'_2$ the distances from their images at any point outside the sphere, the potential at that point will be

$$p = \frac{E'}{R} + e_1\left(\frac{1}{r_1} - \frac{a}{b_1}\frac{1}{r'_1}\right) + e_2\left(\frac{1}{r_2} - \frac{a}{b_2}\frac{1}{r'_2}\right) + \text{\&c.}$$

$$= \frac{E}{R} + \frac{e_1}{b_1}\left(\frac{a}{R} + \frac{b_1}{r_1} - \frac{a}{r'_1}\right) + \frac{e_2}{b_2}\left(\frac{a}{R} + \frac{b_2}{r_2} - \frac{a}{r'_2}\right) + \text{\&c.}$$

This is the value of the potential outside the sphere. At the surface we have

$$R=a \text{ and } \frac{b_1}{r_1}=\frac{a}{r'_1}, \quad \frac{b_2}{r_2}=\frac{a}{r'_2} \text{ \&c.}$$

so that at the surface

$$p=\frac{E}{a}+\frac{e_1}{b_1}+\frac{e_2}{b_2}+\text{\&c.}$$

and this must also be the value of p for any point within the sphere.

For the application of the principle of electrical images the reader is referred to Prof. Thomson's papers in the *Cambridge and Dublin Mathematical Journal*. The only case which we shall consider is that in which $\frac{e_1}{b_1^2}=I$, and b_1 is infinitely distant along axis of x, and $E=0$.

The value p outside the sphere becomes then

$$p=Ix\left(-\frac{a^3}{r^3}\right),$$

and inside $p=0$.

II. *On the effect of a paramagnetic or diamagnetic sphere in a uniform field of magnetic force*.*

The expression for the potential of a small magnet placed at the origin of co-ordinates in the direction of the axis of x is

$$\frac{d}{dx}\left(\frac{m}{r}\right)=-lm\frac{x}{r^3}.$$

The effect of the sphere in disturbing the lines of force may be supposed as a first hypothesis to be similar to that of a small magnet at the origin, whose strength is to be determined. (We shall find this to be accurately true.)

Let the value of the potential undisturbed by the presence of the sphere be

$$p=Ix.$$

Let the sphere produce an additional potential, which for external points is

$$p'=A\frac{a^3}{r^3}x,$$

and let the potential within the sphere be

$$p_1=Bx.$$

Let k' be the coefficient of resistance outside, and k inside the sphere, then the conditions to be fulfilled are, that the interior and exterior potential should coincide at the

* See Prof. Thomson, on the Theory of Magnetic Induction, *Phil. Mag.* March, 1851. The *inductive capacity* of the sphere, according to that paper, is the ratio of the *quantity* of magnetic induction (not the *intensity*) within the sphere to that without. It is therefore equal to $\frac{1}{I}B\frac{k'}{k}=\frac{3k'}{2k+k'}$, according to our notation.

surface, and that the induction through the surface should be the same whether deduced from the external or the internal potential. Putting $x = r\cos\theta$, we have for the external potential

$$p = \left(Ir + A\frac{a^3}{r^2}\right)\cos\theta,$$

and for the internal

$$p_1 = Br\cos\theta,$$

and these must be identical when $r = a$, or

$$I + A = B.$$

The induction through the surface in the external medium is

$$\frac{1}{k'}\frac{dp}{dr_{r=a}} = \frac{1}{k'}(I - 2A)\cos\theta,$$

and that through the interior surface is

$$\frac{1}{k}\frac{dp_1}{dr_{r=a}} = \frac{1}{k}B\cos\theta;$$

$$\text{and } \therefore \frac{1}{k'}(I - 2A) = \frac{1}{k}B.$$

These equations give

$$A = \frac{k - k'}{2k + k'}I, \qquad B = \frac{3k}{2k + k'}I.$$

The effect outside the sphere is equal to that of a little magnet whose length is l and moment ml, provided

$$ml = \frac{k - k'}{2k + k'}a^3 I.$$

Suppose this uniform field to be that due to terrestrial magnetism, then, if k is less than k' as in paramagnetic bodies, the marked end of the equivalent magnet will be turned to the north. If k is greater than k' as in diamagnetic bodies, the unmarked end of the equivalent magnet would be turned to the north.

