Biquaternion Relativity -
Gravitation as an Effect of Spatial Varying Speed of Light

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The biquaternion notation - or more precisely the semi-biquaternion notation - is applied to mechanics and special relativity. The Lorentz transformation as well as electromagnetic field transformations result from a pure multiplication with the biquaternion relative velocity. By a straight forward development of the biquaternion notation, it turns out that the biquaternion velocity plays a crucial role regarding the principle of equivalence. The total time derivation of the four-velocity delivers inertial as well as gravitational acceleration simultaneously. It our model, the cause of gravitation is a spatial varying speed of light.

Issued: January 01, 2011

1 Introduction

A Hamilton quaternion $\boldsymbol{q}$ [1] is an expansion of real numbers by the Hamilton units i, j, k having the magnitude of $\sqrt{-1}$ and forming a four-number as

$$\boldsymbol{q} = q_0 + iq_1 + jq_2 + kq_3 \ ,$$

whereas $q_0 ... q_3$ are real numbers. The Hamilton units constitute the associative but not commutative multiplications

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \ , \\ ij = k \; jk = i \; ki = j \ , \\ ij = -ji \; jk = -kj \; ki = -ik \ . \end{gathered}$$

The real part of a quaternion is often called the scalar part while the other three imaginary parts are called a three-vector. Both together constitute a four vector:

$$\boldsymbol{q} = (q_0, \ \vec{q}) \ ,$$

where $\vec{q} = iq_1 + jq_2 + kq_3$. Later Hamilton [2] presented the biquaternion as an extension of a quaternion by introducing complex numbers $q_n \to q_n + \mathrm{i}p_n$, where i denotes the Gaussian imaginary unit.

$$\boldsymbol{Q} = q_0 + \mathrm{i}p_0 + i\,(q_1 + \mathrm{i}p_1) + j\,(q_2 + \mathrm{i}p_2) + k\,(q_3 + \mathrm{i}p_3) \ .$$

Generally a biquaternion can be grouped into two four-vectors

$$\boldsymbol{Q} = [(q_0, \ \mathrm{i}\vec{q}) + (\mathrm{i}p_0, \ \vec{p})] = q + p \ .$$

Both terms q and p shall be called semi-biquaternions. These two semi-biquaternions do not have the same structure. The first term is very suitable to cast translational physical four-quantities whereas the second may represent a complementary rotational four-quantity, if existing.

For a more convenient readability a biquaternion can be written as the sum of a 'left' and 'right' semi-biquaternion as

$$\boldsymbol{Q} = q + p = \left[\begin{matrix} q_0 \\ \mathrm{i}\vec{q} \end{matrix}\right) + \left(\begin{matrix} \mathrm{i}p_0 \\ \vec{p} \end{matrix}\right] \ .$$

The different brackets shall indicate that the two semi-biquaternions are not of identical structure. The multiplication of two biquaternions

$$\boldsymbol{A} = a + b = \left[\begin{matrix} a_0 \\ \mathrm{i}\vec{a} \end{matrix}\right) + \left(\begin{matrix} \mathrm{i}b_0 \\ \vec{b} \end{matrix}\right]$$

$$\boldsymbol{X} = x + y = \left[\begin{matrix} x_0 \\ \mathrm{i}\vec{x} \end{matrix}\right) + \left(\begin{matrix} \mathrm{i}y_0 \\ \vec{y} \end{matrix}\right]$$

is

$$\begin{aligned} \boldsymbol{AX} = &\left[\begin{matrix} a_0x_0 - b_0y_0 + \vec{a}\cdot\vec{x} - \vec{b}\cdot\vec{y} \\ \mathrm{i}\left(a_0\vec{x} + x_0\vec{a} + b_0\vec{y} + y_0\vec{b} + \vec{a}\times\vec{y} - \vec{b}\times\vec{x}\right) \end{matrix}\right) + \\ &\left(\begin{matrix} \mathrm{i}\left(a_0y_0 + b_0x_0 - \vec{a}\cdot\vec{y} - \vec{b}\cdot\vec{x}\right) \\ a_0\vec{y} + x_0\vec{b} - b_0\vec{x} - y_0\vec{a} - \vec{a}\times\vec{x} + \vec{b}\times\vec{y} \end{matrix}\right] \ . \end{aligned} \tag{1}$$

The biquaternion multiplication is not commutative but associative. For the biquaternion scalar multiplication the Hamilton units are not multiplied and thus

$$\boldsymbol{A}\cdot\boldsymbol{X} = a_0x_0 - \vec{a}\cdot\vec{x} - \left(b_0y_0 - \vec{b}\cdot\vec{y}\right) \ .$$

Thereof follows the magnitude of a biquaternion

$$|\boldsymbol{Q}| = \sqrt{\boldsymbol{Q}\cdot\boldsymbol{Q}} = \sqrt{a_0^2 - \vec{a}\cdot\vec{a} - \left(b_0^2 - \vec{b}\cdot\vec{b}\right)} \ . \tag{2}$$

A special case is given when the second of the two semi-biquaternions is set to zero. With $b = 0$ and $\vec{y} = 0$, the multiplication of two biquaternions reduces to

$$\boldsymbol{AX} = \left[\begin{matrix} a_0x_0 + \vec{a}\cdot\vec{x} \\ \mathrm{i}\,(a_0\vec{x} + x_0\vec{a}) \end{matrix}\right) + \left(\begin{matrix} 0 \\ -\vec{a}\times\vec{x} \end{matrix}\right] \ .$$

The multiplication of a semi-biquaternion with an other one results almost in another semi-biquaternion, except of an additional cross product.

