Chapter 8

Acoustics

By UNO INGARD, Massachusetts Institute of Technology

INTRODUCTION

The field of acoustics as commonly understood deals with the generation and propagation of mechanical vibrations in matter, the application of sound in various fields of science, and its effect on men. Of these branches the general equations of wave motions in solids and fluids have already been treated in Part 3, Chaps. 7 and 4, and we need give only a supplementary summary of some forms of the equations of sound which will be used in the subsequent discussions. Some basic concepts and relations will also be included.

1. Limits of Frequency and Sound Pressure

The range of frequencies encountered in acoustics is quite large; the audible range itself extends over almost 10 octaves (cf. the visual electromagnetic frequency range of about 1 octave). The lower and upper limits of the audible range are approximately 20 and 20,000 cps, respectively, and much of acoustics is concerned with this range.

However, the frequencies in acoustics are by no means limited to the audible range. Frequencies as high as 500 megacycles have been generated, a wavelength of 0.6×10^{-4} cm in air. In liquids and solids the corresponding wavelengths are approximately 2.4×10^{-4} and 8×10^{-4} cm. These wavelengths are of the same order of magnitude as that of visible light.

A gas ceases to behave like a continuum when the wavelength of sound becomes of the order of the mean free path. Strong dispersion and absorption result, and when the sound frequency becomes considerably greater than the collision frequency, the ordered sound motion of the molecules will quickly be transformed into random thermal motion, and no sound propagation can take place [1].† At ordinary atmospheric conditions the mean free path is of the order of 10^{-5} cm, a limit frequency of the order of 10^9 cps.

In solids the assumption of continuum is senseless when the wavelength approaches the intermolecular distance, approximately 10^{-8} cm with corresponding limiting frequency of about 10^{12} cps. The ultimate limit is actually reached when the wavelength is twice the spacing of the unit cell of a crystal. In this region propagation of sound (multiply scattered) resembles diffusion of heat [2].

† Numbers in brackets refer to references at end of chapter.

The range of sound pressure is also considerable. The ear responds to sound pressures from 0.0002 to 2,000 dynes/cm². The lower limit corresponds to an intensity of 10^{-16} watt/cm² at continuous exposure. The least amount of acoustic energy that the ear can detect is of the order of $kT \simeq 10^{-20}$ watt-sec (cf. sensitivity of the eye: about one quantum of light in the middle of the visible region $h\nu \simeq 4 \times 10^{-19}$).

The upper limit of the sound pressure in a pure tone that can be generated in the medium is set approximately by the static pressure. At this pressure the rarefaction part of the sound cycle would create vacuum and breakdown, or cavitation, and the medium could no longer "support" the wave. The intensity in air of a plane sound wave with this limiting pressure equals approximately 1.20×10^3 watts/cm² ($\simeq$191 db). In water at atmospheric pressure the corresponding intensity is 0.36 watt/cm². (The cavitation pressure in a liquid is frequency-dependent and can under certain conditions considerably exceed the static pressure.)

Before the upper pure-tone intensity limit is reached, nonlinearity of the medium causes distortion of the wave: energy is in effect removed from the fundamental frequency and distributed on higher harmonics. Therefore, a large-amplitude wave propagating in air will change waveform and finally, after a certain distance of travel, break into a shock, reaching a stable saw-toothed form. This behavior is somewhat similar to the familiar breaking of waves on a water surface (see Sec. 17).

Sound Pressure and Intensity Levels. In expressing the sound intensity on a logarithmic scale, the reference intensity I_0 is usually taken to be $I_0 = 10^{-16}$ watt/cm², so that

$$\text{Intensity level in decibels} = 10 \log \frac{I}{I_0}$$

Correspondingly, the sound pressure is expressed in decibels by

$$\text{Sound-pressure level in decibels} = 20 \log \frac{p}{p_0}$$

where $p_0 = 0.0002$ dyne/cm², rms value. The reference pressure has been chosen to be approximately the threshold of hearing. The intensity level and the sound-pressure level would be identical if p_0 were the sound pressure that corresponds to the intensity I_0. This is not exactly true, however, except at one temperature T_0 approximately equal to 300°K (20°C). For temperatures higher than T_0 the intensity level as defined above will be larger than the sound-pressure level by an additional term $10 \log (T/T_0)^{1/2}$ db.

2. General Linear Equations of Sound Propagation [3, 4]

From the linearized equations of sound propagation obtained by keeping only first-order terms of the variations **u**, δ, p, and σ in the fluid-field variables, velocity **v**, density ρ, pressure P, and entropy S, respectively, in the general hydrodynamic equations (see Part 3, Chap. 4), one finds the wave equation for a *homogeneous* moving medium

$$\frac{D^2p}{Dt^2} = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)^2 p = c^2\,\nabla^2 p \qquad (8.1)$$

where **v** is the flow velocity of the medium.