III. *Magnetic field of variable Intensity.*

Now suppose the intensity in the undisturbed magnetic field to vary in magnitude and direction from one point to another, and that its components in xyz are represented by $\alpha_1\beta_1\gamma_1$, then, if as a first approximation we regard the intensity within the sphere as sensibly equal to that at the centre, the change of potential outside the sphere arising from the presence of the sphere, disturbing the lines of force, will be the same as that due to three small magnets at the centre, with their axes parallel to x, y, and z, and their moments equal to

$$\frac{k-k'}{2k+k'}\,a^3\alpha, \qquad \frac{k-k'}{2k+k'}\,a^3\beta, \qquad \frac{k-k'}{2k+k'}\,a^3\gamma.$$

The actual distribution of potential within and without the sphere may be conceived as the result of a distribution of imaginary magnetic matter on the surface of the sphere; but since the external effect of this superficial magnetism is exactly the same as that of the three small magnets at the centre, the mechanical effect of external attractions will be the same as if the three magnets really existed.

Now let three small magnets whose lengths are $l_1\,l_2\,l_3$, and strengths $m_1\,m_2\,m_3$ exist at the point $x\,y\,z$ with their axes parallel to the axes of $x\,y\,z$; then, resolving the forces on the three magnets in the direction of X, we have

$$-X = m_1\left\{\begin{matrix} \alpha_1 + \dfrac{d\alpha}{dx}\dfrac{l_1}{2} \\ -\alpha_1 + \dfrac{d\alpha}{dx}\dfrac{l_1}{2} \end{matrix}\right\} + m_2\left\{\begin{matrix} \alpha_1 + \dfrac{d\alpha}{dy}\dfrac{l_2}{2} \\ -\alpha_1 + \dfrac{d\alpha}{dy}\dfrac{l_2}{2} \end{matrix}\right\} + m_3\left\{\begin{matrix} \alpha + \dfrac{d\alpha}{dz}\dfrac{l_3}{2} \\ -\alpha + \dfrac{d\alpha}{dz}\dfrac{l_2}{2} \end{matrix}\right\}$$

$$= m_1 l_1 \frac{d\alpha}{dx} + m_2 l_2 \frac{d\alpha}{dy} + m_3 l_3 \frac{d\alpha}{dz}.$$

Substituting the values of the moments of the imaginary magnets

$$-X = \frac{k-k'}{2k+k'}\,a^3\left(\alpha\frac{d\alpha}{dx} + \beta\frac{d\beta}{dx} + \gamma\frac{d\gamma}{dx}\right) = \frac{k-k'}{2k+k'}\,\frac{a^3}{2}\,\frac{d}{dx}(\alpha^2+\beta^2+\gamma^2).$$

The force impelling the sphere in the direction of x is therefore dependent on the variation of the square of the intensity or $(\alpha^2+\beta^2+\gamma^2)$, as we move along the direction of x, and the same is true for y and z, so that the law is, that the force acting on diamagnetic spheres is from places of greater to places of less intensity of magnetic force, and that in similar distributions of magnetic force it varies as the mass of the sphere and the square of the intensity.

It is easy by means of Laplace's Coefficients to extend the approximation to the value of the potential as far as we please, and to calculate the attraction. For instance, if a north or south magnetic pole whose strength is M, be placed at a distance b from a diamagnetic sphere, radius a, the repulsion will be

$$R = M^2(k-k')\frac{a^3}{b^5}\left(\frac{2\,.\,1}{2k+k'} + \frac{3\,.\,2}{3k+2k'}\frac{a^2}{b^2} + \frac{4\,.\,3}{4k+3k'}\frac{a^4}{b^4} + \text{\&c.}\right)$$

When $\frac{a}{b}$ is small, the first term gives a sufficient approximation. The repulsion is then as the square of the strength of the pole and the mass of the sphere directly and the fifth power of the distance inversely, considering the pole as a point.

IV. *Two Spheres in uniform field.*

Let two spheres of radius a be connected together so that their centres are kept at a distance b, and let them be suspended in a uniform magnetic field, then, although each sphere by itself would have been in equilibrium at any part of the field, the disturbance of the field will produce forces tending to make the balls set in a particular direction.