2 Biquaternion Velocity

We have shown some time ago that such semi-biquaternions can be applied well to electrodynamics [3]. In this paper we focus on mechanics, and especially on special relativity. We start with the position biquaternion representing an event in space-time

$$\boldsymbol{X} = \left[\begin{matrix} ct \\ \mathrm{i}\vec{x} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right] \tag{3}$$

where the constant c is the speed of light. With

$$d\boldsymbol{X} = \left[\begin{matrix} cdt \\ \mathrm{i}d\vec{x} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

and with the proper time interval $d\tau$

$$d\,|\boldsymbol{X}| = \sqrt{c^2(dt)^2 - d\vec{x}\cdot d\vec{x}} = cdt\sqrt{1-\frac{\vec{v}\cdot\vec{v}}{c^2}} = cd\tau\;.$$

follows the biquaternion velocity

$$\boldsymbol{V} = \frac{d\boldsymbol{X}}{d\tau} = \gamma\left[\begin{matrix} c \\ \mathrm{i}\vec{v} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right] \tag{4}$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\;.$$

Independent of the magnitude of $\vec{v}$, the magnitude of the biquaternion velocity is always c. Thus the unity biquaternion velocity is

$$\boldsymbol{V}^0 = \gamma\left[\begin{matrix} 1 \\ \mathrm{i}\frac{\vec{v}}{c} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right] \tag{5}$$

or

$$\frac{d\boldsymbol{X}}{d|\boldsymbol{X}|} = \frac{\boldsymbol{V}}{|\boldsymbol{V}|} = \boldsymbol{V}^0 \tag{6}$$

The inverse of a unity biquaternion velocity is identical to its biconjugate, which in turn is again a unity biquaternion velocity

$$\frac{1}{\boldsymbol{V}^0} = \boldsymbol{V}^{0*} \tag{7}$$

with

$$\boldsymbol{V}^{0*} = \gamma\left[\begin{matrix} 1 \\ -\mathrm{i}\frac{\vec{v}}{c} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right]\;.$$

3 Biquaternion Lorentz Transformation

S shall be the inertial system where an observer is at rest. Another inertial system S' shall move with constant velocity $\vec{v}$ against S. The distance between two events in four-space is invariant for both systems

$$d|\boldsymbol{X}| = d|\boldsymbol{X}'|\;.$$

Also the square of these length elements is an invariant and therefore

$$d\boldsymbol{X}\cdot d\boldsymbol{X} = d\boldsymbol{X}'\cdot d\boldsymbol{X}' = d^2|\boldsymbol{X}| = d^2|\boldsymbol{X}'|\;.$$

With equation (6) follows

$$\boldsymbol{V}^0\cdot\boldsymbol{V}^0 = \frac{d\boldsymbol{X}\cdot d\boldsymbol{X}}{d\boldsymbol{X}'\cdot d\boldsymbol{X}'} \tag{8}$$

This directly leads to the biquaternion coordinate transformation from S to S'

$$\boldsymbol{X}' = \boldsymbol{X}\boldsymbol{V}^{0*} \tag{9}$$

as well as the backward transformation

$$\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{V}^0 \tag{10}$$

Explicitly the forward and backward transformations are

$$\boldsymbol{X}' = \left[\begin{matrix} ct' \\ \mathrm{i}\vec{x}' \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right] = \gamma\left[\begin{matrix} ct - \frac{\vec{v}\cdot\vec{v}}{c} \\ \mathrm{i}\,(\vec{x}-\vec{v}t) \end{matrix}\right) + \left(\begin{matrix} 0 \\ \frac{\vec{x}\times\vec{v}}{c} \end{matrix}\right] \tag{11}$$

$$\boldsymbol{X} = \left[\begin{matrix} ct \\ \mathrm{i}\vec{x} \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right] = \gamma\left[\begin{matrix} ct' + \frac{\vec{v}\cdot\vec{v}}{c} \\ \mathrm{i}\,(\vec{x}' + \vec{v}t') \end{matrix}\right) + \left(\begin{matrix} 0 \\ -\frac{\vec{x}'\times\vec{v}}{c} \end{matrix}\right]\;. \tag{12}$$

Equation (11) and equation (12) reveal, that

$$\vec{x}\times\vec{v} = 0 \quad\text{and}\quad \vec{x}'\times\vec{v} = 0\;.$$

Above transformations are valid for the special case of collinear vectors $\vec{x}\|\vec{v}$ and $\vec{x}'\|\vec{v}$ only. The general case is found by transforming this part of position vector $\boldsymbol{X}$ with a spatial vector parallel to $\vec{v}$ and add the remaining part of $\boldsymbol{X}$ with a spatial vector orthogonal to $\vec{v}$. With

$$\boldsymbol{X}_\| = \left[\begin{matrix} ct \\ \mathrm{i}\left(\frac{\vec{x}\cdot\vec{v}}{v}\frac{\vec{v}}{v}\right) \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

the general forward transformation is

$$\boldsymbol{X}' = \boldsymbol{X}_\perp + \boldsymbol{X}_\|\boldsymbol{V}^{0*} = \boldsymbol{X} - \boldsymbol{X}_\|\left(1-\boldsymbol{V}^{0*}\right)\,, \tag{13}$$

or explicitly

$$\boldsymbol{X}' = \left[\begin{matrix} \gamma\left(ct - \frac{\vec{v}\cdot\vec{v}}{c}\right) \\ \mathrm{i}\left(\vec{x} - (1-\gamma)\,\frac{\vec{x}\cdot\vec{v}}{v}\frac{\vec{v}}{v} - \gamma\vec{v}t\right) \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right]\,,$$