If the medium is *inhomogeneous*, the basic equations become considerably more complicated [4]:

$$\frac{\partial\delta}{\partial t} + \mathbf{v}\cdot\nabla\delta + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} + \delta\nabla\cdot\mathbf{v} = 0 \qquad \text{(conservation of mass)} \quad (8.2)$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\nabla\times\mathbf{v})\times\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{v} + \nabla(\mathbf{v}\cdot\mathbf{u}) = -\frac{\nabla p}{\rho} + \frac{(\nabla P)\delta}{\rho^2} \qquad \text{(conservation of momentum)} \quad (8.3)$$

$$\frac{\partial\sigma}{\partial t} + \mathbf{v}\cdot\nabla\sigma + \mathbf{u}\cdot\nabla S = 0 \qquad \text{(conservation of energy)} \quad (8.4)$$

$$p = c^2\delta + h\sigma \qquad \text{(equation of state)} \quad (8.5)$$

where

$$c = \text{velocity of sound} = \left(\frac{\partial P}{\partial\rho}\right)_S \qquad h = \left(\frac{\partial P}{\partial S}\right)_\rho$$

The effect of dissipation in the medium due to heat conduction and viscosity has not been accounted for in these equations.

Irrotational and Isentropic Flow ($\nabla\times\mathbf{v} = 0$, $\nabla S = 0$). In this particular case the sound-particle velocity **u** is irrotational, so that, with $\mathbf{u} = -\nabla\phi$, Eqs. (8.2) to (8.5) reduce to [4]

$$\frac{D^2\phi}{Dt^2} = c^2\,\nabla^2\phi + (\nabla\pi_0)\cdot\nabla\phi + \frac{D\phi}{Dt}\mathbf{v}\cdot\nabla\ln c^2 \qquad (8.6)$$

where $\pi_0 = \int dp/\rho$ is the enthalpy of the original flow. The second term on the right-hand side essentially expresses the effect of a density variation of the medium, and the third term the variation of velocity of propagation (as does the factor c^2 in the first term). For a homogeneous medium Eq. (8.6) reduces to (8.1). If the flow is directed along the x axis, the expanded version of this equation is

$$\nabla^{*2}\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2\beta}{c\sqrt{1-\beta^2}}\frac{\partial^2\phi}{\partial t\,\partial x^*} = 0 \qquad (8.7)$$

where

$$\nabla^* = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$x^* = \frac{x}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

Rotational Flow ($\nabla\times\mathbf{v} \neq 0$). The sound particle velocity **u** is no longer irrotational but contains a vector potential in addition to the scalar potential. Neglecting terms of second order in v/c and $|\nabla\times\mathbf{v}|/\omega$ and their products, the velocity **u** can be expressed in terms of a single quantity ψ, so that [4]

$$\mathbf{u} = -\nabla\psi + \int^t (\nabla\times\mathbf{v})\times\nabla\psi\,dt \qquad (8.8)$$

and the Eqs. (8.1) to (8.5) now lead to the wave equation

$$\frac{D^2\psi}{Dt^2} - c^2\,\nabla^2\psi = (\nabla\pi_0)\cdot\nabla\psi + \frac{D\psi}{Dt}\mathbf{v}\cdot\nabla\ln c^2 + c^2\int^t(\nabla\psi)\cdot\nabla^2\mathbf{v}\,dt - (\nabla\pi_0)\cdot\int_0^t(\nabla\times\mathbf{v})\times\nabla\psi\,dt \qquad (8.9)$$

where

$$\nabla^2\mathbf{v} = -\nabla\times(\nabla\times\mathbf{v})$$

3. Kirchhoff's Formula in a Moving Medium [4]

The velocity potential ϕ at a fixed point, in terms of the values of ϕ and $\partial\phi/\partial n$ on fixed surfaces bounding the region under consideration, becomes, in the case of a moving medium [4],

$$\phi(t) = \frac{1}{4\pi}\int\left\{\frac{1}{R^*}\left[\frac{\partial\phi}{\partial n}\right] - \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)[\phi] + \frac{1}{c}\frac{1}{R^*}\frac{\partial R}{\partial n}\left[\frac{\partial\phi}{\partial t}\right]\right\}dS + \frac{1}{4\pi}\frac{\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\int\frac{1}{R^*}\left[\frac{\partial\phi}{\partial t}\right]dS_x \qquad (8.10)$$

The flow is here assumed to be in the positive x direction. $R^{*2} = x^{*2} + y^2 + z^2$ (see Sec. 2) is approximately the distance from P to the "source" point on the surface. The brackets indicate retarded values, $[\phi] = \phi(t - R/c)$, where

$$R = \frac{-\beta x^* + R^*}{\sqrt{1-\beta^2}}$$

The surface element dS_x in the last integral is the projection of dS upon the direction of the flow (the x axis).