Let the centre of one of the spheres be taken as origin, then the undisturbed potential is

$$p = I r \cos\theta,$$

and the potential due to the sphere is

$$p' = I \frac{k - k'}{2k + k'} \frac{a^3}{r^2} \cos\theta.$$

The whole potential is therefore equal to

$$I\left(r + \frac{k - k'}{2k + k'} \frac{a^3}{r^2}\right)\cos\theta = p,$$

$$\frac{dp}{dr} = I\left(1 - 2\frac{k - k'}{2k + k'} \frac{a^3}{r^3}\right)\cos\theta,$$

$$\frac{1}{r}\frac{dp}{d\theta} = -I\left(1 + \frac{k - k'}{2k + k'} \frac{a^3}{r^3}\right)\sin\theta, \qquad \frac{dp}{d\phi} = 0,$$

$$\therefore i^2 = \overline{\frac{dp}{dr}}\Big|^2 + \frac{1}{r^2}\overline{\frac{dp}{d\theta}}\Big|^2 + \frac{1}{r^2\sin^2\theta}\overline{\frac{dp}{d\phi}}\Big|^2 = I^2\left\{1 + \frac{k - k'}{2k + k'}\frac{a^3}{r^3}(1 - 3\cos^2\theta) + \overline{\frac{k - k'}{2k + k'}}\Big|^2 \frac{a^6}{r^6}(1 + 3\cos^2\theta)\right\}.$$

This is the value of the square of the intensity at any point. The moment of the couple tending to turn the combination of balls in the direction of the original force

$$L = \tfrac{1}{2}\frac{d}{d\theta} i^2 \left(\frac{k - k'}{2k + k'} a^3\right) \text{ when } r = b,$$

$$L = \tfrac{3}{2} I^2 \overline{\frac{k - k'}{2k + k'}}\Big|^2 \frac{a^6}{b^3}\left(1 - \frac{k - k'}{2k + k'}\frac{a^3}{b^3}\right)\sin 2\theta.$$

This expression, which must be positive, since b is greater than a, gives the moment of a force tending to turn the line joining the centres of the spheres towards the original lines of force.

Whether the spheres are magnetic or diamagnetic they tend to set in the *axial* direction, and that without distinction of north and south. If, however, one sphere be magnetic and the other diamagnetic, the line of centres will set equatoreally. The magnitude of the force depends on the square of $(k - k')$, and is therefore quite insensible except in iron *.

V. *Two Spheres between the poles of a Magnet.*

Let us next take the case of the same balls placed not in a uniform field but between a north and a south pole, $\pm M$, distant $2c$ from each other in the direction of x.

* See Prof. Thomson in *Phil. Mag.* March, 1851.

The expression for the potential, the middle of the line joining the poles being the origin, is

$$p = M\left(\frac{1}{\sqrt{c^2+r^2-2\cos\theta cr}} - \frac{1}{\sqrt{c^2+r^2+2\cos\theta cr}}\right).$$

From this we find as the value of I^2,

$$I^2 = \frac{4M^2}{c^4}\left(1 - 3\frac{r^2}{c^2} + 9\frac{r^2}{c^2}\cos^2\theta\right);$$

$$\therefore\ I\frac{dI}{d\theta} = -18\frac{M^2}{c^6}r^2\sin 2\theta,$$

and the moment to turn a pair of spheres (radius a, distance $2b$) in the direction in which θ is increased is

$$-36\frac{k-k'}{2k+k'},\frac{M^2a^3b^2}{c^6}\sin 2\theta.$$

This force, which tends to turn the line of centres equatoreally for diamagnetic and axially for magnetic spheres, varies directly as the square of the strength of the magnet, the cube of the radius of the spheres and the square of the distance of their centres, and inversely as the sixth power of the distance of the poles of the magnet, considered as points. As long as these poles are near each other this action of the poles will be much stronger than the mutual action of the spheres, so that as a general rule we may say that elongated bodies set axially or equatoreally between the poles of a magnet according as they are magnetic or diamagnetic. If, instead of being placed between two poles very near to each other, they had been placed in a uniform field such as that of terrestrial magnetism or that produced by a spherical electro-magnet (see Ex. VIII.), an elongated body would set axially whether magnetic or diamagnetic.

In all these cases the phenomena depend on $k - k'$, so that the sphere conducts itself magnetically or diamagnetically according as it is more or less magnetic, or less or more diamagnetic than the medium in which it is placed.

VI. *On the Magnetic Phenomena of a Sphere cut from a substance whose coefficient of resistance is different in different directions.*

Let the axes of magnetic resistance be parallel throughout the sphere, and let them be taken for the axes of x, y, z. Let k_1, k_2, k_3, be the coefficients of resistance in these three directions, and let k' be that of the external medium, and a the radius of the sphere. Let I be the undisturbed magnetic intensity of the field into which the sphere is introduced, and let its direction-cosines be l, m, n.