whereas the general backward transformation is

$$\boldsymbol{X} = \boldsymbol{X}'_\perp + \boldsymbol{X}'_\|\boldsymbol{V}^0 = \boldsymbol{X}' - \boldsymbol{X}'_\|\left(1-\boldsymbol{V}^0\right)\,, \tag{14}$$

or explicitly

$$\boldsymbol{X} = \left[\begin{matrix} \gamma\left(ct + \frac{\vec{v}\cdot\vec{v}}{c}\right) \\ \mathrm{i}\left(\vec{x}' - (1+\gamma)\,\frac{\vec{x}'\cdot\vec{v}}{v}\frac{\vec{v}}{v} + \gamma\vec{v}t\right) \end{matrix}\right) + \left(\begin{matrix} 0 \\ 0 \end{matrix}\right]\;.$$

No new Lorentz transformation has been described above. But it might be of some interest to see that the Lorentz transformation can be described with a multiplication of the position biquaternion with the velocity biquaternion of relative movement. The component of the position biquaternion parallel to the velocity biquaternion is

$$\left(\boldsymbol{X}\cdot\boldsymbol{V}^0\right)\boldsymbol{V}^0\;.$$

Its Lorentz transformation is

$$\left(\boldsymbol{X}\cdot\boldsymbol{V}^0\right)\boldsymbol{V}^0\boldsymbol{V}^{0^*} = \gamma\left(ct-\frac{\vec{x}\cdot\vec{v}}{c}\right) .$$

The result has no spatial position vector at all and has a pure timely character in all cases, i.e. independent of the direction of the unprimed position biquaternion or of its spatial vector. Thus, for every observer in S, the direction of the time vector (scalar part) in any moving system S' coincides with the direction $\boldsymbol{V}^0$ of the relative four-velocity.

To find the addition law of two biquaternion velocities we denote the relative biquaternion velocity between two systems S and S' as $\boldsymbol{V}$, and the velocity of a moving body measured in S' shall be denoted as $\boldsymbol{U}'$. The total biquaternion velocity $\boldsymbol{W}$ measured in S is found by the backward transformation of $\boldsymbol{U}'$ as

$$\boldsymbol{W} = \boldsymbol{U}'\boldsymbol{V}^0 = \gamma_u\gamma_v\left[\begin{matrix} c+\frac{\vec{u}'\cdot\vec{v}}{c} \\ \mathrm{i}\left(\vec{u}'+\vec{v}\right)\end{matrix}\right] + \gamma_u\gamma_v\left[\begin{matrix} 0 \\ \frac{\vec{u}'\times\vec{v}}{c}\end{matrix}\right]$$

with

$$\gamma_u = \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} \quad \text{and} \quad \gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} .$$

Some calculation directly leads to the total velocity

$$\boldsymbol{W} = \gamma_w\left[\begin{matrix} c \\ \mathrm{i}\vec{w}\end{matrix}\right) + \left(\begin{matrix} 0 \\ -\frac{\vec{w}\times\vec{v}}{c}\end{matrix}\right]$$

with

$$\gamma_w = \frac{1}{\sqrt{1-\frac{w^2}{c^2}}}$$

and the law of velocity addition known from Special Relativity

$$\vec{w} = \frac{\vec{u}'+\vec{v}}{1+\frac{\vec{u}'\cdot\vec{v}}{c^2}} .$$

Again, the magnitude of $\boldsymbol{W}$ is equal to the speed of light c.

4 Biquaternion Field Transformations

The biquaternion electromagnetic potential

$$\boldsymbol{A} \equiv \left[\begin{matrix} \frac{1}{c}\varphi \\ \vec{A}\end{matrix}\right) + \left(\begin{matrix} 0 \\ 0\end{matrix}\right] \tag{15}$$

is transformed exactly in the same way as the biquaternion position vector. For completeness, the forward transformation is

$$\boldsymbol{A}' = \gamma\left[\begin{matrix} \frac{\varphi}{c}-\frac{\vec{v}\cdot\vec{A}}{c} \\ \mathrm{i}\left(\vec{A}-\frac{\varphi}{c^2}\vec{v}\right)\end{matrix}\right) + \left(\begin{matrix} 0 \\ \frac{\vec{A}\times\vec{v}}{c}\end{matrix}\right] ,$$

which again is valid for $\vec{A}\parallel\vec{v}$ only. The general case is found with

$$\boldsymbol{A}' = \left[\begin{matrix} \gamma\left(\frac{\varphi}{c}-\frac{\vec{A}\cdot\vec{v}}{c}\right) \\ \mathrm{i}\left(\vec{A}-(1-\gamma)\,\frac{\vec{A}\cdot\vec{v}}{v}\frac{\vec{v}}{v}-\gamma\frac{\varphi}{c^2}\vec{v}\right)\end{matrix}\right) + \left(\begin{matrix} 0 \\ 0\end{matrix}\right] . \tag{16}$$