For harmonic time dependence $\phi = \phi_1 e^{-i\omega t}$, the formula reduces to

$$\phi_1 = \frac{1}{4\pi}\int\left\{\frac{\partial\phi}{\partial n}\frac{e^{ikR}}{R^*} - \phi_1\frac{\partial}{\partial n}\left(\frac{e^{ikR}}{R^*}\right)\right\}dS - \frac{1}{4\pi}\frac{2i\beta k}{\sqrt{1-\beta^2}}\int\frac{e^{ikR}}{R^*}\,dS_x \qquad (8.10a)$$

If the medium in addition contains a volume-source distribution $Q = Q_1 e^{-i\omega t}$, an additional term

$$\int Q_1 e^{ikR}/R^* \, dv$$

appears in the right-hand side of (8.10*a*).

The generalized Kirchhoff's theorem is applicable not only for surfaces and sources at rest and the medium moving but also for the reversed case (see Secs. 9 and 10).

4. Boundary Conditions. Impedance and Absorption Coefficients

Accompanying the differential equations (8.2) to (8.9) or Kirchhoff's formula are the boundary conditions of *continuity of velocity normal to the boundary* and *continuity of pressure.*

The ratio between the particle velocity at a point in a field and the sound pressure is termed the *specific acoustic admittance.*

$$\frac{\mathbf{n}_i}{\rho c} = \frac{\mathbf{u}_i}{p} \qquad i = 1, 2, 3$$

It is a vector with the same direction as $\mathbf{u}_i$. The inverse of $\mathbf{n}/\rho c$ in a given direction is termed the *specific acoustic impedance*

$$\mathbf{z} = \zeta\rho c = \left(\frac{\mathbf{n}}{\rho c}\right)^{-1} = (\theta - i\chi)\rho c$$

For a plane wave in the x direction the specific acoustic impedance or the *characteristic impedance of the medium* is real and equals

$$\mathbf{Z} = \frac{p}{u} = \rho c$$

(for air $\simeq$ 41.5 cgs at 20°C) [5], where ρ = density, c = velocity of sound ($\simeq$340 m/sec at 20°C in air). The *radiation impedance* of a vibrating surface is the ratio of the pressure at the boundary and the particle velocity of the surface.

The ratio between the pressure and normal velocity at a boundary is referred to as the *normal impedance* of the boundary. In general this quantity is not known a priori and can be determined first after the field has been found, utilizing the boundary conditions mentioned above. The normal impedance will then in general be a function of the angle of incidence. However, for some special material (a "wall" with pores normal to the surface) the particle velocity is always normal to the boundary and will depend only on the local pressure at the point under consideration. For such a *locally reacting* or *point-reacting* boundary the normal impedance will be independent of the sound field and can be specified in advance as a characteristic property of the boundary. Under those conditions the analysis of many field problems, sound waves in rooms, etc., is considerably simplified [6]. Many materials met in practice are approximately locally reacting, e.g., perforated porous tiles, dense porous homogeneous material, cavity-resonator arrangements, etc.

The *absorption coefficient* of a plane boundary exposed to a plane wave with an angle of incidence ϕ is

$$\alpha(\phi) = 1 - \left|\frac{\zeta \cos\phi - 1}{\zeta\cos\phi + 1}\right|^2 = \frac{4\theta\cos\phi}{(\theta\cos\phi + 1)^2 + (\chi\cos\phi)^2} \qquad (8.11)$$

In a diffuse sound field the probability $B(\phi)$ that an elementary plane wave has an angle of incidence ϕ is proportional to the solid angle $2\pi \sin\phi \, d\phi$, so that the average absorption coefficient

$$[\int B(\phi)\alpha(\phi) \cos\phi \, d\phi]/[\int B(\phi)\cos\phi \, d\phi]$$

becomes

$$\bar{\alpha} = 2\int_0^{\pi/2} \alpha(\phi) \sin\phi \cos\phi \, d\phi$$

This coefficient is usually referred to as the *statistical-average absorption coefficient.* With $\alpha(\phi)$ given by (8.11), $\bar{\alpha}$ in general cannot be expressed in closed form. However, if ζ is independent of ϕ, the integral reduces to [7, p. 388]

$$\bar{\alpha} = 8\mu\left[1 + \frac{\mu^2 - \sigma^2}{\sigma}\tan^{-1}\frac{\sigma}{\sigma^2 + \mu^2 + \mu} - \mu \ln\frac{(\mu + 1)^2 + \sigma^2}{\mu^2 + \sigma^2}\right] \qquad (8.12)$$

$$\mu + i\sigma = \zeta^{-1} = (\theta - i\chi)^{-1}$$

For relatively hard boundaries, $(\mu,\sigma) \ll 1$, we get $\bar{\alpha} \simeq 8\mu$. Graphical representation of (8.12) is available [8].