Let us now take the case of a homogeneous sphere whose coefficient is k_1 placed in a uniform magnetic field whose intensity is lI in the direction of x. The resultant potential outside the sphere would be

$$p' = lI\left(1 + \frac{k_1 - k'}{2k_1 + k'}\frac{a^3}{r^3}\right)x,$$

and for internal points

$$p_1 = lI \frac{3k_1}{2k_1 + k'} x.$$

So that in the interior of the sphere the magnetization is entirely in the direction of x. It is therefore quite independent of the coefficients of resistance in the directions of x and y, which may be changed from k_1 into k_2 and k_3 without disturbing this distribution of magnetism. We may therefore treat the sphere as homogeneous for each of the three components of I, but we must use a different coefficient for each. We find for external points

$$p' = I\left\{lx + my + nz + \left(\frac{k_1 - k'}{2k_1 + k'} lx + \frac{k_2 - k'}{2k_2 + k'} my + \frac{k_3 - k'}{2k_3 + k'} nz\right)\frac{a^3}{r^3}\right\},$$

and for internal points

$$p_1 = I\left(\frac{3k_1}{2k_1 + k'} lx + \frac{3k_2}{2k_2 + k'} my + \frac{3k_3}{2k_3 + k'} nz\right).$$

The external effect is the same as that which would have been produced if the small magnet whose moments are

$$\frac{k_1 - k'}{2k_1 + k'} lIa^3, \qquad \frac{k_2 - k'}{2k_2 + k'} mIa^3, \qquad \frac{k_3 - k'}{2k_3 + k'} nIa^3$$

had been placed at the origin with their directions coinciding with the axes of x, y, z. The effect of the original force I in turning the sphere about the axis of x may be found by taking the moments of the components of that force on these equivalent magnets. The moment of the force in the direction of y acting on the third magnet is

$$\frac{k_3 - k'}{2k_3 + k'} mnI^2a^3,$$

and that of the force in z on the second magnet is

$$-\frac{k_2 - k'}{2k_2 + k'} mnI^2a^3.$$

The whole couple about the axis of x is therefore

$$\frac{3k'(k_3 - k_2)}{(2k_3 + k')(2k_2 + k')} mnI^2a^3,$$

tending to turn the sphere round from the axis of y towards that of z. Suppose the sphere to be suspended so that the axis of x is vertical, and let I be horizontal, then if θ be the angle which the axis of y makes with the direction of I, $m = \cos\theta$, $n = -\sin\theta$, and the expression for the moment becomes

$$\frac{3}{2}\frac{k'(k_2 - k_3)}{(2k_2 + k')(2k_3 + k')} I^2a^3 \sin 2\theta$$

tending to increase θ. The axis of least resistance therefore sets axially, but with either end indifferently towards the north.

Since in all bodies, except iron, the values of k are nearly the same as in a vacuum, the coefficient of this quantity can be but little altered by changing the value of k' to k, the value in space. The expression then becomes

$$\frac{1}{6}\,\frac{k_2 - k_3}{k}\,I^2 a^3 \sin 2\theta,$$

independent of the external medium *.

VII. *Permanent magnetism in a spherical shell.*

The case of a homogeneous shell of a diamagnetic or paramagnetic substance presents no difficulty. The intensity within the shell is less than what it would have been if the shell were away, whether the substance of the shell be diamagnetic or paramagnetic. When the resistance of the shell is infinite, and when it vanishes, the intensity within the shell is zero.

In the case of no resistance the entire effect of the shell on any point, internal or external, may be represented by supposing a superficial stratum of magnetic matter spread over the outer surface, the density being given by the equation

$$\rho = 3I \cos\theta.$$

Suppose the shell now to be converted into a permanent magnet, so that the distribution of imaginary magnetic matter is invariable, then the external potential due to the shell will be

$$p' = -I\frac{a^3}{r^2}\cos\theta,$$

and the internal potential $\qquad p_1 = -Ir\cos\theta.$

Now let us investigate the effect of filling up the shell with some substance of which the resistance is k, the resistance in the external medium being k'. The thickness of the magnetized shell may be neglected. Let the magnetic moment of the permanent magnetism be Ia^3, and that of the imaginary superficial distribution due to the medium $k = Aa^3$. Then the potentials are

$$\text{external } p' = (I + A)\frac{a^3}{r^2}\cos\theta, \quad \text{internal } p_1 = (I + A)\, r\cos\theta.$$

The distribution of real magnetism is the same before and after the introduction of the medium k, so that

$$\frac{1}{k'}I + \frac{2}{k'}I = \frac{1}{k}(I + A) + \frac{2}{k}(I + A),$$

$$\text{or } A = \frac{k - k'}{2k + k'}\,I.$$

The external effect of the magnetized shell is increased or diminished according as k is greater or less than k'. It is therefore increased by filling up the shell with diamagnetic matter, and diminished by filling it with paramagnetic matter, such as iron.