The electromagnetic fields follow from the differentiation of the potential fields. With the definition of the biquaternion differential operator

$$\nabla \equiv \frac{\partial}{\partial\boldsymbol{X}} = \left[\begin{matrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla}\end{matrix}\right) + \left(\begin{matrix} 0 \\ 0\end{matrix}\right] . \tag{17}$$

follow the biquaternion electric and magnetic fields

$$\boldsymbol{E} = c\nabla\boldsymbol{A} = \left[\begin{matrix} c\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t}-\vec{\nabla}\cdot\vec{A}\right) \\ \mathrm{i}\vec{E}\end{matrix}\right) + \left(\begin{matrix} 0 \\ c\vec{B}\end{matrix}\right] ,$$

$$\boldsymbol{B} = -\mathrm{i}\nabla\boldsymbol{A} = \left[\begin{matrix} 0 \\ \mathrm{i}\vec{B}\end{matrix}\right) + \left(\begin{matrix} \mathrm{i}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t}-\vec{\nabla}\cdot\vec{A}\right) \\ -\frac{\vec{E}}{c}\end{matrix}\right] ,$$

where $\boldsymbol{E} = -\mathrm{i}c\boldsymbol{B}$. With the Lorentz gauge

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t}-\vec{\nabla}\cdot\vec{A} = 0$$

the biquaternion electric and magnetic fields without a scalar part are found

$$\boldsymbol{E} = \left[\begin{matrix} 0 \\ \mathrm{i}\vec{E}\end{matrix}\right) + \left(\begin{matrix} 0 \\ c\vec{B}\end{matrix}\right] , \tag{18}$$

$$\boldsymbol{B} = \left[\begin{matrix} 0 \\ \mathrm{i}\vec{B}\end{matrix}\right) + \left(\begin{matrix} 0 \\ -\frac{\vec{E}}{c}\end{matrix}\right] . \tag{19}$$

With the electric field biquaternion (18), the forward transformation is

$$\begin{aligned}\boldsymbol{E}' &= \left[\begin{matrix} 0 \\ \mathrm{i}\vec{E}'\end{matrix}\right) + \left(\begin{matrix} 0 \\ \vec{B}'\end{matrix}\right] \\ &= \boldsymbol{E}\,\boldsymbol{V}^{0^*} = \gamma\left[\begin{matrix} -\frac{1}{c}\vec{v}\cdot\vec{E} \\ \mathrm{i}\left(\vec{E}+\vec{v}\times\vec{B}\right)\end{matrix}\right) + \gamma\left(\begin{matrix} \vec{v}\cdot\vec{B} \\ -\frac{\vec{E}}{c}\end{matrix}\right] .\end{aligned} \tag{20}$$

Obviously this transformation is only applicable for

$$\vec{v}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{v}\cdot\vec{B} = 0 .$$

Thus only the field components orthogonal to the spatial velocity vector are transformed, but not the collinear field components. With

$$\boldsymbol{E}_{\parallel} = \left[\begin{matrix} 0 \\ \mathrm{i}\left(\frac{\vec{v}\cdot\vec{E}}{v}\frac{\vec{v}}{v}\right)\end{matrix}\right) + \left(\begin{matrix} 0 \\ 0\end{matrix}\right]$$

we find the general transformation for electromagnetic fields

$$\boldsymbol{E}' = \boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}\,\boldsymbol{V}^{0^*} = \boldsymbol{E}\,\boldsymbol{V}^{0^*} + \boldsymbol{E}_{\parallel}\left(1-\boldsymbol{V}^{0^*}\right) . \tag{21}$$

5 Biquaternion Total Time Derivative

With any biquaternion field

$$\boldsymbol{A} = \left[\begin{matrix} a_0 \\ \mathrm{i}\vec{a}\end{matrix}\right) + \left(\begin{matrix} 0 \\ 0\end{matrix}\right]$$

that is a function of a biquaternion position

$$\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{X}) = \boldsymbol{A}(\boldsymbol{X}(ct,\ x_1,\ x_2,\ x_3))\ .$$

the total time differential is

$$d\boldsymbol{A} = \frac{\partial \boldsymbol{A}(\boldsymbol{X})}{\partial \boldsymbol{X}} d\boldsymbol{X} = \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial t} + \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial \vec{x}} d\vec{x}\ .$$

With

$$\frac{\partial}{\partial \boldsymbol{X}} = \nabla \quad \text{and} \quad \frac{\partial}{\partial \vec{x}} = \vec{\nabla}$$

follows after some calculation

$$\frac{d\boldsymbol{A}}{dt} = \left[\begin{pmatrix} \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \\ \mathrm{i}\left(c\vec{\nabla} + \frac{\vec{v}}{c} \circ \frac{\partial}{\partial t}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ -\vec{v} \circ \vec{\nabla} \end{pmatrix} \right] \boldsymbol{A}\ ,$$

where the multiplication symbol $\circ$ denotes that for the scalar part the scalar product and for the vector part the cross product must be applied. Explicitly this becomes

$$\frac{d\boldsymbol{A}}{dt} = \left[\begin{pmatrix} \frac{\partial a_0}{\partial t} + \vec{v}\cdot\vec{\nabla}a_0 + c\vec{\nabla}\cdot\vec{a} + \vec{v}\cdot\frac{\vec{a}}{\partial t} \\ \mathrm{i}\left(\frac{\partial\vec{a}}{\partial t} + \vec{v}\left(\vec{\nabla}\cdot\vec{a}\right) + c\vec{\nabla}a_0 + \frac{\vec{v}}{c}\frac{\partial a_0}{\partial t} - \vec{v}\times\left(\vec{\nabla}\times\vec{a}\right)\right) \end{pmatrix} + \begin{pmatrix} \mathrm{i}\vec{v}\cdot\left(\vec{\nabla}\times\vec{a}\right) \\ -c\vec{\nabla}\times\vec{a} - \vec{v}\times\left(\frac{\partial\vec{a}}{c\partial t} + \vec{\nabla}a_0\right) \end{pmatrix} \right]\ . \tag{22}$$