5. Second-order Quantities

Any quantity that contains the product of two first-order variables of the field is of second order and hence of the same order of magnitude as the terms originally neglected in linearizing the fundamental equations. In the calculation of such quantities as energy flow, mass flow, and time averages like radiation pressure the contribution of the originally neglected terms has to be evaluated.

Energy Flow. Such an investigation [9] of the energy flow in the wave leads to an expression that involves only first-order quantities, so that the sound intensity equals

$$I(t) = pu \qquad (8.13)$$

With $\mathbf{u} = -\nabla\phi$ and $\phi = \phi_1 e^{-i\omega t}$, the time average of the intensity is

$$I = \tfrac{1}{2}\,\mathrm{Re}\,(p\tilde{u}_1) = \frac{i\omega\rho_0}{4}(\tilde{\phi}\nabla\phi - \phi\nabla\tilde{\phi}) \qquad (8.14)$$

where $\tilde{u}_1$ is the complex conjugate of $\mathbf{u}_1$ (Re = real part of).

Radiation Pressure. Correspondingly, the radiation pressure of a plane wave incident on a perfect absorber equals *exactly* twice the mean kinetic energy density of the wave motion. For *small amplitudes* this expression can be written as the mean *total* energy density,

$$p_{\text{rad}} = \text{energy density} = \frac{\rho_0|u_1|^2}{2} = \frac{I}{c}$$

i.e., the same expression as in the electromagnetic case. For a perfect reflector the radiation pressure is *twice* as large. A thorough discussion of subtle questions regarding the effect of large amplitudes, the infinite extension of the sound beam, the angle of incidence, etc., can be found in Borgnis [10].

6. Electromechanical Analogues

The electrical analogue of a mechanical system is the electrical network (or field) which is described by the same (Lagrangian) equations of motion as the mechanical system. As an illustration consider the simple mechanical system shown in Fig. 8.1*a*, consisting of a horizontal bar, the vertical motion of which is impeded by an attached mass M, a spring with spring constant K, and a friction force Dv.

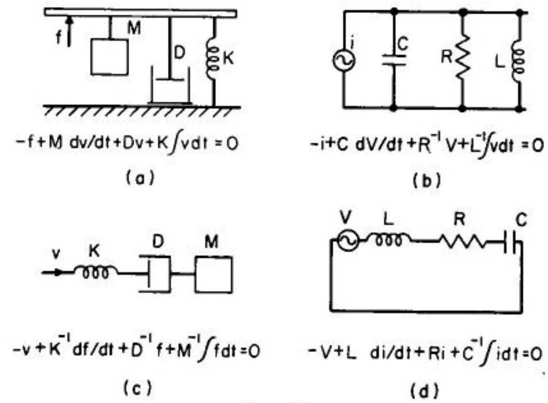

FIG. 8.1

There are two electrical analogues of this system: one, a series circuit shown in Fig. 8.1*d*, and the other, the *dual* of Fig. 8.1*d*, a parallel circuit shown in Fig. 8.1*b*. In the first of these analogues, often called the *classical*, we have the following correspondence between mechanical and electrical quantities: force—voltage, velocity—current, mass—inductance, compressibility (K^{-1})—capacitance, mechanical resistance (D)—electrical resistance. In the second analogue, often called *mobility* analogue, we get correspondingly: force—current, velocity—voltage, mass—capacitance, compressibility—inductance, and mechanical resistance—inverse of electrical resistance. Le Corbeiller and Yeung [11] have pointed out that the picture is not complete without introducing the *mechanical dual* shown in Fig. 8.1*c*. This system is topologically similar to Fig. 8.1*d*. The choice of representation most convenient for a particular problem has been the subject of many studies [12].

SOUND SOURCES AND THEIR FIELDS

7. The "Natural" Sources of Sound [13]

The linear equations governing the propagation of sound in air are

$$\frac{\partial(\rho u_i)}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(conservation of mass)} \qquad (8.15)$$

$$\frac{\partial(\rho u_i)}{\partial t} + c^2 \frac{\partial \rho}{\partial x_i} = 0 \qquad \text{(conservation of momentum)} \qquad (8.16)$$

$$dU = c_v\, dT + P\, dv \qquad \text{(conservation of energy)} \qquad (8.17)$$

$$P = (c_p - c_v)\rho T \qquad \text{(equation of state)} \qquad (8.18)$$

The first of these is the *exact* equation of conservation of mass, and the second an *approximation* of conservation of momentum. The two last equations combine, under isentropic conditions, to the relation $p = (\gamma P_0/\rho_0)\delta = c^2\delta$ between the sound pressure and the density $\delta = \rho - \rho_0$. The three resulting equations lead to the wave equation

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

with the *source term* on the right-hand side being zero. The simplest physical means required to bring about a source term is found directly by inspection of the equations above, and amounts to introduction of

a. Mass at a rate Q per unit volume in the medium. This means an addition of the term Q on the right-hand side of Eq. (8.15).

b. Force F_i per unit volume in the medium, which enters as an additional term on the right-hand side of Eq. (8.16).

c. An addition $\partial(\rho v_i v_j)/\partial x_j$ per unit volume of the rate of change of momentum. This term, introduced by Lighthill, is due to fluctuations (turbulence), $\rho v_i v_j$ being the Reynolds stress tensor. The term enters in the left-hand side of Eq. (8.16).

d. Heat, at a rate ρH per unit volume. Appears in the left-hand side of Eq. (8.17) (after time differentiation of this equation).