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* Taking the more general case of magnetic induction referred to in Art. (28), we find, in the expression for the moment of the magnetic forces, a constant term depending on T, besides those terms which depend on sines and cosines of θ. The result is, that in every complete revolution in the negative direction round the axis of T, a certain positive amount of work is gained; but, since no inexhaustible source of work can exist in nature, we must admit that $T = 0$ in all substances, with respect to magnetic induction. This argument does not hold in the case of electric conduction, or in the case of a body through which heat or electricity is passing, for such states are maintained by the continual expenditure of work. See Prof. Thomson, *Phil. Mag.* March, 1851, p. 186.

VIII. *Electro-magnetic spherical shell.*

Let us take as an example of the magnetic effects of electric currents, an electro-magnet in the form of a thin spherical shell. Let its radius be a, and its thickness t, and let its external effect be that of a magnet whose moment is Ia^3. Both within and without the shell the magnetic effect may be represented by a potential, but within the substance of the shell, where there are electric currents, the magnetic effects cannot be represented by a potential. Let p', p_1 be the external and internal potentials,

$$p' = I\frac{a^3}{r^2}\cos\theta, \qquad p_1 = Ar\cos\theta,$$

and since there is no permanent magnetism, $\frac{dp'}{dr} = \frac{dp_1}{dr}$, when $r = a$,

$$A = -2I.$$

If we draw any closed curve cutting the shell at the equator, and at some other point for which θ is known, then the total magnetic intensity round this curve will be $3Ia\cos\theta$, and as this is a measure of the total electric current which flows through it, the quantity of the current at any point may be found by differentiation. The quantity which flows through the element $td\theta$ is $-3Ia\sin\theta d\theta$, so that the quantity of the current referred to unit of area of section is

$$-3I\frac{a}{t}\sin\theta.$$

If the shell be composed of a wire coiled round the sphere so that the number of coils to the inch varies as the sine of θ, then the external effect will be nearly the same as if the shell had been made of a uniform conducting substance, and the currents had been distributed according to the law we have just given.

If a wire conducting a current of strength I_2 be wound round a sphere of radius a so that the distance between successive coils measured along the axis of x is $\frac{2a}{n}$, then there will be n coils altogether, and the value of I_1 for the resulting electro-magnet will be

$$I_1 = \frac{n}{6a}I_2.$$

The potentials, external and internal, will be

$$p' = I_2\frac{n}{6}\frac{a^2}{r^2}\cos\theta, \qquad p_1 = -2I_2\frac{n}{6}\frac{r}{a}\cos\theta.$$

The interior of the shell is therefore a uniform magnetic field.

IX. *Effect of the core of the electro-magnet.*

Now let us suppose a sphere of diamagnetic or paramagnetic matter introduced into the electro-magnetic coil. The result may be obtained as in the last case, and the potentials become

$$p' = I_2\frac{n}{6}\frac{3k'}{2k+k'}\frac{a^2}{r^2}\cos\theta, \qquad p_1 = -2I_2\frac{n}{6}\frac{3k}{2k+k'}\frac{r}{a}\cos\theta.$$

The external effect is greater or less than before, according as k' is greater or less than k, that is, according as the interior of the sphere is magnetic or diamagnetic with respect to the external medium, and the internal effect is altered in the opposite direction, being greatest for a diamagnetic medium.

This investigation explains the effect of introducing an iron core into an electro-magnet. If the value of k for the core were to vanish altogether, the effect of the electro-magnet would be three times that which it has without the core. As k has always a finite value, the effect of the core is less than this.

In the interior of the electro-magnet we have a uniform field of magnetic force, the intensity of which may be increased by surrounding the coil with a shell of iron. If $k' = 0$, and the shell infinitely thick, the effect on internal points would be tripled.

The effect of the core is greater in the case of a cylindric magnet, and greatest of all when the core is a ring of soft iron.

X. *Electro-tonic functions in spherical electro-magnet.*

Let us now find the electro-tonic functions due to this electro-magnet.