Interestingly this is completely equivalent to

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{\gamma}\boldsymbol{V}\nabla\boldsymbol{A}$$

such that the relativistic total time derivative operator in biquaternion form is

$$\frac{d}{d\tau} = \boldsymbol{V}\nabla\ . \tag{23}$$

6 Biquaternion Acceleration and Force

With $c = \text{constant}$ and with the relativistic spatial acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \tag{24}$$

the relativistic biquaternion acceleration $\boldsymbol{a}$ can be written directly as

$$\boldsymbol{a} \equiv \left[\begin{pmatrix} a_0 \\ \mathrm{i}\vec{a} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \frac{d\boldsymbol{V}}{d\tau} = \left[\begin{pmatrix} \gamma^4 \frac{\vec{v}\cdot\vec{a}}{c} \\ \mathrm{i}\gamma^2\left(\gamma^2\frac{\vec{v}\cdot\vec{a}}{c}\frac{\vec{v}}{c} + \vec{a}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]\ .$$

Instead of using equation (24), the operator (23) can be applied to find for the relativistic biquaternion acceleration

$$\boldsymbol{a} \equiv \frac{d\boldsymbol{V}}{d\tau} = \boldsymbol{V}\nabla\boldsymbol{V}\ . \tag{25}$$

We will not evaluate this equation for relativistic velocities in this paper. With the non-relativistic velocity $\boldsymbol{U}$, the non-relativistic acceleration becomes

$$\boldsymbol{a} = \frac{d\boldsymbol{U}}{dt} = \boldsymbol{U}\nabla\boldsymbol{U} \quad \text{with} \quad \boldsymbol{U} \equiv \left[\begin{pmatrix} c \\ \mathrm{i}\frac{c_0}{c}\vec{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]\ .$$

The biquaternion force to accelerate a mass with non-relativistic velocities is then

$$\boldsymbol{F} \equiv m\boldsymbol{a} = m\left[\begin{pmatrix} P \\ \mathrm{i}\vec{F} \end{pmatrix} + \begin{pmatrix} 0 \\ c\vec{\Omega} \end{pmatrix}\right]\ , \tag{26}$$

or explicitly

$$\boldsymbol{F} = m\left[\begin{pmatrix} \frac{\partial c}{\partial t} + \vec{v}\cdot\vec{\nabla}c + c\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\frac{\vec{v}}{\partial t} \\ \mathrm{i}\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\left(\vec{\nabla}\cdot\vec{v}\right) + c\vec{\nabla}c + \frac{\vec{v}}{c}\frac{\partial c}{\partial t} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\right) \end{pmatrix} + \begin{pmatrix} \mathrm{i}\vec{v}\cdot\left(\vec{\nabla}\times\vec{v}\right) \\ -c\vec{\nabla}\times\vec{v} - \vec{v}\times\left(\frac{\partial\vec{v}}{c\partial t} + \vec{\nabla}c\right) \end{pmatrix} \right]\ . \tag{27}$$

The non-relativistic spatial Force $\vec{F}$ becomes

$$\vec{F} = m\vec{a} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\left(\vec{\nabla}\cdot\vec{v}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) + c\vec{\nabla}c + \frac{\vec{v}}{c}\frac{\partial c}{\partial t}\ . \tag{28}$$

With $c = \text{constant}$ and replacing the second term with a vector identity follows

$$\vec{a} = \frac{\partial\vec{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)\ ,$$

which corresponds to the well known acceleration field in fluid mechanics [4].

7 Gravitation Potential

In equation (28) we find two other terms that are of particular interest:

$$\vec{F} \sim c\vec{\nabla}c + \frac{\vec{v}}{c}\frac{\partial c}{\partial t}\ .$$

With $c = \text{constant}$, thus independent of space and time, both terms would be zero. But there are already theories of time varying speed of light (VSL theories), which have been introduced to explain the horizon problem of cosmology and propose an alternative to cosmic inflation (see for example [5] and citations therein). As a straight forward approach we assume that $c = c(\vec{x})$. To still have a four-space $(x_0, \vec{x})$ with four independent dimensions, the time interval dx_0 needs to be independent of spatial interval $d\vec{x}$ and vice-versa. With $c \neq$ constant, the interval dx_0 shall be invariant to the local speed of light. Thus we have

$$dx_0 = c_0 dt = c dt_c\ ,$$

where c_0 is the constant speed of light in vacuum as commonly defined. The time interval in a region with $c \neq$ constant changes according to

$$dt_c = \frac{c_0}{c}dt\ . \tag{29}$$

Thus the interval of time dimension x_0 is independent of c. The velocity in a region with $c \neq$ constant is measured with local time interval

$$\vec{v} = \frac{d\vec{x}}{dt_c} = \frac{c}{c_0}\frac{d\vec{x}}{dt}\ .$$

This satisfies the principle of causality where local velocity $\vec{v}$ cannot be higher than local speed of light c. The invariant length element of four-space is now

$$d\boldsymbol{X}_c = dt\left[\begin{pmatrix} c \\ \mathrm{i}\frac{c_0}{c}\vec{v}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = cdt\sqrt{1-\frac{c_0^2}{c^4}v^2} = cd\tau_c \qquad (30)$$

and the biquaternion velocity (4) changes to

$$\boldsymbol{V}_c = \frac{d\boldsymbol{X}_c}{d\tau_c} = \gamma_c\left[\begin{pmatrix} c \\ \mathrm{i}\frac{c_0}{c}\vec{v}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] \qquad (31)$$