By consideration of these additional terms, the resulting wave equation becomes [13]

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\left(\dot{Q} + \frac{\gamma - 1}{c^2}\rho \dot{H}\right) + \frac{\partial F_i}{\partial x_i} - \frac{\partial^2 T_{ij}}{\partial x_i\, \partial x_j} \qquad (8.19)$$

where $\dot{Q} = \dfrac{\partial Q}{\partial t} \qquad \dot{H} = \dfrac{\partial H}{\partial t} \qquad T_{ij} = \rho v_i v_j$

The three source terms in this equation are those of a simple source, a dipole, and a quadrupole distribution, respectively. The heat source is equivalent to a mass source of strength $\rho H(\gamma - 1)/c^2$.

The Simple Source. The first term in (8.19) leads to a pressure field given by

$$p = \frac{1}{4\pi}\int_{v'} \frac{\dot{Q}(t - r/c)}{r}\, dv' \qquad (8.20)$$

$$r^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2$$

where x_i' = source point, x_i = field point. In the case where the source is concentrated in a point with the flow strength q the field becomes

$$p = \frac{\dot{q}(t - r/c)}{4\pi r} \qquad (8.21)$$

or

$$p = \frac{-i\omega q_0}{4\pi r} e^{ikr} e^{-i\omega t}$$

if $q = q_0 e^{-i\omega t}$. The total radiated power is

$$W(t) = \frac{1}{4\pi\rho c}\left[\dot{q}\left(t - \frac{r}{c}\right)\right]^2$$
$$\overline{W(t)} = \frac{\omega^2|q_0|^2}{8\pi\rho c,} \qquad \text{when } q = q_0 e^{-i\omega t} \tag{8.22}$$

The simple-source field (8.21) is realized by means of a pulsating sphere having a radial velocity $U_0 e^{-i\omega t}$ such that $q = 4\pi a^2 U_0 \rho$. At low frequencies the simple-source field represents the major contribution to the far field of *any* finite source with a net flow strength different from zero.

The Dipole. The second source term in (8.19) corresponds to a *dipole* distribution equal to the *vector field* F_i. The source term $\partial f_i/\partial x_i'$ of a concentrated force f_i can be considered as the sum of two simple sources

$$-\frac{1}{\Delta x_i'} f_i(x_i' + \Delta x_i') \quad \text{and} \quad \frac{1}{\Delta x_i'} f_i(x_i')$$

of opposite sign, a distance $\Delta x_i'$ apart, each giving rise to a field (8.21). In the limit $\Delta x_i' = 0$ the resulting field will then be that of a dipole of strength f_i,

$$p = -\frac{1}{4\pi}\frac{\partial}{\partial x_i}\left[\frac{f_i(t - r/c)}{r}\right] \tag{8.23}$$

If the force is in the x_1 direction, the far field becomes

$$p = \frac{1}{4\pi rc}\cos\theta \frac{\partial}{\partial t}\left[f_1\left(t - \frac{r}{c}\right)\right]$$

or $$p = \frac{-i\omega f_1}{4\pi rc}\cos\theta\, e^{ikr}e^{-i\omega t} \tag{8.24}$$

if $f_i = f_1 e^{-i\omega t}$ and $x_1/r = \cos\theta$. The total radiated power is

$$W(t) = \frac{1}{12\pi c^3 \rho}\left[\frac{\partial}{\partial t} f_1\left(t - \frac{r}{c}\right)\right]^2$$

or $$\overline{W(t)} = \frac{\omega^2|f_1|^2}{24\pi c^3 \rho} \tag{8.25}$$

if $f_i = f_1 e^{-i\omega t}$. In general for a continuous force (or dipole) distribution of strength F_i the field becomes

$$p = \frac{1}{4\pi}\int_{v'}\frac{\partial}{\partial x_i'}\left[\frac{F_i(t - r/c)}{r}\right]dv' \tag{8.26}$$

or in the far field

$$p = \frac{1}{4\pi c}\int \frac{x_i - x_i'}{r^2}\frac{\partial}{\partial t}\left[F_i\left(t - \frac{r}{c}\right)\right]dv' \tag{8.27}$$

The dipole field (8.24) is obtained, for example, from an oscillating sphere having a velocity $U_1 e^{-i\omega t}$ corresponding to the dipole strength $f_1 = -i\omega(\pi a^3 \rho)U_1$, where a = sphere radius.