They will be of the form

$$\alpha_0 = 0, \qquad \beta_0 = \omega z, \qquad \gamma_0 = -\omega y,$$

where ω is some function of r. Where there are no electric currents, we must have a_2, b_2, c_2 each $= 0$, and this implies

$$\frac{d}{dr}\left(3\omega + r\frac{d\omega}{dr}\right) = 0,$$

the solution of which is

$$\omega = C_1 + \frac{C_2}{r^3}.$$

Within the shell ω cannot become infinite; therefore $\omega = C_1$ is the solution, and outside a must vanish at an infinite distance, so that

$$\omega = \frac{C_2}{r_3}$$

is the solution outside. The magnetic quantity within the shell is found by last article to be

$$-2I_2\frac{n}{6a}\frac{3}{2k+k'} = a_1 = \frac{d\beta_0}{dr} - \frac{d\gamma_0}{dy} = 2C_1;$$

therefore within the sphere

$$\omega_0 = -\frac{I_2 n}{2a}\frac{1}{3k+k'}.$$

Outside the sphere we must determine ω so as to coincide at the surface with the internal value. The external value is therefore

$$\omega = -\frac{I_2 n}{2a}\frac{1}{3k+k'}\frac{a^3}{r^3},$$

where the shell containing the currents is made up of n coils of wire, conducting a current of total quantity I_2.

Let another wire be coiled round the shell according to the same law, and let the total number of coils be n'; then the total electro-tonic intensity EI_2 round the second coil is found by integrating

$$EI_2 = \int_0^{2\pi} \omega a \sin\theta\, ds,$$

along the whole length of the wire. The equation of the wire is

$$\cos\theta = \frac{\phi}{n'\pi},$$

where n is a large number; and therefore

$$ds = a \sin\theta d\phi,$$
$$= - an'\pi \sin^2\theta d\theta,$$
$$\therefore\ EI_2 = \frac{4\pi}{3} \omega a^2 n' = - \frac{2\pi}{3} ann' I \frac{1}{3k + k'}.$$

E may be called the electro-tonic coefficient for the particular wire.

XI. *Spherical electro-magnetic Coil-Machine.*

We have now obtained the electro-tonic function which defines the action of the one coil on the other. The action of each coil on itself is found by putting n^2 or n'^2 for nn'. Let the first coil be connected with an apparatus producing a variable electro-motive force F. Let us find the effects on both wires, supposing their total resistances to be R and R', and the quantity of the currents I and I'.

Let N stand for $\frac{2\pi}{3}\frac{a}{(3k+k')}$, then the electro-motive force of the first wire on the second is

$$- Nnn' \frac{dI}{dt}.$$

That of the second on itself is

$$- Nn'^2 \frac{dI'}{dt}.$$

The equation of the current in the second wire is therefore

$$- Nnn' \frac{dI}{dt} - Nn'^2 \frac{dI'}{dt} = R'I' \quad \text{..............} (1)$$

The equation of the current in the first wire is

$$- Nn^2 \frac{dI}{dt} - Nnn' \frac{dI'}{dt} + F = RI \quad \text{...........} (2)$$

Eliminating the differential coefficients, we get

$$\frac{R}{n} I - \frac{R'}{n'} I' = \frac{F}{n},$$

and $$N \left(\frac{n^2}{R} + \frac{n'^2}{R'}\right) \frac{dI}{dt} + I = \frac{F}{R} + N \frac{n'^2}{R'} \frac{dF}{dt} \quad \text{......} (3)$$

from which to find I and I'. For this purpose we require to know the value of F in terms of t.

Let us first take the case in which F is constant and I and I' initially $= 0$. This is the case of an electro-magnetic coil-machine at the moment when the connexion is made with the galvanic trough.

Putting $\frac{1}{2}\tau$ for $N\left(\frac{n^2}{R}+\frac{n'^2}{R'}\right)$ we find

$$I=\frac{F}{R}\left(1-\epsilon^{-\frac{2t}{\tau}}\right),$$

$$I'=-F\frac{n'}{R'n}\epsilon^{-\frac{2t}{\tau}}.$$

The primary current increases very rapidly from O to $\frac{F}{R}$, and the secondary commences at $-\frac{F}{R'}\frac{n'}{n}$ and speedily vanishes, owing to the value of τ being generally very small.