with

$$\gamma_c = \frac{1}{\sqrt{1-\frac{c_0^2}{c^4}v^2}} .$$

Also the biquaternion differential operator (17) changes to

$$\nabla_c \equiv \frac{\partial}{\partial \boldsymbol{X}_c} = \left[\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] , \qquad (32)$$

where c is not a constant anymore. The total time derivative operator (23) further changes to

$$\frac{d}{d\tau_c} = \boldsymbol{V}_c \nabla_c . \qquad (33)$$

In non-relativistic case, the acceleration $\boldsymbol{a}$ of a non-relativistic velocity $\boldsymbol{U}$ can then be written as

$$\boldsymbol{a} = \frac{d\boldsymbol{U}}{dt} = \boldsymbol{U}\nabla\boldsymbol{U} \approx \boldsymbol{V}_c\nabla_c\boldsymbol{V}_c = \frac{d\boldsymbol{V}_c}{d\tau_c} = \boldsymbol{a}_c .$$

In non-relativistic case this still corresponds to equation (27). The inertia force to a mass m just becomes $\boldsymbol{F}_g = -m\boldsymbol{a}$. Since we are only interested in derivations of c, above equation can be reduced to

$$\boldsymbol{g} = \frac{dc}{dt} = \boldsymbol{U}\nabla c . \qquad (34)$$

The inertia force $\boldsymbol{F}_g$ acting to an accelerated mass m is then

$$\boldsymbol{F}_g = -m\boldsymbol{g} = -mc\vec{\nabla}c - m\frac{\vec{v}}{c}\frac{\partial c}{\partial t} .$$

Considering only the spatial variation, thus with c constant in time, above equation reduces further to

$$\boldsymbol{F}_g = -mc\vec{\nabla}c = -m\nabla\frac{c^2}{2} = -m\nabla\phi . \qquad (35)$$

The scalar ϕ can be interpreted as a gravity potential with units $[m^2/s^2]$ of any mass M as

$$\phi(\vec{r}) = \frac{c(M,\vec{r})^2}{2} . \qquad (36)$$

In our model, the non-relativistic gravity potential energy V of a mass m in any gravity field ϕ is then

$$V = m\phi = \frac{1}{2}mc^2 , \qquad (37)$$

what could be interpreted as the 'kinetic energy' of a mass m 'moving' with local speed of light c along the time coordinate axis x_0.

8 Gravitation of two Body System

In a system with two point masses $M \gg m$, where M is the central mass, m the orbiting mass, and ϕ the gravity potential of M, the speed of light shall be a function of M as well as of the distance $\vec{r}$ measured from M to m, i.e.

$$c = c_0 f(M,\vec{r}) .$$

The function $f(M,\vec{r})$ shall fulfill these boundary conditions:

- $0 \le f(M,\vec{r}) \le 1$ for $0 \le r \le \infty$, and
- $f(M,\vec{r}) = 1$ with $r \to \infty$, and
- the spatial variation of c must be below the today's precision of light speed measurements.

In plain spherical coordinates with radial symmetry, where M is in the origin, the inertia force $\boldsymbol{F}_g$ to mass m is

$$\boldsymbol{F}_g = -mc\vec{\nabla}c = mc_0^2 f(M,r)\frac{\partial f(M,r)}{\partial \vec{r}} ,$$

where r denotes the radius measured from M to m. Above equation shall be equal to the Newton gravity force

$$\boldsymbol{F}_g = -G\frac{Mm_c}{r^2}\vec{r}^0 .$$

Here the proper mass m_c has been used, since - with conservation of total energy-momentum - the mass is also subject to $f(M,\vec{r})$ as

$$E = \gamma_c mc^2 \approx mc^2 = mc_0^2 f^2(M,r) = m_c c_0^2 . \qquad (38)$$

Thus we have

$$mc_0^2 f(M,r)\frac{\partial f(M,r)}{\partial \vec{r}} = -G\frac{Mm}{r^2}f^2(M,r)\vec{r}^0 .$$

By integration and using above boundary conditions the explicit solution becomes

$$f(M,r) = e^{-\frac{GM}{c_0^2 r}} ,$$

and thereof the spatial varying speed of light around M to

$$c(M,r) = c_0\, e^{-\frac{GM}{c_0^2 r}} = e^{-\frac{r_S}{2r}} , \qquad (39)$$

where r_S is the Schwarzschild radius. The same spatial variation of c has been proposed by Puthoff [6]. Newton's law of gravity attraction between two bodies changes slightly to

$$\boldsymbol{F}_g = -G\frac{Mm_c}{r^2}\, e^{-\frac{GM}{c_0^2 r}}\vec{r}^0 \quad \text{with} \quad \vec{r} = \vec{r}_m - \vec{r}_M .$$

For r$\gg r_S$ the deviation of $1/r^2$ is far beyond today's measurement capabilities, but below r_S the gravity attraction for point masses turns to zero instead to infinity.

Using 39 and the first terms of the corresponding Taylor series, the total energy of a mass m becomes approximately

$$E = \gamma_c mc^2 = \gamma_c mc_0^2\, e^{\frac{-GM}{c_0 r}} \approx mc_0^2 + \frac{1}{2}mv^2 - m\frac{GM}{r} + \ldots ,$$

where the last term represents the traditional gravity potential energy of m in gravity field of M.