It follows from (8.24) that the dipole pressure field will contain one more factor of ω than do the simple-source fields from which it is made up. The total radiated acoustic power, being proportional to ω^2 for the simple source of a given flow strength, will therefore be proportional to ω^4 for the dipole.

The mounting of a loudspeaker or an oscillating piston in an infinite wall or in a closed box will in effect convert the radiator from a dipole to a simple source and hence improve the low-frequency efficiency.

The Quadrupole. In complete analogy with the derivation of the field from the force distribution, it follows that the field caused by the third term in (8.19) is that of a quadrupole distribution equal to the stress *tensor* T_{ij} [13]. The pressure distribution becomes

$$p = \frac{1}{4\pi}\int_{v'}\frac{\partial^2}{\partial x_i'\,\partial x_j'}\frac{T_{ij}(t - r/c)}{r}\,dv'$$

or in the far field

$$p = \frac{1}{4\pi c^2}\frac{x_i x_j}{r^3}\int_{v'}\frac{\partial^2}{\partial t^2}T_{ij}\left(t - \frac{r}{c}\right)dv' \tag{8.28}$$

It follows from (8.28) that the quadrupole pressure field will contain one more factor of ω than do the dipole fields from which it is composed. If the total acoustic power from the dipole is proportional to ω^4, it will be proportional to ω^6 for the quadrupole.

For a *longitudinal* quadrupole of total strength t_{11} concentrated at $r = 0$ and with both axes in the x_1 direction, the field is

$$p = \frac{1}{4\pi c^2}\frac{1}{r}\cos^2\theta\frac{\partial^2}{\partial t^2}t_{11}\left(t - \frac{r}{c}\right)$$
$$p = -\frac{\omega^2}{4\pi c^2}\frac{1}{r}\cos^2\theta\, t_{11}e^{ikr}e^{-i\omega t} \tag{8.29}$$

if $t_{ii} = t_{11}e^{-i\omega t}$, with the corresponding radiated total power

$$\overline{W} = \frac{\omega^4|t_{11}|^2}{40\pi\rho c^5} \tag{8.30}$$

if $t_{ii} = t_{11}e^{-i\omega t}$.

For a *lateral* quadrupole the expression for the pressure is the same except for a factor $\sin\theta$ replacing one of the factors $\cos\theta$ in (8.29). The total radiated power is

$$\overline{W} = \frac{\omega^4|t_{12}|^2}{120\pi\rho c^5} \tag{8.31}$$

if $t_{ij} = t_{12}e^{-i\omega t}$. Although the quadrupole source has little importance in general, it is the sole contribution in the generation of sound by turbulence in free space.

Sources of higher order can be built and superimposed correspondingly to represent (in a *multipole expansion*) the field from an arbitrary finite source.

8. Generation of Sound by Turbulent Flow [13, 14]

As shown by Lighthill [13], the generation of sound by turbulent flow is due to the (Reynolds) stress-tensor (quadrupole) source distribution discussed above, leading to a sound field given by (8.28). As far as the dependence on the flow parameters is concerned, the radiated power is of the form given by (8.30) or (8.31). Since $t_{ij} \sim \rho v^2 l^3$ and $\omega \sim v/l$, it follows that

$$\text{Acoustic power from turbulence} \sim \rho v^8 c^{-5} l^2 \tag{8.32}$$

In other words, since the rate of kinetic energy of the flow entering a region is $l^2 \rho v^3$, it follows that the *efficiency of sound generation by turbulence goes as the fifth power of the Mach number* $M = v/c$.

9. Radiation from a Simple Source in a Moving Medium [4, 15, 16]

An important question concerns the effect of steady motion of the medium on the field distribution from a stationary simple source of sound located in free space. In the absence of motion of the medium the sound field will be spherically symmetrical, and the surfaces of constant phase will coincide with the surfaces of constant amplitude. Motion of the medium will split this coincidence.

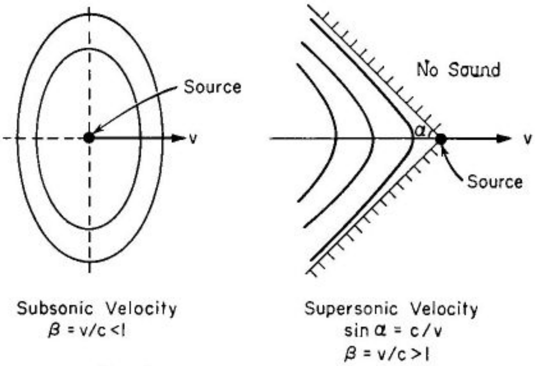

FIG. 8.2. Equal sound pressure contours from sound source in motion or for a stationary source in a moving medium. The contours are in both cases measured in a coordinate system attached to the source.