The whole work done by either current in heating the wire or in any other kind of action is found from the expression

$$\int_0^\infty I^2 R dt.$$

The total quantity of current is

$$\int_0^\infty I dt.$$

For the secondary current we find

$$\int_0^\infty I'^2 R' dt=\frac{F^2 n'^2}{R' n^2}\frac{\tau}{4}, \qquad \int_0^\infty I' dt=\frac{F n'}{R' n}\frac{\tau}{2}.$$

The work done and the quantity of the current are therefore the same as if a current of quantity $I'=\frac{Fn'}{2R'n}$ had passed through the wire for a time τ, where

$$\tau=2N\left(\frac{n^2}{R}+\frac{n'^2}{R'}\right).$$

This method of considering a variable current of short duration is due to Weber, whose experimental methods render the determination of the equivalent current a matter of great precision.

Now let the electro-motive force F suddenly cease while the current in the primary wire is I_0 and in the secondary $=0$. Then we shall have for the subsequent time

$$I=I_0\epsilon^{-\frac{2t}{\tau}}, \qquad I'=\frac{I_0}{R'}\frac{Rn'}{n}\epsilon^{-\frac{2t}{\tau}}.$$

The equivalent currents are $\frac{1}{2}I_0$ and $\frac{1}{2}I_0\frac{R}{R'}\frac{n'}{n}$, and their duration is τ.

When the communication with the source of the current is cut off, there will be a change of R. This will produce a change in the value of τ, so that if R be suddenly increased, the strength of the secondary current will be increased, and its duration diminished. This is the case in the ordinary coil-machines. The quantity N depends on the form of the machine, and may be determined by experiment for a machine of any shape.

XII. *Spherical shell revolving in magnetic field.*

Let us next take the case of a revolving shell of conducting matter under the influence of a uniform field of magnetic force. The phenomena are explained by Faraday in his *Experimental Researches*, Series II., and references are there given to previous experiments.

Let the axis of z be the axis of revolution, and let the angular velocity be ω. Let the magnetism of the field be represented in quantity by I, inclined at an angle θ to the direction of z, in the plane of zx.

Let R be the radius of the spherical shell, and T the thickness. Let the quantities $\alpha_0, \beta_0, \gamma_0$, be the electro-tonic functions at any point of space; $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$ symbols of magnetic quantity and intensity; $a_2, b_2, c_2, \alpha_2, \beta_2, \gamma_2$ of electric quantity and intensity. Let p_2 be the electric tension at any point,

$$\left.\begin{aligned} \alpha_2 &= \frac{dp_2}{dx} + ka_2 \\ \beta_2 &= \frac{dp_2}{dy} + kb_2 \\ \gamma_2 &= \frac{dp_2}{dz} + kc_2 \end{aligned}\right\} \quad \text{..........................(1)}$$

$$\frac{da_2}{dx} + \frac{db_2}{dy} + \frac{dc_2}{dz} = 0 \text{ .... (2);}$$

$$\therefore \frac{d\alpha_2}{dx} + \frac{d\beta_2}{dy} + \frac{d\gamma_2}{dz} = \nabla^2 p.$$

The expressions for $\alpha_0, \beta_0, \gamma_0$ due to the magnetism of the field are

$$\alpha_0 = A_0 + \frac{I}{2} y \cos\theta,$$

$$\beta_0 = B_0 + \frac{I}{2}(z\sin\theta - x\cos\theta),$$

$$\gamma_0 = C_0 - \frac{I}{2} y \sin\theta,$$

A_0, B_0, C_0 being constants; and the velocities of the particles of the revolving sphere are

$$\frac{dx}{dt} = -\omega y, \qquad \frac{dy}{dt} = \omega x, \qquad \frac{dz}{dt} = 0.$$

We have therefore for the electro-motive forces

$$\alpha_2 = -\frac{1}{4\pi}\frac{d\alpha_0}{dt} = -\frac{1}{4\pi}\frac{I}{2}\cos\theta\,\omega x,$$

$$\beta_2 = -\frac{1}{4\pi}\frac{d\beta_0}{dt} = \frac{1}{4\pi}\frac{I}{2}\cos\theta\omega y,$$

$$\gamma_2 = -\frac{1}{4\pi}\frac{d\gamma_0}{dt} = \frac{1}{4\pi}\frac{I}{2}\sin\theta\omega x.$$

Returning to equations (1), we get

$$k\left(\frac{db_2}{dz}-\frac{dc_2}{dy}\right)=\frac{d\beta_2}{dz}-\frac{d\gamma_2}{dy}=0,$$

$$k\left(\frac{dc_2}{dx}-\frac{da_2}{dz}\right)=\frac{d\gamma_2}{dx}-\frac{da_2}{dz}=\frac{1}{4\pi}\frac{I}{2}\sin\theta\omega,$$

$$k\left(\frac{da_2}{dy}-\frac{db_2}{dx}\right)=\frac{da_2}{dy}-\frac{d\beta_2}{dx}=0.$$