9 Planet Orbit with new Gravity Potential

The biquaternion momentum is

$$\boldsymbol{p}_c = m\boldsymbol{V}_c = m\left[\begin{pmatrix} c \\ \mathrm{i}\frac{c_0}{c}\vec{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] . \tag{40}$$

For a constant speed of light, the magnitude of any four-dimensional speed is constantly c and the four-dimensional momentum of a body with mass m is always conserved. With presence of gravitation, this momentum shall still be conserved. This is achieved by equalizing the magnitudes of four-dimensional velocity without and with gravity field. We start with the squared magnitude of four-dimensional velocity

$$c_0^2 = \gamma_c^2 \left(c^2 - \frac{c_0^2}{c^2} v^2\right) .$$

For non-relativistic velocities and with (39) this can be written as

$$mc_0^2 = mc_0^2\, e^{-\frac{GM}{c_0^2 r}} - mv^2\, e^{\frac{GM}{c_0^2 r}} .$$

Introducing plain polar coordinates (r: radius, φ: rotation angle) and replacing the exponential functions by the first terms of its Taylor series, it follows

$$mc_0^2 = mc_0^2 \left(1 - \frac{r_S}{r}\right) - m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\left(1 - \frac{r_S}{r}\right)^{-1} .$$

For radial symmetric and isotropic conditions this reduces further to

$$m\dot{r}^2 + \frac{L^2}{mr^2} - \frac{L^2}{mr^2}\frac{r_S}{r} - mc_0^2\frac{r_S}{r} = mc_0^2\left(1 - \frac{r_S}{r}\right)^2 - mc_0^2 ,$$

where the angular momentum $L = mr^2\dot{\varphi}$ has been used. The angular momentum is conserved for central force fields. Finally, by using the potential energy

$$E = mc^2 \approx mc_0^2\left(1 - \frac{r_S}{r}\right)$$

the equation of motion for m around M known from General Relativity is found as

$$\frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r} + \frac{L^2}{2mr^2} - G\frac{L^2 M}{mc_0^2 r^3} = \frac{E^2}{2mc_0^2} - \frac{mc_0^2}{2} . \tag{41}$$

The perihelion precession can now be calculated as done in standard text books. Finally, by using the four-dimensional length element (30) together with equation (39), whereas the first terms of its Taylor series are used again, it follows

$$dX^2 = dt^2 c_0^2\left(1 - \frac{2GM}{c_0^2 r}\right) - dx^2\left(1 - \frac{2GM}{c_0^2 r}\right)^{-1} .$$

Using polar coordinates and again assuming radial symmetry and isotropic conditions follows directly the Schwarzschild metric

$$dX^2 = dt^2 c_0^2\left(1 - \frac{2GM}{c_0^2 r}\right) - dr^2\left(1 - \frac{2GM}{c_0^2 r}\right)^{-1} - r^2 d\varphi^2 - r^2\sin^2\varphi\, d\theta . \tag{42}$$

10 Gravitation Time Dilation and Redshift

Using equation (39) with (29), the time dilation in a gravity field of M is found approximatively with Taylor series as

$$dt_c = e^{-\frac{GM}{c_0^2 r}}\, dt_0 \approx \left(1 - \frac{GM}{c_0^2 r}\right) dt_0 \approx \sqrt{1 - \frac{2GM}{c_0^2 r}}\, dt_0 . \tag{43}$$

The gravitational redshift is defined to

$$z = \frac{\lambda_0}{\lambda_c} - 1 ,$$

where the wavelength λ_0 is measured at a distant point of observation and λ_c closer to the mass M that causes the gravity potential. The closer point usually is regarded as the emitting point (of the sun or other celestial bodies) and the distant point as the receiving point (on Earth). For simplicity we assume, that emitter and receiver do maintain a constant radial distance r. The photon energy does not change along its way according to (38). Thus, the frequency at emitting point must be identical to receiver point (i.e. there is no doppler effect). With $\lambda = c/\nu$ we directly find

$$z = \frac{c_0}{c} - 1 = \frac{1}{e^{-\frac{GM}{c_0^2 r}}} - 1 \approx \frac{1}{\sqrt{1 - \frac{r_S}{r}}} - 1 . \tag{44}$$

This is the gravity redshift well known from General Relativity. But in our model the photon frequency and thus photon energy (four-dimensional momentum) is conserved, where the speed of light changes instead.

11 Generalized Biquaternion Inertia and Gravitation

A generalization of inertia and gravitation can be achieved by consequently interpret the four-velocity $\boldsymbol{V}_c$ as a 'potential field for inertia and gravity'.

With the derivation of four-velocity $\boldsymbol{V}_c$

$$\nabla_c \boldsymbol{V}_c = \left[\begin{pmatrix} \frac{1}{c}\frac{\partial c}{\partial t} + \vec{\nabla}\cdot\vec{u} \\ \mathrm{i}\left(\vec{\nabla}c + \frac{1}{c}\frac{\partial\vec{u}}{\partial t}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{\nabla}\times\vec{u} \end{pmatrix}\right] . \tag{45}$$

and with the substitutions

$$g_0 = -\frac{1}{c}\frac{\partial c}{\partial t} - \vec{\nabla}\cdot\vec{u}$$
$$\vec{g} = -c\vec{\nabla}c - \frac{\partial\vec{u}}{\partial t}$$
$$\vec{\omega} = \vec{\nabla}\times\vec{u}$$

the a biquaternion 'acceleration field' can be identified to

$$\boldsymbol{g} = -c\nabla\boldsymbol{U} = \left[\begin{pmatrix} cg_0 \\ \mathrm{i}\vec{g} \end{pmatrix} + \begin{pmatrix} 0 \\ c\vec{\omega} \end{pmatrix}\right] . \tag{46}$$

The similarity between $\vec{g}$ and the electric field $\vec{E}$ as well as between $\vec{\omega}$ and the magnetic induction $\vec{B}$ is obvious. The vector field $\vec{g}$ is an acceleration field. The vector field $\vec{\omega}$ might be visualized as a spin (or rotation) field.