The field distribution of a stationary source in a moving medium measured in the stationary coordinate system is the same as that of a moving source in a stationary medium measured in the frame connected with the source. The field from a point source located at the origin of the stationary coordinate system xyz in which the medium moves with a constant velocity v in the direction of the x axis is

$$p(x,y,z,t) = \frac{\dot{q}(t - R/c)}{4\pi R^* \sqrt{1-\beta^2}} \qquad (8.33)$$

where
$$R = \frac{-\beta x^* + R^*}{\sqrt{1-\beta^2}}$$

$$R^{*2} = x^{*2} + y^2 + z^2 \qquad x^* = \frac{x}{\sqrt{1-\beta^2}}$$

which can readily be seen to satisfy Eqs. (8.7).

The surfaces of *constant phase,* given by R = constant, are spheres of radius $R\sqrt{1+\beta^2}$ with the origin at $x = R\beta$. This can be easily seen in an elementary way by calculating the time it takes for a pulse of sound to reach x, y, z. The surfaces of *constant sound pressure,* on the other hand, are given by R^* = constant, which corresponds to the ellipsoid $x^2/(1-\beta^2) + y^2 + z^2$ = constant = R^{*2}, as pictured in Fig. 8.2. It is interesting to notice that the field is the same up and down wind and that the intensity is larger in a direction at right angles to the flow. Physically the decrease of sound pressure in the directions with and against the wind can be explained as follows. Down wind the space occupied by a pulse of energy of certain length is "stretched" out, and the energy density is correspondingly decreased. Up wind the wave has effectively to travel further to reach the point of observation, and the effect of spherical divergence will be comparatively larger.

10. Radiation from a Moving Sound Source [4, 15, 16]

Consider a simple source of strength q moving in an arbitrary path defined by the coordinates $X(t)$, $Y(t)$, and $Z(t)$ with respect to the stationary coordinate system xyz. In a way similar to that used in the derivation of the field from a moving charge, it can be shown that the sound-pressure field becomes

$$p(x,y,z,t) = \frac{\dot{q}(t - R/c)}{R(1 - [v_R]/c)4\pi} \qquad (8.34)$$

where the distance R is found from the equation

$$f(R) = \left[x - X\left(t - \frac{R}{c}\right)\right]^2 + \left[y - Y\left(t - \frac{R}{c}\right)\right]^2 + \left[z - Z\left(t - \frac{R}{c}\right)\right]^2 - R^2 = 0 \qquad (8.35)$$

and $[v_R]$ is the projection of the velocity at time $t - R/c$ upon R.

Subsonic Velocity. For velocities $v < c$ the equation for determination of R has only one root, so that at time t there is effectively only one point on the path of the source which contributes to the field at x, y, z. In the particular case of source motion in the x direction with a constant velocity v ($\beta = v/c$), the distance R becomes

$$R = \frac{\beta\xi^* + R^*}{\sqrt{1-\beta^2}} \qquad R^{*2} = \xi^{*2} + y^2 + z^2$$
$$\xi^* = \frac{x - vt}{\sqrt{1-\beta^2}} \qquad (8.36)$$

and
$$R\left(1 - \frac{[v_R]}{c}\right) = R^* \sqrt{1-\beta^2}$$

In the coordinate system attached to the source $\xi = x - vt$, $\eta = y$, and $\zeta = z$, the sound-pressure field becomes $p(\xi,\eta,\zeta,t) = \dot{q}(t - R/c)/R^* 4\pi \sqrt{1-\beta^2}$ so that the surfaces of *constant phase* given by

$$t - \frac{R}{c} = \text{constant}$$

become circles with their centers displaced along the ξ axis (see Sec. 9), and the *surfaces of constant pressure* are the ellipsoids $\xi^2/(1-\beta^2) + \eta^2 + \zeta^2$ = constant. Hence the sound pressure is higher in a direction at right angles to the direction of motion (see Fig. 8.2).