From which with equation (2) we find

$$a_2=-\frac{1}{k}\frac{1}{4\pi}\frac{I}{4}\sin\theta\omega z,$$

$$b_2=0,$$

$$c_2=\frac{1}{k}\frac{1}{4\pi}\frac{I}{4}\sin\theta\omega x,$$

$$p_2=\frac{1}{16\pi}I\omega\left\{(x^2+y^2)\cos\theta-xz\sin\theta\right\}.$$

These expressions would determine completely the motion of electricity in a revolving sphere if we neglect the action of these currents on themselves. They express a system of circular currents about the axis of y, the quantity of current at any point being proportional to the distance from that axis. The external magnetic effect will be that of a small magnet whose moment is $\frac{TR^3}{48\pi k}\omega I\sin\theta$, with its direction along the axis of y, so that the magnetism of the field would tend to turn it back to the axis of x*.

The existence of these currents will of course alter the distribution of the electro-tonic functions, and so they will react on themselves. Let the final result of this action be a system of currents about an axis in the plane of xy inclined to the axis of x at an angle ϕ and producing an external effect equal to that of a magnet whose moment is $I'R^3$.

The magnetic inductive components within the shell are

$$I_1\sin\theta-2I'\cos\phi \text{ in } x,$$

$$-2I'\sin\phi \text{ in } y,$$

$$I_1\cos\theta \quad \text{ in } z.$$

Each of these would produce its own system of currents when the sphere is in motion, and these would give rise to new distributions of magnetism, which, when the velocity is uniform, must be the same as the original distribution,

$$(I_1\sin\theta-2I'\cos\phi)\text{ in } x \text{ produces } 2\frac{T}{48\pi k}\omega\,(I_1\sin\theta-2I'\cos\phi)\text{ in } y,$$

$$(-2I'\sin\phi)\text{ in } y \text{ produces } 2\frac{T}{48\pi k}\omega\,(2I'\sin\phi)\text{ in } x;$$

$I_1\cos\theta$ in z produces no currents.

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* The expression for p_2 indicates a variable electric tension in the shell, so that currents might be collected by wires touching it at the equator and poles.

We must therefore have the following equations, since the state of the shell is the same at every instant,

$$I_1 \sin\theta - 2I' \cos\phi = I_1 \sin\theta + \frac{T}{24\pi k}\,\omega 2I' \sin\phi$$

$$-2I' \sin\phi = \frac{T}{24\pi k}\,\omega(I_1 \sin\theta - 2I' \cos\phi),$$

whence

$$\cot\phi = -\frac{TR^3}{24\pi k}\,\omega, \quad I' = \frac{1}{2}\,\frac{\frac{T}{24\pi k}\,\omega}{\sqrt{1 + \left(\frac{T}{24\pi k}\,\omega\right)^2}}\,I_1 \sin\theta.$$

To understand the meaning of these expressions let us take a particular case.

Let the axis of the revolving shell be vertical, and let the revolution be from north to west. Let I be the total intensity of the terrestrial magnetism, and let the dip be θ, then $I\cos\theta$ is the horizontal component in the direction of magnetic north.

The result of the rotation is to produce currents in the shell about an axis inclined at a small angle $= \tan^{-1}\frac{T}{24\pi k}\,\omega$ to the south of magnetic west, and the external effect of these currents is the same as that of a magnet whose moment is

$$\frac{1}{2}\,\frac{T\omega}{\sqrt{\overline{24\pi k}|^2 + T^2 u^2}}\,R^3 I \cos\theta.$$

The moment of the couple due to terrestrial magnetism tending to stop the rotation is

$$\frac{24\pi k}{2}\,\frac{T\omega}{\overline{24\pi k}|^2 + T^2\omega^2}\,R^3 I^2 \cos^2\theta,$$

and the loss of work due to this in unit of time is

$$\frac{24\pi k}{2}\,\frac{T\omega^2}{\overline{24\pi k}| + T^2\omega^2}\,R^3 I^2 \cos^2\theta.$$

This loss of work is made up by an evolution of heat in the substance of the shell, as is proved by a recent experiment of M. Foucault, (see *Comptes Rendus*, XLI. p. 450).