With the biquaternion d'Alembert operator

$$\Delta_c \equiv |\nabla_c|^2 = \nabla_c^* \nabla_c = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla} \tag{47}$$

equation (45) can be expanded further to

$$\nabla_c^* \nabla_c \boldsymbol{V}_c = \frac{G}{c^2}\boldsymbol{P} = \frac{G}{c^2}\rho \boldsymbol{V}_c ,$$

where G is the gravity constant, ρ is the mass density and $\boldsymbol{P}$ is the biquaternion momentum density

$$\boldsymbol{P} \equiv \rho \boldsymbol{V}_c = \rho \left[\begin{pmatrix} c \\ \mathrm{i}\frac{c_0}{c}\vec{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \quad \text{with} \quad \rho = \gamma_c \rho_0 . \tag{48}$$

With the 'kinetic Lorentz condition' g_0=0 follow four equations similar to Maxwell's equations as

$$\begin{aligned} \vec{\nabla} \cdot \vec{\omega} &= 0 , \\ \vec{\nabla} \cdot \vec{g} &= G\rho , \\ \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times \vec{g} &= 0 , \\ \vec{\nabla} \times \vec{\omega} - \frac{1}{c^2}\frac{\partial \vec{g}}{\partial t} &= \frac{G}{c^2}\rho \vec{u} . \end{aligned}$$

The further analysis of these equations is beyond the scope of this paper. They might be interesting for fluid mechanics as well as for cosmology, for example the analysis of large cosmic structures. With

$$\Delta \boldsymbol{V}_c = \frac{G}{c^2}\rho \boldsymbol{V}_c \quad \text{and} \quad g_0 = 0$$

follow the inhomogeneous wave equations

$$\begin{aligned} \frac{1}{c^2}\frac{\partial \vec{g}}{\partial t} - \vec{\nabla}^2 \vec{g} &= -\frac{G}{c^2}\left(\vec{\nabla}(\rho c) + \frac{\partial(\rho \vec{u})}{\partial t} \right) , \\ \frac{1}{c^2}\frac{\partial \vec{\omega}}{\partial t} - \vec{\nabla}^2 \vec{\omega} &= \frac{G}{c^2}\vec{\nabla} \times (\rho \vec{u}) . \end{aligned}$$

Thereof follow, in absence of mass densities, the homogenous wave equations. The result is the prediction of free transverse gravitation waves with propagation velocity c.

12 Electric Field of a Point Charge

Since c is not constant, also the permittivity ϵ_0 and permeability μ_0 of the vacuum must change together with c

$$c_0^2 f^2(\vec{x}) = \frac{f(\vec{x})}{\epsilon_0}\frac{f(\vec{x})}{\mu_0} .$$

Thus both vacuum properties ϵ_0 and μ_0 change inversely to c, whereas the vacuum impedance remains constant [6]. For example, the electric potential of a single point charge is

$$\varphi = \frac{e}{4\pi\epsilon_0 r} e^{-\frac{Gm}{c_0^2 r}} \vec{r}^0$$

where m is the particle's rest mass. To find the electric field of a point charge, the modified biquaternion electric potential

$$\boldsymbol{A} \equiv \left[\begin{pmatrix} \frac{1}{c}\varphi \\ \mathrm{i}\vec{A} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \quad \text{with} \quad c = c_0 f(\vec{x}) . \tag{49}$$

is required. The electric field biquaternion follows directly as

$$\boldsymbol{E} = -c\nabla \boldsymbol{A} = \left[\begin{pmatrix} \frac{1}{c}\frac{\partial \varphi}{\partial t} + c\vec{\nabla} \cdot \vec{A} - \frac{\varphi}{c^2}\frac{\partial c}{\partial t} \\ \mathrm{i}\left(\vec{\nabla}\varphi + \frac{\partial \vec{A}}{\partial t} - \frac{\varphi}{c}\vec{\nabla} c \right) \end{pmatrix} + \begin{pmatrix} 0 \\ c\vec{\nabla} \times \vec{A} \end{pmatrix} \right] .$$

Thereof follows the electric field of an electric potential φ as

$$\vec{E} = -\vec{\nabla}\varphi + \frac{\varphi}{c}\vec{\nabla} c .$$

In particular, for a point charge, this becomes

$$\vec{E} = \frac{e}{4\pi\epsilon_0 r^2} e^{-\frac{Gm}{c_0^2 r}} \vec{r}^0 . \tag{50}$$

The electric field of a charged point particle has the same characteristics as it's gravity field.

13 Resume

Casting physical quantities with biquaternions is not essential as such, but it allows a straight-forward, 'intuitive' approach to describe physical models with 'natural numbers' just using the basic arithmetics of addition and multiplication together with some calculus algebra.

The four-dimensional velocity plays a key role to understand special relativity, field transformations, energy-momentum conservation as well as gravitation.

In our proposed model, the speed of light is not a constant but depends on masses (or localized energy densities). As we have roughly shown, this model can offer an alternative to understand the cause of gravity. The principle of equivalence, that laid the foundation of General Relativity, has been enhanced by a - let's say - 'principle of identity'. Both, inertia and gravitation have the same identical origin, namely the total time derivative of the velocity four-vector.

Issued on January 01, 2011.

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