Supersonic Velocity. If we carry through a formal solution for the case of $v > c$, there are two solutions to Eq. (8.35), R_1 and R_2, and the pressure field becomes

$$p(x,y,z,t) = \frac{\dot{q}(t - R_1/c)}{4\pi R_1|1 - [v_{R_1}]/c|} + \frac{\dot{q}(t - R_2/c)}{4\pi R_2|1 - [v_{R_2}]/c|} \tag{8.37}$$

For a rectilinear motion in the x direction the two values of R are

$$R_{1,2} = \frac{\pm R^* - \beta\xi^*}{\beta^2 - 1} \qquad R^{*2} = \xi^{*2} - (y^2 + z^2)$$

$$\xi^* = \frac{x - vt}{\sqrt{\beta^2 - 1}} = \frac{\xi}{\sqrt{\beta^2 - 1}}$$

and $R_1\left|1 - \frac{[v_{R_1}]}{c}\right| = R_2\left|1 - \frac{[v_{R_2}]}{c}\right| = R^*\sqrt{\beta^2 - 1}$

Hence, the surfaces of constant pressure are, in the coordinate system $\xi\eta\zeta$ connected with the source, hyperboloids

$$R^{*2} = \text{constant} = \frac{\xi^2}{\beta^2 - 1} - \eta^2 - \zeta^2 > 0$$

as shown in Fig. 8.2. The limiting curve, for $R^* = 0$, corresponds to the so-called Mach cone

$$\eta^2 + \zeta^2 = \frac{\xi^2}{\beta^2 - 1}$$

with the half angle $\alpha = \sin^{-1}(c/v)$. In this idealized case with a point source of sound the pressure goes to infinity at the origin and hence along the whole Mach cone. In front of the cone there will be no sound.

For a sound source of finite size there will be complicated disturbance in the medium and a shock wave extending from the front (or any discontinuity) on the source. This will change the local properties of the medium, which will affect the sound propagation.

11. The Doppler Effect

When a sound source of frequency ω passes a stationary receiver, the sound observed will in general have a continuous spectrum. At any instance, however, frequency can be defined as the time ratio of change of the phase $\omega = (d/dt)(\omega t - R/c)$. The frequencies thus obtained in the case of subsonic and supersonic velocities are:

Subsonic Velocity.

$$\omega' = \omega\left[\frac{(1 + \beta\xi^*/R^*)}{(1 - \beta^2)}\right] \simeq \omega(1 + \beta\cos\theta)$$

where the distance between the source and observer is $(x - vt)^2 + y^2 + z^2$, $\xi^* = (x - vt)/\sqrt{1 - \beta^2}$, and $R^* = \xi^{*2} + y^2 + z^2$. When the observer is on the x axis in front of the source, $\omega' = \omega/(1 - \beta)$, and when behind the source, $\omega' = \omega/(1 + \beta)$.

Supersonic Velocity. In this case we have two instantaneous frequencies

$$|\omega'| = \omega\frac{\beta\xi^*/R^* - 1}{\beta^2 - 1} \qquad |\omega''| = \omega\frac{\beta\xi^*/R^* + 1}{\beta^2 - 1}$$

which reduce to

$$\omega' = \frac{\omega}{\beta + 1} \qquad \text{and} \qquad \omega'' = \frac{\omega}{\beta - 1}$$

when the observer is in the x axis. If $1 < \beta < 2$, both ω' and ω'' are less than ω.

12. Radiation and Scattering

In addition to the basic fields from point sources described above, a number of other radiation fields from finite sources are known. With a given velocity distribution on the surface of the source, the field is formally given by the Kirchhoff theorem in Eq. (8.10). Evaluation of the field leads often to great analytical difficulties; some important radiation problems solved in the literature are indicated below. Generally only the "far" field is of interest. However, to find the reactive part of the radiation impedance of the source, the field over the surface of the source must be evaluated. The radiation *resistance* can readily be found directly from the far field by calculation of the radiated power. Among the large number of papers in this field we find solutions of, for example, the following problems: pulsating and oscillating spheres and cylinders [7, p. 244], piston in a sphere or cylinder [7, p. 244], piston in an infinite wall in a medium at rest [7, p. 244] and in a moving medium [18], vibrating piston in free space [19], radiation from open end of a pipe [20].

The scattering problem is closely related to the radiation problem with the additional difficulty that the field distribution on the scatterer is not known a priori. The resulting integral equations must often be solved approximately. Work on scattering includes spheres and cylinders [7, p. 244], spherical aperture in an infinite screen [21], spherical disk [17], straight edge [22], resonators [23], absorbing strips [24], cylindrical vortex [25] using general equations in Sec. 2.

13. Technical Aspects of Sound Generation [5, 26]

The methods used in the generation of sound can essentially be divided into two groups: (1) conversion of electrical oscillations into mechanical ones, and (2) conversion of nonoscillatory mechanical energy (or heat) into oscillatory motion. As a third group one could perhaps specify (3) explosions, electric sparks, and similar effects.

In the first group the means of making the conversion is usually a *linear* system, an electromechanical transducer, whereas in the second group *nonlinear* mechanisms are essential. A sound source like a bell, in which the eigenoscillations are excited, requires for continuous operation an oscillatory driving force and can therefore be considered to belong to the sources in group 1.

In general it is impossible to cover the entire acoustic spectrum by one and the same transducer, and several different mechanisms have been developed for different frequency regions. In the *first group* are membrane and piston vibrators (loudspeakers) driven electromagnetically or electrostatically [27],